Chapter 4 Applications of the Derivative

4.1 Related Rates

Concepts and Vocabulary

1. If a spherical balloon of volume V is inflated at a rate of 10 m^3/min, where t is the time (in minutes), then the rate of change of V with respect to t is

$$\frac{dV}{dt} = \boxed{10\ \text{m}^3/\text{min}}.$$

3. Let $x^2 + y^2 = 25$. Then, differentiating with respect to t yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}.$$

Given that $x = 3$, $y = 4$ and $\dfrac{dy}{dt} = 2$, it follows that

$$\frac{dx}{dt} = -\frac{4}{3} \cdot 2 = \boxed{-\frac{8}{3}}.$$

Skill Building

5. Let $V = \dfrac{1}{12}\pi h^3$. Then, differentiating with respect to t yields

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}.$$

Given that $\dfrac{dh}{dt} = \dfrac{5\pi}{16}$ when $h = 8$, it follows that when $h = 8$,

$$\frac{dV}{dt} = \frac{1}{4}\pi(8)^2 \cdot \frac{5\pi}{16} = \boxed{5\pi^2}.$$

7. Let $V = 80h^2$. Then, differentiating with respect to t yields

$$\frac{dV}{dt} = 160h\frac{dh}{dt}.$$

Given that $\dfrac{dh}{dt} = \dfrac{1}{12}$ when $h = 3$, it follows that when $h = 3$,

$$\frac{dV}{dt} = 160(3) \cdot \frac{1}{12} = \boxed{40}.$$

9. The volume V of a cube with edge length x is $V = x^3$. Differentiating with respect to time t yields

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 3$ cm/s, it follows that when $x = 10$ cm,

$$\frac{dV}{dt} = 3(10)^2 \cdot 3 = 900.$$

When the length of a side of the cube is 10 cm, the volume is increasing at a rate of $\boxed{900 \text{ cm}^3/\text{s}}$.

11. The surface area S of a sphere of radius r is $S = 4\pi r^2$. Differentiating with respect to time t yields

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} \quad \text{so that} \quad \frac{dr}{dt} = \frac{1}{8\pi r}\frac{dS}{dt}.$$

Given that $\frac{dS}{dt} = -0.1$ cm^2/h, it follows that when $r = \frac{20}{\pi}$ cm,

$$\frac{dr}{dt} = \frac{1}{8\pi(20/\pi)} \cdot -0.1 = -\frac{1}{1600} = -0.000625.$$

When the radius of the sphere is $\frac{20}{\pi}$ cm, the radius is decreasing at a rate of $\boxed{0.000625 \text{ cm/h}}$.

13. Let x and y denote the lengths of the legs of a right triangle with hypotenuse of length 45 cm. By the Pythagorean theorem, $x^2 + y^2 = 45^2$. Differentiating with respect to time t yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 2$ cm/min, it follows that when $x = 4$ cm,

$$y = \sqrt{45^2 - 4^2} = \sqrt{2009} \text{ cm},$$

and

$$\frac{dy}{dt} = -\frac{4}{\sqrt{2009}} \cdot 2 = -\frac{8}{\sqrt{2009}}.$$

When $x = 4$ cm, y is decreasing at a rate of $\boxed{\frac{8}{\sqrt{2009}} \text{ cm/min} \approx 0.178 \text{ cm/min}}$.

15. The area of an isosceles triangle with equal sides of length 4 cm and an included angle θ is $A = \frac{1}{2}4^2 \sin\theta = 8\sin\theta$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = 8\cos\theta\frac{d\theta}{dt}.$$

Given that $\frac{d\theta}{dt} = 2°/\text{min} = \frac{\pi}{90}$ radians/min, it follows that when $\theta = 30°$

$$\frac{dA}{dt} = 8\cos 30° \frac{\pi}{90} = \frac{2\sqrt{3}\pi}{45}.$$

When $\theta = 30°$, the area of the triangle is increasing at a rate of $\boxed{\frac{2\sqrt{3}\pi}{45} \text{ cm}^2/\text{min} \approx 0.242 \text{ cm}^2/\text{min}}$.

17. The volume V and surface area S of a sphere of radius r are $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Differentiating both formulas with respect to time t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Solving the former equation for $\frac{dr}{dt}$ and substituting that expression into the latter formula gives

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} \quad \text{so that} \quad \frac{dS}{dt} = \frac{8\pi r}{4\pi r^2}\frac{dV}{dt} = \frac{2}{r}\frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = -1.5 \text{ m}^3/\text{min}$, it follows that when $r = 4$ m,

$$\frac{dS}{dt} = \frac{2}{4}(-1.5) = -0.75.$$

When the radius of the balloon is 4 m, the surface area is shrinking at a rate of $\boxed{0.75 \text{ m}^2/\text{min}}$.

Applications and Extensions

19. Let x denote the distance from the lower end of the ladder to the wall against which the ladder is leaning. Then

$$\cos\theta = \frac{x}{5} \quad \text{and} \quad -\sin\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 0.5$ m/s, it follows that when $x = 4$ m,

$$\cos\theta = \frac{4}{5} \quad \text{so that} \quad \sin\theta = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

and

$$-\frac{3}{5}\frac{d\theta}{dt} = \frac{1}{5}(0.5).$$

Therefore,

$$\frac{d\theta}{dt} = -\frac{1}{6}.$$

When the lower end of the ladder is 4 m from the wall, the inclination θ of the ladder is deceasing at a rate of $\boxed{\frac{1}{6} \text{ rad/s}}$.

21. Let h denote the depth of the water in the pool at the deep end, and let L denote the horizontal length of the surface of the water at depth h (see the diagram below). By similar triangles

$$\frac{L}{h} = \frac{30}{2} = 15 \quad \text{so that} \quad L = 15h$$

and the volume of water in the pool is

$$V = \left(\frac{1}{2}Lh\right)(5) = \frac{5}{2}(15h)h = \frac{75}{2}h^2.$$

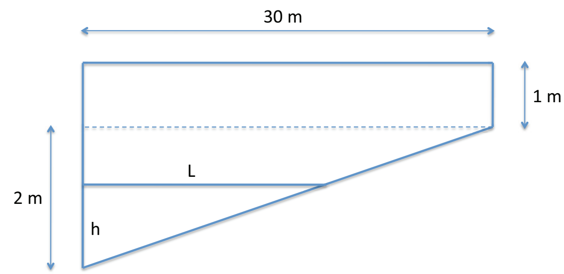
Differentiating with respect to time t yields

$$\frac{dV}{dt} = 75h\frac{dh}{dt} \quad \text{so} \quad \frac{dh}{dt} = \frac{1}{75h}\frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 15$ m^3/min, it follows that when $h = 1$ m,

$$\frac{dh}{dt} = \frac{1}{75(1)} 15 = \frac{1}{5} = 0.2.$$

When the water level is 1 m deep at the deep end of the pool, the water level is rising at a rate of $\boxed{0.2 \text{ m/min}}$.



23. The volume V of water in the shape of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Consider the diagram on the right in the text. By similar triangles

$$\frac{r}{h} = \frac{4}{16} = \frac{1}{4} \quad \text{so that} \quad r = \frac{1}{4}h$$

and

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h = \frac{1}{48}\pi h^3.$$

Differentiating with respect to time t yields

$$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{16}{\pi h^2}\frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 16$ m^3/min, it follows that when $h = 8$ m,

$$\frac{dh}{dt} = \frac{16}{\pi(8)^2}(16) = \frac{4}{\pi}.$$

When the water is 8 m deep, the water level is rising at a rate of $\boxed{\frac{4}{\pi} \text{ m/min} \approx 1.273 \text{ m/min}}$.

25. (a) The volume V of water in the shape of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. By similar triangles (see the diagram below),

$$\frac{r}{h} = \frac{1}{4} \quad \text{so that} \quad r = \frac{1}{4}h$$

and

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h = \frac{1}{48}\pi h^3.$$

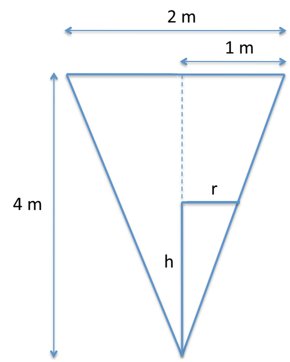
Differentiating with respect to time t yields

$$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{16}{\pi h^2}\frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 3$ m^3/min, it follows that when $h = 3$ m,

$$\frac{dh}{dt} = \frac{16}{\pi(3)^2}(3) = \frac{16}{3\pi}.$$

When the water in the tank is 3 m deep, the water level is rising at a rate of $\boxed{\frac{16}{3\pi} \text{ m/min} \approx 1.698 \text{ m/min}}$.



(b) Based on the observation that the depth of the water is increasing at only 0.5 m/min when the depth is 3 m, the rate at which the volume of water in the tank is increasing is

$$\frac{dV}{dt} = \frac{1}{16}\pi(3)^2(0.5) = \frac{9\pi}{32} \text{ m}^3\text{/min} \approx 0.884 \text{ m}^3\text{/min}.$$

Because water is entering the tank at a rate of 3 m^3/min, the rate at which water is leaking from the tank is

$$3 - \frac{9\pi}{32} = \boxed{\frac{96 - 9\pi}{32} \text{ m}^3\text{/min} \approx 2.116 \text{ m}^3\text{/min}}.$$

27. Let (x, y) be a point on the graph of the parabola $y^2 = 4(3 - x) = 12 - 4x$. The distance D from the point (x, y) to the origin is

$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 - 4x + 12}.$$

Differentiating both the equation of the parabola and the distance formula with respect to time t yields

$$2y\frac{dy}{dt} = -4\frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = -\frac{y}{2}\frac{dy}{dt}$$

and

$$\frac{dD}{dt} = \frac{1}{2}(x^2 - 4x + 12)^{-1/2}(2x - 4)\frac{dx}{dt} = \frac{x - 2}{\sqrt{x^2 - 4x + 12}}\frac{dx}{dt}.$$

Given that $\frac{dy}{dt} = 3$ units per second when the object is at the point $(-1, 4)$, it follows that when the object is at the point $(-1, 4)$,

$$\frac{dx}{dt} = -\frac{4}{2}(3) = -6 \text{ units per second,}$$

and

$$\frac{dD}{dt} = \frac{-1 - 2}{\sqrt{1 + 4 + 12}}(-6) = \frac{18}{\sqrt{17}}.$$

When the object is at the point $(-1, 4)$, the distance between the object and the origin is increasing at a rate of $\boxed{\frac{18}{\sqrt{17}} \text{ units per second} \approx 4.366 \text{ units per second}}$.

29. The distance x between the ball and first base is

$$x = \sqrt{s^2 + 90^2} = \sqrt{s^2 + 8100},$$

where s is the distance between the ball and home plate along the third-base line. Differentiating with respect to time t yields

$$\frac{dx}{dt} = \frac{1}{2}(s^2 + 8100)^{-1/2}(2s)\frac{ds}{dt} = \frac{s}{\sqrt{s^2 + 8100}}\frac{ds}{dt}.$$

Given that $\frac{ds}{dt} = 100$ ft/s, it follows that when the ball crosses third base ($s = 90$ ft),

$$\frac{dx}{dt} = \frac{90}{\sqrt{90^2 + 8100}}(100) = \frac{100}{\sqrt{2}} = 50\sqrt{2}.$$

When the ball crosses third base, the distance from the ball to first base is increasing at a rate of $\boxed{50\sqrt{2} \text{ ft/s} \approx 70.711 \text{ ft/s}}$.

31. Differentiating the equation $PV^{1.4} = k$ with respect to time t yields

$$P \cdot 1.4V^{0.4}\frac{dV}{dt} + V^{1.4}\frac{dP}{dt} = 0$$

so that

$$\frac{dP}{dt} = -\frac{1.4PV^{0.4}}{V^{1.4}}\frac{dV}{dt} = -\frac{1.4P}{V}\frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = -2 \text{ cm}^3/\text{min}$ at the instant when $P = 20 \text{ kg/cm}^2$ and $V = 32 \text{ cm}^3$, it follows that

$$\frac{dP}{dt} = -\frac{1.4(20)}{32}(-2) = \frac{28}{16} = 1.75.$$

At the instant when $P = 20 \text{ kg/cm}^2$ and $V = 32 \text{ cm}^3$, the pressure is increasing at a rate of $\boxed{1.75 \frac{\text{kg/cm}^2}{\text{min}}}$.

33. The area A of a circle of radius r is $A = \pi r^2$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$

Given that $\frac{dr}{dt} = 0.42$ ft/min, it follows that when $r = 120$ ft,

$$\frac{dA}{dt} = 2\pi(120)(0.42) = 100.8\pi.$$

When the radius of the oil spill is 120 ft, the area of the spill is increasing at a rate of $\boxed{100.8\pi \text{ ft}^2/\text{min} \approx 316.673 \text{ ft}^2/\text{min}}$.

35. Let x denote the distance from the base of the ladder to the wall and y denote the distance up the wall from the top of the ladder to the ground. Then $x^2 + y^2 = 8^2 = 64$. Differentiating with respect to time t yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

(a) Given that $\dfrac{dx}{dt} = 0.5$ m/s, it follows that when $x = 3$ m,

$$y = \sqrt{64 - 3^2} = \sqrt{55} \text{ m},$$

and

$$\frac{dy}{dt} = -\frac{3}{\sqrt{55}}(0.5) = -\frac{1.5}{\sqrt{55}}.$$

The top of the ladder is moving down the wall at a rate of $\boxed{\dfrac{1.5}{\sqrt{55}} \text{ m/s} \approx 0.202 \text{ m/s}}$.

(b) Given that $\dfrac{dx}{dt} = 0.5$ m/s, it follows that when $x = 4$ m,

$$y = \sqrt{64 - 4^2} = \sqrt{48} = 4\sqrt{3} \text{ m},$$

and

$$\frac{dy}{dt} = -\frac{4}{4\sqrt{3}}(0.5) = -\frac{0.5}{\sqrt{3}}.$$

The top of the ladder is moving down the wall at a rate of $\boxed{\dfrac{0.5}{\sqrt{3}} \text{ m/s} \approx 0.289 \text{ m/s}}$.

(c) Given that $\dfrac{dx}{dt} = 0.5$ m/s, it follows that when $x = 6$ m,

$$y = \sqrt{64 - 6^2} = \sqrt{28} = 2\sqrt{7} \text{ m},$$

and

$$\frac{dy}{dt} = -\frac{6}{2\sqrt{7}}(0.5) = -\frac{1.5}{\sqrt{7}}.$$

The top of the ladder is moving down the wall at a rate of $\boxed{\dfrac{1.5}{\sqrt{7}} \text{ m/s} \approx 0.567 \text{ m/s}}$.

37. Let x denote the distance from the point on the shore opposite the ship to the point where the radar beam intersects the shoreline, and let θ denote the angle made by the beam and the perpendicular from the ship to the shoreline (see the diagram below). Then

$$\frac{x}{6} = \tan\theta \quad \text{so that} \quad \frac{1}{6}\frac{dx}{dt} = \sec^2\theta \frac{d\theta}{dt}.$$

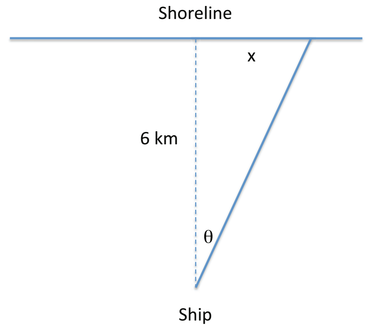
Given that

$$\frac{d\theta}{dt} = \frac{1 \text{ revolution}}{5 \text{ s}} \cdot \frac{2\pi \text{ rad}}{\text{revolution}} = \frac{2\pi}{5} \frac{\text{rad}}{\text{s}},$$

and $\theta = 45°$, it follows that

$$\frac{dx}{dt} = 6\sec^2 45° \cdot \frac{2\pi}{5} = \frac{24\pi}{5}.$$

When the beam makes an angle of 45° with the shore, the beam is moving along the shore at a rate of $\boxed{\dfrac{24\pi}{5} \text{ km/s} \approx 15.080 \text{ km/s}}$.



39. Let x denote the horizontal distance from the boy to the street lamp, and let s denote the length of the boy's shadow (see the diagram below). By similar triangles

$$\frac{s}{1} = \frac{x+s}{6} \quad \text{so that} \quad 6s = x+s \quad \text{or} \quad s = \frac{1}{5}x.$$

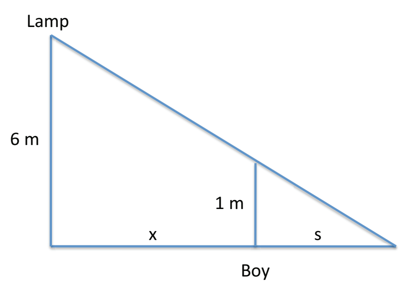
Therefore,

$$\frac{ds}{dt} = \frac{1}{5}\frac{dx}{dt},$$

and with $\frac{dx}{dt} = 20$ m/min,

$$\frac{ds}{dt} = \frac{1}{5}(20) = 4.$$

The child's shadow is lengthening at a rate of $\boxed{4 \text{ m/min}}$.



41. Place the center of the ferris wheel at the point $(0, 30)$, and suppose that the ferris wheel rotates counterclockwise. Then the x- and y-coordinates of a passenger on the ferris wheel can be modeled by

$$x = 25\cos\theta \quad \text{and} \quad y = 30 + 25\sin\theta.$$

When a passenger is 42.5 ft above the ground,

$$42.5 = 30 + 25\sin\theta \quad \text{so that} \quad \sin\theta = \frac{1}{2}.$$

Given that the passenger is rising, $\theta = 30°$. Moreover, with

$$\frac{d\theta}{dt} = \frac{1 \text{ revolution}}{2 \text{ min}} \cdot \frac{2\pi \text{ rad}}{\text{revolution}} = \pi \frac{\text{rad}}{\text{min}},$$

it follows that

$$\frac{dy}{dt} = 25\cos\theta\frac{d\theta}{dt} = 25\pi\cos 30° = \frac{25\pi\sqrt{3}}{2} \text{ ft/min} \approx 68.017 \text{ ft/min}.$$

Additionally,

$$\frac{dx}{dt} = -25\sin\theta\frac{d\theta}{dt} = -25\pi\sin 30^\circ = -\frac{25\pi}{2}\text{ ft/min} \approx -39.270\text{ ft/min}.$$

The passenger is rising at a rate of $\boxed{\frac{25\pi\sqrt{3}}{2}\text{ ft/min} \approx 68.017\text{ ft/min}}$ and is moving horizontally backward at a rate of $\boxed{\frac{25\pi}{2}\text{ ft/min} \approx 39.270\text{ ft/min}}$.

43. Let x denote the horizontal distance between the delivery truck and the elevator, and let h denote the elevation of the elevator. The distance D between the delivery truck and the elevator is then given by $D = \sqrt{x^2 + h^2}$, and

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + h^2)^{-1/2}\left(2x\frac{dx}{dt} + 2h\frac{dh}{dt}\right) = \frac{x\frac{dx}{dt} + h\frac{dh}{dt}}{\sqrt{x^2 + h^2}}.$$

One second after the elevator and delivery truck begin moving, $x = 8$ m and $h = 20$ m. Given that $\frac{dx}{dt} = 8$ m/s and $\frac{dh}{dt} = -5$ m/s, it follows that

$$\frac{dD}{dt} = \frac{8\cdot 8 + 20\cdot -5}{\sqrt{8^2 + 20^2}} = -\frac{9}{\sqrt{29}} = -\frac{9\sqrt{29}}{\sqrt{29}}\text{ m/s} \approx -1.671\text{ m/s}.$$

The elevator and the delivery truck are $\boxed{\text{separating at a rate of } -\frac{9\sqrt{29}}{\sqrt{29}}\text{ m/s} \approx -1.671\text{ m/s}}$; that is, they are approaching one another at a rate of $\frac{9\sqrt{29}}{\sqrt{29}}$ m/s ≈ 1.671 m/s.

45. (a) Differentiating $C = 10,000 + 3x$ with respect to time t yields

$$\frac{dC}{dt} = 3\frac{dx}{dt}.$$

When $x = 1000$ switches and $\frac{dx}{dt} = 50$ switches per day,

$$\frac{dC}{dt} = 3(50) = 150.$$

When production is 1000 switches per day, cost increases at a rate of $\boxed{\$150\text{/day}}$.

(b) Differentiating

$$R = 5x - \frac{x^2}{2000}$$

with respect to time t yields

$$\frac{dR}{dt} = \left(5 - \frac{x}{1000}\right)\frac{dx}{dt}.$$

When $x = 1000$ switches and $\frac{dx}{dt} = 50$ switches per day,

$$\frac{dR}{dt} = \left(5 - \frac{1000}{1000}\right)\cdot 50 = 200.$$

When production is 1000 switches per day, revenue increases at a rate of $\boxed{\$200\text{/day}}$.

(c) Let P denote profit, so that $P = R - C$ and

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}.$$

When $x = 1000$ switches and $\frac{dx}{dt} = 50$ switches per day,

$$\frac{dP}{dt} = 200 - 150 = 50,$$

based on the answers from parts (a) and (b). When production is 1000 switches per day, profit increases at a rate of $\boxed{\$50/\text{day}}$.

47. Because the object weighs 1000 lb on Earth's surface, $K = 1000$, and

$$W = 1000\left(\frac{3960}{3960+R}\right)^2 = 1000(3960)^2(3960+R)^{-2}.$$

Differentiating with respect to time t then yields

$$\frac{dW}{dt} = -2000(3960)^2(3960+R)^{-3}\frac{dR}{dt}.$$

Given that $\frac{dR}{dt} = 10$ mi/s, it follows that when $R = 50$ mi,

$$\frac{dW}{dt} = -2000\frac{3960^2}{(3960+50)^3}(10) \approx -4.864.$$

When the object is 50 mi above Earth's surface, the weight of the object is decreasing at a rate of approximately $\boxed{-4.864 \text{ lb/s}}$.

49. Using the diagram in the text,

$$\cot\theta = \frac{x}{4500},$$

so that

$$-\csc^2\theta\frac{d\theta}{dt} = \frac{1}{4500}\frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = -4500\csc^2\theta\frac{d\theta}{dt}.$$

Given that $\frac{d\theta}{dt} = 1^\circ/\text{s} = \frac{\pi}{180}$ rad/s, it follows that when $\theta = 30^\circ$,

$$\frac{dx}{dt} = -4500 \cdot 4 \cdot \frac{\pi}{180} = -100\pi.$$

The plane is approaching the battery at a rate of $\boxed{100\pi \text{ ft/s} \approx 314.159 \text{ ft/s}}$.

51. Let x denote the horizontal distance between the searchlight and the plane. Then

$$\cot\theta = \frac{x}{3000 \text{ ft}} \cdot \frac{5280 \text{ ft}}{\text{mi}} = 1.76x,$$

so that

$$-\csc^2\theta\frac{d\theta}{dt} = 1.76\frac{dx}{dt} \quad \text{or} \quad \frac{d\theta}{dt} = -\frac{1.76}{\csc^2\theta}\frac{dx}{dt}.$$

When the distance between the plane and the searchlight is 5000 ft, then $x = 4000$ ft and $\cot\theta = 4/3$ so

$$\csc^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}.$$

Given that $\frac{dx}{dt} = 500$ mi/h, it follows that

$$\frac{d\theta}{dt} = -\frac{1.76}{25/9}(500) = -316.8.$$

When the distance between the searchlight and the plane is 5000 ft, the searchlight is turning at a rate of $\boxed{-316.8 \text{ rad/h} = -0.088 \text{ rad/s}}$.

53. Let $V = hx^2$.

(a) If h decreases with time but x remains constant, then

$$\frac{dV}{dt} = \boxed{x^2 \frac{dh}{dt}},$$

so that the volume of the box decreases at a rate x^2 time the rate of decrease in the height.

(b) If both h and x change with time, then

$$\frac{dV}{dt} = h \cdot 2x\frac{dx}{dt} + x^2\frac{dh}{dt} = \boxed{2hx\frac{dx}{dt} + x^2\frac{dh}{dt}}.$$

Answers will now vary. For example, note that the rate of change of the height is multiplied by the area of the square base while the rate of change of the side length of the base is multiplied by twice the area of the vertical sides.

Challenge Problems

55. Place the origin at the center of the dome, so the surface of the dome over which the shadow travels can be modeled by the equation $x^2 + y^2 = 30^2 = 900$. As the shadow of the ball moves along the surface of the dome,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt},$$

and the speed with which the shadow is moving is

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{1 + \frac{y^2}{x^2}}\left|\frac{dy}{dt}\right| \\ &= \sqrt{\frac{x^2+y^2}{x^2}}\left|\frac{dy}{dt}\right| = \frac{30}{\sqrt{900 - y^2}}\left|\frac{dy}{dt}\right|. \end{aligned}$$

Because it is sunset, the rays from the sun are traveling parallel to the ground, and the y-coordinate of the shadow is the same as the y-coordinate of the ball itself. Let $t = 0$ denote the time when the ball begins falling. Then the y-coordinate of the ball (and of the shadow) is given by $y = 30 - 16t^2$, and

$$\frac{dy}{dt} = -32t.$$

At $t = 1/2$ s, $y = 30 - 16(1/4) = 26$ ft, $\frac{dy}{dt} = -16$ ft/s, and the speed of the shadow is

$$\frac{30}{\sqrt{900 - 26^2}}|-16| \approx \boxed{32.071 \text{ ft/s}}.$$

57. The area A of the triangle OAB is

$$A = \frac{1}{2}(2)(3)\sin\theta = 3\sin\theta,$$

where θ is the angle between the two hands of the clock. Now the hour hand moves one-twelfth of the circle in 60 minutes, so it moves at a rate of

$$\frac{1}{12} \cdot 2\pi \cdot \frac{1}{60} = \frac{\pi}{360} \text{ rad/min},$$

while the minute hand makes one revolution of the circle in 60 minutes, so it moves at a rate of

$$\frac{2\pi}{60} = \frac{\pi}{30} \text{ rad/min}.$$

At noon, the hour hand and the minute hand overlay, so ten minutes later the angle between the hands is

$$\theta = \frac{\pi}{30} \cdot 10 - \frac{\pi}{360} \cdot 10 = \frac{\pi}{3} - \frac{\pi}{36} = \frac{11\pi}{36},$$

and the rate at which the angle θ is changing is

$$\frac{d\theta}{dt} = \frac{\pi}{30} - \frac{\pi}{360} = \frac{11\pi}{360} \text{ rad/min}.$$

At 12:10 p.m.,

$$\frac{dA}{dt} = 3\cos\theta\frac{d\theta}{dt} = 3\cos\frac{11\pi}{36} \cdot \frac{11\pi}{360} \approx 0.165;$$

that is, the area of the triangle is increasing at a rate of approximately $\boxed{0.165 \text{ in}^2/\text{min}}$.

AP® Practice Problems

1. The volume, V, of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Differentiating with respect to time t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Given that $\frac{dV}{dt} = 50\,\text{m}^3/\text{min}$ and $r = \frac{D}{2} = 10\,\text{m}$, it follows that

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = \frac{1}{4\pi(10)^2} \cdot (50) = \frac{1}{8\pi}.$$

When the radius of the balloon is 10 m, the radius is increasing at the rate of $\boxed{\frac{1}{8\pi} \text{ m/min}}$.

CHOICE C

3. From $A = \pi r^2$, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. From $A = \pi r^2 = 25\pi$, $r = 5$, so given that $\frac{dr}{dt} = -2$ it follows that $\frac{dA}{dt} = 2\pi(5)(-2) = \boxed{-20\pi \text{ in}^2/\text{min}}$.

CHOICE A

5. $\frac{dy}{dt} = 3\frac{dx}{dt}$

Differentiating $x^2 + y^2 = c^2$ with respect to time, t, yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2c\frac{dc}{dt}$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = c\frac{dc}{dt}$$

Given that $x = 6$ and $y = 8$, it follows from $x^2 + y^2 = c^2$ that $c^2 = (6)^2 + (8)^8 = 100$, so $c = 10$.

For $x = 6$, $y = 8$, $c = 10$, and $\frac{dc}{dt} = 1$, it follows from $x\frac{dx}{dt} + y\frac{dy}{dt} = c\frac{dc}{dt}$ that

$$(6)\frac{dx}{dt} + (8)3\frac{dx}{dt} = (10)(1)$$
$$30\frac{dx}{dt} = 10$$
$$\frac{dx}{dt} = \boxed{\frac{1}{3}}.$$

CHOICE B

7. Differentiating $V = \frac{1}{3}\pi r^2 h$ with respect to time, t, yields

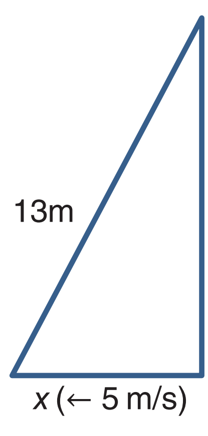
$$\frac{dV}{dt} = \frac{\pi}{3}\left[2r\frac{dr}{dt} \cdot h + r^2\frac{dh}{dt}\right].$$

Given that $\frac{dr}{dt} = 2$, $\frac{dh}{dt} = 2$, $r = 6$, and $h = 15$, it follows that

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2(6)(2)(15) + (6)^2(2)\right] = \boxed{144\pi \text{ cm}^3/\text{h}}.$$

CHOICE C

9.



(a) Differentiating $x^2 + y^2 = 13^2$ with respect to time, t, yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

At $y = 5$, $x^2 + 5^2 = 13^2$, so $x = 12$

From $x = 12$, $y = 5$, $\frac{dx}{dt} = 5$, and $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$, it follows that

$$(12)(5) + (5)\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = \boxed{-12 \text{ m/s}}.$$

(b) Differentiating $A = \frac{xy}{2}$ with respect to time, t, yields

$$\frac{dA}{dt} = \frac{1}{2}\left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)$$

From $y = 5$, $x = 12$, $\frac{dx}{dt} = 5$, and $\frac{dy}{dt} = -12$,

$$\frac{dA}{dt} = \frac{1}{2}\left[(5)(5) + (12)(-12)\right] = \frac{-119}{2} = \boxed{-59.5 \text{ m}^2/\text{s}}.$$

(c) Differentiating $\sin\theta = \frac{y}{13}$ with respect to time, t, yields

$$\cos\theta\frac{d\theta}{dt} = \frac{1}{13}\cdot\frac{dy}{dt}$$
$$\frac{d\theta}{dt} = \frac{1}{13\cos\theta}\cdot\frac{dy}{dt}$$

From $\cos\theta = \frac{12}{13}$ and $\frac{dy}{dt} = -12$,

$$\frac{d\theta}{dt} = \frac{(-12)}{13\left(\frac{12}{13}\right)} = \boxed{-1 \text{ rad/s}}.$$

4.2 Maximum and Minimum Values; Critical Numbers

Concepts and Vocabulary

1. $\boxed{\text{False}}$. Any function f that is defined **and continuous** on a closed interval $[a, b]$ will have both an absolute maximum value and an absolute minimum value.

3. $\boxed{\text{False}}$. At a critical number, there **may be** a local extreme value.

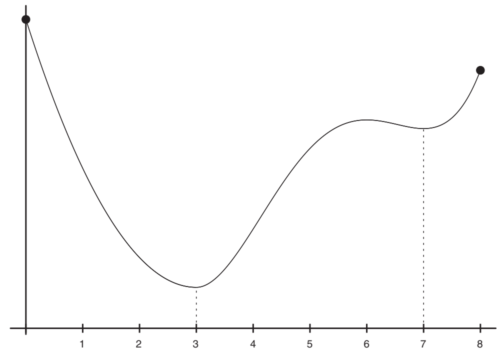
5. $\boxed{\text{False}}$. The Extreme Value Theorem tells us when the absolute maximum and absolute minimum **exist**.

Skill Building

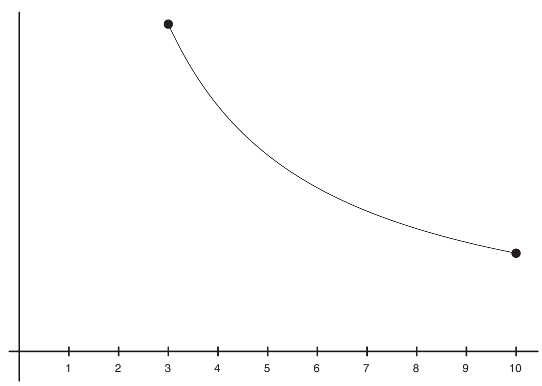
7. For the function given in the graph,

x_1:	neither
x_2:	local maximum
x_3:	local minimum and absolute minimum
x_4:	neither
x_5:	local maximum
x_6:	neither
x_7:	local minimum
x_8:	absolute maximum

9. The figure below displays the graph of a function with the following properties: domain $[0, 8]$, absolute maximum at 0, absolute minimum at 3, local minimum at 7.



11. The figure below displays the graph of a function with the following properties: domain $[3, 10]$ and no local extreme values.



13. Let $f(x) = x^2 - 8x$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 2x - 8 = 2(x - 4) = 0$$

when $x = 4$. Therefore, $\boxed{4}$ is the only critical number of f.

15. Let $f(x) = x^3 - 3x^2$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$$

when $x = 0$ and when $x = 2$. Therefore, $\boxed{0 \text{ and } 2}$ are the critical numbers of f.

17. Let $f(x) = x^4 - 2x^2 + 1$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0$$

when $x = 0$ and when $x = \pm 1$. Therefore, $\boxed{-1, 0, \text{ and } 1}$ are the critical numbers of f.

19. Let $f(x) = x^{2/3}$. The domain of f is the set of all real numbers, and $f'(x) = \dfrac{2}{3}x^{-1/3}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f, 0 is a critical number of f. Therefore, $\boxed{0}$ is the only critical number of f.

21. Let $f(x) = 2\sqrt{x}$. The domain of f is the set $\{x | x \geq 0\}$, and $f'(x) = \dfrac{1}{\sqrt{x}}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f, 0 is a critical number of f. Therefore, $\boxed{0}$ is the only critical number of f.

23. Let $f(x) = x + \sin x$, and consider the interval $0 \leq x \leq \pi$. Because f is differentiable on $0 \leq x \leq \pi$, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 1 + \cos x = 0$$

when $\cos x = -1$. On the interval $0 \leq x \leq \pi$, $\cos x = -1$ when $x = \pi$. Therefore, $\boxed{\pi}$ is the only critical number of f on the interval $0 \leq x \leq \pi$.

25. Let $f(x) = x\sqrt{1 - x^2}$. The domain of f is the set $\{x | -1 \leq x \leq 1\}$, and

$$\begin{aligned} f'(x) &= x \cdot \frac{1}{2}(1 - x^2)^{-1/2}(-2x) + \sqrt{1 - x^2} \\ &= -\frac{x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} = \frac{-x^2 + 1 - x^2}{\sqrt{1 - x^2}} = \frac{1 - 2x^2}{\sqrt{1 - x^2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $1 - 2x^2 = 0$, or when $x = \pm\dfrac{\sqrt{2}}{2}$, and $f'(x)$ does not exist when $1 - x^2 = 0$, or when $x = \pm 1$. All four of these numbers are in the domain of f, so all four are critical numbers. Therefore, $\boxed{\pm 1 \text{ and } \pm\dfrac{\sqrt{2}}{2}}$ are critical numbers of f.

27. Let $f(x) = \dfrac{x^2}{x - 1}$. The domain of f is the set $\{x | x \neq 1\}$, and

$$f'(x) = \frac{(x - 1)(2x) - x^2}{(x - 1)^2} = \frac{2x^2 - 2x - x^2}{(x - 1)^2} = \frac{x^2 - 2x}{(x - 1)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x^2 - 2x = x(x - 2) = 0$, or when $x = 0$ or $x = 2$. On the other hand, $f'(x)$ does not exist when $x - 1 = 0$, or when $x = 1$. As $x = 1$ is not in the domain of f, 1 is not a critical number. Therefore, $\boxed{0 \text{ and } 2}$ are the critical numbers of f.

29. Let $f(x) = (x + 3)^2(x - 1)^{2/3}$. The domain of f is the set of all real numbers, and

$$\begin{aligned} f'(x) &= (x + 3)^2 \cdot \frac{2}{3}(x - 1)^{-1/3} + (x - 1)^{2/3} \cdot 2(x + 3) \\ &= \frac{2(x + 3)^2}{3(x - 1)^{1/3}} + 2(x + 3)(x - 1)^{2/3} = \frac{2(x + 3)^2 + 6(x + 3)(x - 1)}{3(x - 1)^{1/3}} \\ &= \frac{2(x + 3)[x + 3 + 3(x - 1)]}{3(x - 1)^{1/3}} = \frac{8x(x + 3)}{3(x - 1)^{1/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $8x(x + 3) = 0$, or when $x = 0$ or $x = -3$. On the other hand, $f'(x)$ does not exist when $x - 1 = 0$, or when $x = 1$. All three of these numbers are in the domain of f, so all three are critical numbers. Therefore, $\boxed{-3, 0, \text{ and } 1}$ are critical numbers of f.

31. Let $f(x) = \dfrac{(x-3)^{1/3}}{x-1}$. The domain of f is the set $\{x|x \neq 1\}$, and

$$\begin{aligned} f'(x) &= \frac{(x-1)\cdot \frac{1}{3}(x-3)^{-2/3} - (x-3)^{1/3}}{(x-1)^2} \cdot \frac{3(x-3)^{2/3}}{3(x-3)^{2/3}} \\ &= \frac{(x-1)-3(x-3)}{3(x-1)^2(x-3)^{2/3}} = \frac{8-2x}{3(x-1)^2(x-3)^{2/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $8 - 2x = 0$, or when $x = 4$. On the other hand, $f'(x)$ does not exist when $(x-1)^2(x-3)^{2/3} = 0$, or when $x = 1$ or $x = 3$. As $x = 1$ is not in the domain of f, 1 is not a critical number. Therefore, $\boxed{3 \text{ and } 4}$ are critical numbers of f.

33. Let $f(x) = \dfrac{\sqrt[3]{x^2-9}}{x}$. The domain of f is the set $\{x|x \neq 0\}$, and

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{3}(x^2-9)^{-2/3}(2x) - \sqrt[3]{x^2-9}}{x^2} \cdot \frac{3(x^2-9)^{2/3}}{3(x^2-9)^{2/3}} \\ &= \frac{2x^2 - 3(x^2-9)}{3x^2(x^2-9)^{2/3}} = \frac{27-x^2}{3x^2(x^2-9)^{2/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $27 - x^2 = 0$, or when $x = \pm\sqrt{27} = \pm 3\sqrt{3}$. On the other hand, $f'(x)$ does not exist when $x^2(x^2-9)^{2/3} = 0$, or when $x = 0$ or $x = \pm 3$. As $x = 0$ is not in the domain of f, 0 is not a critical number. Therefore, $\boxed{\pm 3 \text{ and } \pm 3\sqrt{3}}$ are critical numbers of f.

35. On the interval $[0, 1)$, $f(x) = 3x$ so $f'(x) = 3$. Because $f'(x)$ exists everywhere on the interval $[0, 1)$ and is never equal to 0, f has no critical numbers in $[0, 1)$. On the interval $(1, 2]$, $f(x) = 4 - x$, so $f'(x) = -1$. Because $f'(x)$ exists everywhere on the interval $(1, 2]$ and is never equal to 0, f has no critical numbers in $(1, 2]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{3x-3}{x-1} = \lim_{x\to 1^-} \frac{3(x-1)}{x-1} = \lim_{x\to 1^-} 3 = 3,$$

and

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{(4-x)-3}{x-1} = \lim_{x\to 1^+} \frac{-(x-1)}{x-1} = \lim_{x\to 1^+} -1 = -1.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 1$. Therefore, $\boxed{1}$ is the only critical number of f.

37. Let $f(x) = x^2 - 8x$, and consider the closed interval $[-1, 10]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 10]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 2x - 8 = 2(x-4) = 0$$

when $x = 4$. Therefore, 4 is the only critical number of f. Evaluating f at the endpoints of the interval $[-1, 10]$ and at the critical number 4 yields

$$\begin{aligned} f(-1) &= (-1)^2 - 8(-1) = 9 \\ f(4) &= 4^2 - 8(4) = -16 \\ f(10) &= 10^2 - 8(10) = 20. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 10]$ is $\boxed{20}$ (and this occurs at the endpoint $x = 10$), while the absolute minimum value of f is $\boxed{-16}$ (and this occurs at the critical number $x = 4$).

39. Let $f(x) = x^3 - 3x^2$, and consider the closed interval $[1, 4]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[1, 4]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$$

when $x = 0$ or $x = 2$. Therefore, 0 and 2 are the critical numbers of f; however, $x = 0$ is not inside the interval $(1, 4)$, so this critical number can be excluded. Evaluating f at the endpoints of the interval $[1, 4]$ and at the critical number 2 yields

$$\begin{aligned} f(1) &= 1^3 - 3(1)^2 = -2 \\ f(2) &= 2^3 - 3(2)^2 = -4 \\ f(4) &= 4^3 - 3(4)^2 = 16. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[1, 4]$ is $\boxed{16}$ (and this occurs at the endpoint $x = 4$), while the absolute minimum value of f is $\boxed{-4}$ (and this occurs at critical number $x = 2$).

41. Let $f(x) = x^4 - 2x^2 + 1$, and consider the closed interval $[0, 2]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 2]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0$$

when $x = 0$ or $x = \pm 1$. Therefore, -1, 0 and 1 are the critical numbers of f; however, of these numbers, only $x = 1$ is inside the open interval $(0, 2)$. Evaluating f at the endpoints of the interval $[0, 2]$ and at the critical number 1 yields

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1^4 - 2(1)^2 + 1 = 0 \\ f(2) &= 2^4 - 2(2)^2 + 1 = 9. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 2]$ is $\boxed{9}$ (and this occurs at the endpoint $x = 2$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical number $x = 1$).

43. Let $f(x) = x^{2/3}$, and consider the closed interval $[-1, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{2}{3}x^{-1/3},$$

so $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f, 0 is a critical number of f. Evaluating f at the endpoints of the interval $[-1, 1]$ and the critical number 0 yields

$$\begin{aligned} f(-1) &= (-1)^{2/3} = 1 \\ f(0) &= 0 \\ f(1) &= 1^{2/3} = 1. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{1}$ (and this occurs at both endpoints $x = \pm 1$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical number $x = 0$).

45. Let $f(x) = 2\sqrt{x}$, and consider the closed interval $[1, 4]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[1, 4]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{1}{\sqrt{x}},$$

so $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f, 0 is a critical number of f; however, $x = 0$ is not inside the interval $(1, 4)$, so this critical number can be excluded, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[1, 4]$ yields

$$\begin{aligned} f(1) &= 2\sqrt{1} = 2 \\ f(4) &= 2\sqrt{4} = 4. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[1, 4]$ is $\boxed{4}$ (and this occurs at the endpoint $x = 4$), while the absolute minimum value of f is $\boxed{2}$ (and this occurs at the endpoint $x = 1$).

47. Let $f(x) = x + \sin x$, and consider the closed interval $[0, \pi]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, \pi]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 1 + \cos x = 0$$

when $\cos x = -1$. On the interval $0 \le x \le \pi$, $\cos x = -1$ when $x = \pi$; however, $x = \pi$ is not inside the interval $(0, \pi)$, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[0, \pi]$ yields

$$\begin{aligned} f(0) &= \sin 0 = 0 \\ f(\pi) &= \pi + \sin \pi = \pi. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, \pi]$ is $\boxed{\pi}$ (and this occurs at the endpoint $x = \pi$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the endpoint $x = 0$).

49. Let $f(x) = x\sqrt{1 - x^2}$, and consider the closed interval $[-1, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside

the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) + \sqrt{1-x^2} \\ &= -\frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \frac{-x^2+1-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}, \end{aligned}$$

so $f'(x) = 0$ when $1-2x^2 = 0$, or when $x = \pm\frac{\sqrt{2}}{2}$, and $f'(x)$ does not exist when $1-x^2 = 0$, or when $x = \pm 1$. All four of these numbers are in the domain of f, so all four are critical numbers; however, of these numbers, only $x = \pm\frac{\sqrt{2}}{2}$ are inside the open interval $(-1, 1)$. Evaluating f at the endpoints of the interval $[-1, 1]$ and the critical numbers $\pm\frac{\sqrt{2}}{2}$ yields

$$\begin{aligned} f(-1) &= 0 \\ f\left(-\frac{\sqrt{2}}{2}\right) &= -\frac{\sqrt{2}}{2}\sqrt{1-\frac{1}{2}} = -\frac{1}{2} \\ f\left(\frac{\sqrt{2}}{2}\right) &= \frac{\sqrt{2}}{2}\sqrt{1-\frac{1}{2}} = \frac{1}{2} \\ f(1) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{\frac{1}{2}}$ (and this occurs at the critical number $x = \frac{\sqrt{2}}{2}$), while the absolute minimum value of f is $\boxed{-\frac{1}{2}}$ (and this occurs at the critical number $x = -\frac{\sqrt{2}}{2}$).

51. Let $f(x) = \frac{x^2}{x-1}$, and consider the closed interval $\left[-1, \frac{1}{2}\right]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $\left[-1, \frac{1}{2}\right]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2},$$

so $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$, or when $x = 0$ or $x = 2$. On the other hand, $f'(x)$ does not exist when $x - 1 = 0$, or when $x = 1$. As $x = 1$ is not in the domain of f, 1 is not a critical number. Therefore, 0 and 2 are the critical numbers of f; however, $x = 2$ is not inside the open interval $\left(-1, \frac{1}{2}\right)$, so this critical number can be excluded. Evaluating f at the endpoints of the interval $\left[-1, \frac{1}{2}\right]$ and the critical number 0 yields

$$\begin{aligned} f(-1) &= \frac{(-1)^2}{-1-1} = -\frac{1}{2} \\ f(0) &= 0 \\ f\left(\frac{1}{2}\right) &= \frac{1/4}{-1/2} = -\frac{1}{2}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $\left[-1, \frac{1}{2}\right]$ is $\boxed{0}$ (and this occurs at the critical number $x = 0$), while the absolute minimum value of f is $\boxed{-\frac{1}{2}}$ (and this occurs at both endpoints $x = -1$ and $x = \frac{1}{2}$).

53. Let $f(x) = (x+3)^2(x-1)^{2/3}$, and consider the closed interval $[-4, 5]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-4, 5]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= (x+3)^2 \cdot \frac{2}{3}(x-1)^{-1/3} + (x-1)^{2/3} \cdot 2(x+3) \\ &= \frac{2(x+3)^2}{3(x-1)^{1/3}} + 2(x+3)(x-1)^{2/3} = \frac{2(x+3)^2 + 6(x+3)(x-1)}{3(x-1)^{1/3}} \\ &= \frac{2(x+3)[x+3+3(x-1)]}{3(x-1)^{1/3}} = \frac{8x(x+3)}{3(x-1)^{1/3}}, \end{aligned}$$

so $f'(x) = 0$ when $8x(x+3) = 0$, or when $x = 0$ or $x = -3$. On the other hand, $f'(x)$ does not exist when $x - 1 = 0$, or when $x = 1$. All three of these numbers are in the domain of f, so all three are critical numbers. Evaluating f at the endpoints of the interval $[-4, 5]$ and the critical numbers -3, 0, and 1 yields

$$\begin{aligned} f(-4) &= (-1)^2(-5)^{2/3} = 25^{1/3} = \sqrt[3]{25} \approx 2.924 \\ f(-3) &= 0 \\ f(0) &= 3^2(-1)^{2/3} = 9 \\ f(1) &= 0 \\ f(5) &= 8^2(4)^{2/3} = 64(4)^{2/3} = 64\sqrt[3]{16} \approx 161.270. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-4, 5]$ is $\boxed{64\sqrt[3]{16}}$ (and this occurs at the endpoint $x = 5$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical numbers $x = -3$ and $x = 1$).

55. Let $f(x) = \frac{(x-3)^{1/3}}{x-1}$, and consider the closed interval $[2, 11]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[2, 11]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{(x-1) \cdot \frac{1}{3}(x-3)^{-2/3} - (x-3)^{1/3}}{(x-1)^2} \cdot \frac{3(x-3)^{2/3}}{3(x-3)^{2/3}} \\ &= \frac{(x-1) - 3(x-3)}{3(x-1)^2(x-3)^{2/3}} = \frac{8-2x}{3(x-1)^2(x-3)^{2/3}}, \end{aligned}$$

so $f'(x) = 0$ when $8 - 2x = 0$, or when $x = 4$. On the other hand, $f'(x)$ does not exist when $(x-1)^2(x-3)^{2/3} = 0$, or when $x = 1$ or $x = 3$.As $x = 1$ is not in the domain of f, 1 is not a critical number. Therefore, 3 and 4 are critical numbers of f. Evaluating f at the endpoints of the interval $[2, 11]$ and the critical numbers 3 and 4 yields

$$\begin{aligned} f(2) &= \frac{(-1)^{1/3}}{1} = -1 \\ f(3) &= 0 \\ f(4) &= \frac{1^{1/3}}{3} = \frac{1}{3} \\ f(11) &= \frac{8^{1/3}}{10} = \frac{1}{5}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[2, 11]$ is $\boxed{\frac{1}{3}}$ (and this occurs at the critical number $x = 4$), while the absolute minimum value of f is $\boxed{-1}$ (and this occurs at the endpoint $x = 2$).

57. Let $f(x) = \dfrac{\sqrt[3]{x^2 - 9}}{x}$, and consider the closed interval $[3, 6]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[3, 6]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{3}(x^2 - 9)^{-2/3}(2x) - \sqrt[3]{x^2 - 9}}{x^2} \cdot \frac{3(x^2 - 9)^{2/3}}{3(x^2 - 9)^{2/3}} \\ &= \frac{2x^2 - 3(x^2 - 9)}{3x^2(x^2 - 9)^{2/3}} = \frac{27 - x^2}{3x^2(x^2 - 9)^{2/3}}, \end{aligned}$$

so $f'(x) = 0$ when $27 - x^2 = 0$, or when $x = \pm\sqrt{27} = \pm 3\sqrt{3}$. On the other hand, $f'(x)$ does not exist when $x^2(x^2 - 9)^{2/3} = 0$, or when $x = 0$ or $x = \pm 3$. As $x = 0$ is not in the domain of f, 0 is not a critical number. Therefore, ± 3 and $\pm 3\sqrt{3}$ are critical numbers of f; however, of these numbers, only $3\sqrt{3}$ is inside the open interval $(3, 6)$. Evaluating f at the endpoints of the interval $[3, 6]$ and the critical number $3\sqrt{3}$ yields

$$\begin{aligned} f(3) &= 0 \\ f(3\sqrt{3}) &= \frac{\sqrt[3]{18}}{3\sqrt{3}} \approx 0.504 \\ f(6) &= \frac{\sqrt[3]{27}}{6} = \frac{1}{2}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[3, 6]$ is $\boxed{\frac{\sqrt[3]{18}}{3\sqrt{3}}}$ (and this occurs at the critical number $x = 3\sqrt{3}$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the endpoint $x = 3$).

59. Let $f(x) = e^x - 3x$, and consider the closed interval $[0, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = e^x - 3 = 0$$

when $x = \ln 3$. Therefore, $\ln 3$ is the only critical number of f; however, $x = \ln 3 \approx 1.099$ is not inside the interval $(0, 1)$, so this critical number can be excluded, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[0, 1]$ yields

$$\begin{aligned} f(0) &= e^0 - 3(0) = 1 \\ f(1) &= e^1 - 3(1) = e - 3 \approx -0.282. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 1]$ is $\boxed{1}$ (and this occurs at the endpoint $x = 0$), while the absolute minimum value of f is $\boxed{e - 3}$ (and this occurs at the endpoint $x = 1$).

61. The function f is continuous on the intervals $[0, 1)$ and $(1, 3]$ because the component functions $2x + 1$ and $3x$ are continuous on these intervals, respectively. At $x = 1$,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 1) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x) = 3,$$

so that $\lim_{x \to 1} f(x)$ exists and is equal to 3. As $f(1) = 3(1) = 3$, it follows that f is continuous at $x = 1$. Hence, f is continuous on the closed interval $[0, 3]$.

Now, on the interval $[0, 1)$, $f(x) = 2x + 1$ so $f'(x) = 2$. Because $f'(x)$ exists everywhere on the interval $[0, 1)$ and is never equal to 0, f has no critical numbers in $[0, 1)$. On the interval $(1, 3]$, $f(x) = 3x$, so $f'(x) = 3$. Because $f'(x)$ exists everywhere on the interval $(1, 3]$ and is never equal to 0, f has no critical numbers in $(1, 3]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(2x + 1) - 3}{x - 1} = \lim_{x \to 1^-} \frac{2(x - 1)}{x - 1} = \lim_{x \to 1^-} 2 = 2,$$

and

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{3x - 3}{x - 1} = \lim_{x \to 1^+} \frac{3(x - 1)}{x - 1} = \lim_{x \to 1^+} 3 = 3.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 1$. Therefore, 1 is the only critical number of f. Evaluating f at the endpoints of the interval $[0, 3]$ and the critical number 1 yields

$$\begin{aligned} f(0) &= 2(0) + 1 = 1 \\ f(1) &= 3(1) = 3 \\ f(3) &= 3(3) = 9. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 3]$ is $\boxed{9}$ (and this occurs at the endpoint $x = 3$), while the absolute minimum value of f is $\boxed{1}$ (and this occurs at the endpoint $x = 0$).

63. The function f is continuous on the intervals $[-2, 1)$ and $(1, 2]$ because the component functions x^2 and x^3 are continuous on these intervals, respectively. At $x = 1$,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 = 1,$$

so that $\lim_{x \to 1} f(x)$ exists and is equal to 1. As $f(1) = 1^3 = 1$, it follows that f is continuous at $x = 1$. Hence, f is continuous on the closed interval $[-2, 2]$.

Now, on the interval $[-2, 1)$, $f(x) = x^2$ so $f'(x) = 2x$. Because $f'(x)$ exists everywhere on the interval $[-2, 1)$ but is equal to 0 at $x = 0$, 0 is a critical number in $[-2, 1)$. On the interval $(1, 2]$, $f(x) = x^3$, so $f'(x) = 3x^2$. Because $f'(x)$ exists everywhere and is never equal to 0 on the interval $(1, 2]$, f has no critical numbers in $(1, 2]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{x - 1} = \lim_{x\to 1^-} (x+1) = 2,$$

and

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1^+} \frac{(x-1)(x^2+x+1)}{x - 1} = \lim_{x\to 1^+} (x^2+x+1) = 3.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 1$. Therefore, 1 is a critical number of f. Evaluating f at the endpoints of the interval $[-2, 2]$ and the critical numbers 0 and 1 yields

$$\begin{aligned} f(-2) &= (-2)^2 = 4 \\ f(0) &= 0^2 = 0 \\ f(1) &= 1^3 = 1 \\ f(2) &= 2^3 = 8. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-2, 2]$ is $\boxed{8}$ (and this occurs at the endpoint $x = 2$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical number $x = 0$).

Applications and Extensions

65. Let $f(x) = 3x^4 - 2x^3 - 21x^2 + 36x$.

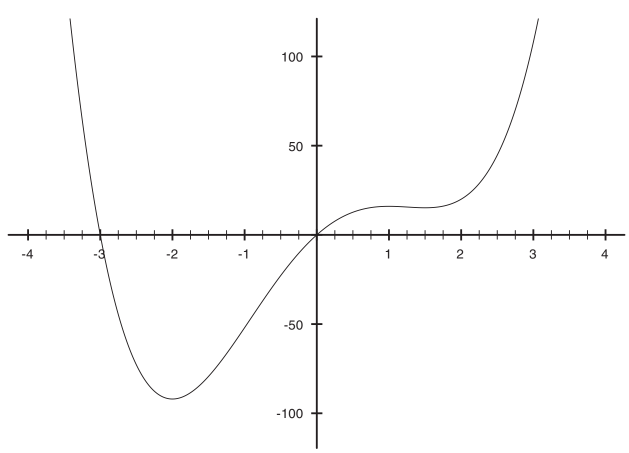
   (a) Then $\boxed{f'(x) = 12x^3 - 6x^2 - 42x + 36}$.

   (b) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Using the computer algebra system *Maple*,

   $$12x^3 - 6x^2 - 42x + 36 = 0$$

   when $x = -2$, $x = 1$, and $x = \frac{3}{2}$. Therefore, $\boxed{-2,\ 1,\text{ and } \frac{3}{2}}$ are the critical numbers of f.

   (c) The figure below displays the graph of f. The graph suggests that f has an $\boxed{\text{absolute minimum at } -2}$, a $\boxed{\text{local maximum at } 1}$, and a $\boxed{\text{local minimum at } \frac{3}{2}}$.



67. Let $f(x) = \dfrac{(x^2-5x+2)\sqrt{x+5}}{\sqrt{x^2+2}}$.

(a) Then

$$\ln f(x) = \ln\left(\frac{(x^2-5x+2)\sqrt{x+5}}{\sqrt{x^2+2}}\right) = \ln(x^2-5x+2) + \frac{1}{2}\ln(x+5) - \frac{1}{2}\ln(x^2+2)$$

and

$$\begin{aligned}
\frac{d}{dx}\ln f(x) &= \frac{d}{dx}\left(\ln(x^2-5x+2) + \frac{1}{2}\ln(x+5) - \frac{1}{2}\ln(x^2+2)\right)\\
\frac{1}{f(x)}\frac{df}{dx} &= \frac{1}{x^2-5x+2}\cdot(2x-5) + \frac{1}{2}\cdot\frac{1}{x+5} - \frac{1}{2}\cdot\frac{1}{x^2+2}\cdot 2x\\
&= \frac{2x-5}{x^2-5x+2} + \frac{1}{2x+10} - \frac{x}{x^2+2}\\
&= \frac{(2x-5)(2x+10)(x^2+2) + (x^2-5x+2)(x^2+2) - x(x^2-5x+2)(2x+10)}{2(x^2-5x+2)(x+5)(x^2+2)}\\
&= \frac{(4x^4+10x^3-42x^2+20x-100) + (x^4-5x^3+4x^2-10x+4) - (2x^4-46x^2+20x)}{2(x^2-5x+2)(x+5)(x^2+2)}\\
&= \frac{3x^4+5x^3+8x^2-10x-96}{2(x^2-5x+2)(x+5)(x^2+2)}.
\end{aligned}$$

Therefore,

$$f'(x) = \frac{(x^2-5x+2)\sqrt{x+5}}{\sqrt{x^2+2}}\cdot\frac{3x^4+5x^3+8x^2-10x-96}{2(x^2-5x+2)(x+5)(x^2+2)} = \boxed{\frac{3x^4+5x^3+8x^2-10x-96}{2\sqrt{x+5}(x^2+2)^{3/2}}}.$$

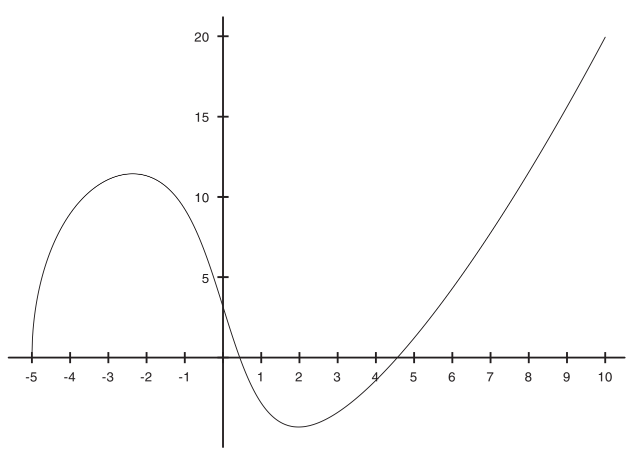
(b) The critical numbers of f occur where $f'(x)=0$ or where $f'(x)$ does not exist. Now, $f'(x)$ does not exist at $x=-5$. Using the computer algebra system *Maple*,

$$\frac{3x^4+5x^3+8x^2-10x-96}{2\sqrt{x+5}(x^2+2)^{3/2}} = 0$$

when $x \approx -2.364$ and $x \approx 1.977$. Therefore, $\boxed{-5,\ -2.364, \text{ and } 1.977}$ are the critical numbers of f.

(c) The figure below displays the graph of f. The graph suggests that f has an $\boxed{\text{absolute minimum at } 1.977}$, a $\boxed{\text{local maximum at } -2.364}$, and

$$\boxed{\text{neither a local maximum nor a local minimum at } -5}.$$



69. Let $f(x) = x^4 - 12.4x^3 + 49.24x^2 - 68.64x$.

(a) Then $\boxed{f'(x) = 4x^3 - 37.2x^2 + 98.48x - 68.64}$.

(b) Consider the closed interval $[0, 5]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 5]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Using the computer algebra system *Maple*,

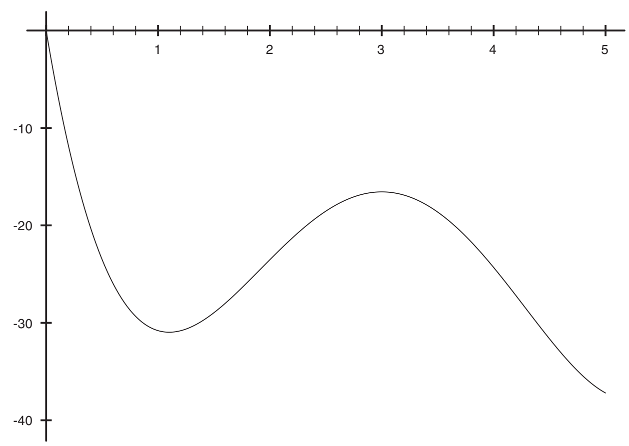
$$4x^3 - 37.2x^2 + 98.48x - 68.64 = 0$$

when $x = 1.1$, when $x = 3$, and when $x = 5.2$. Therefore, 1.1, 3, and 5.2 are the critical numbers of f; however, $x = 5.2$ is not inside the open interval $(0, 5)$, so this critical number can be excluded. Evaluating f at the endpoints of the interval $[0, 5]$ and at the critical numbers 1.1 and 3 yields

$$\begin{aligned} f(0) &= 0 \\ f(1.1) &= -30.9639 \\ f(3) &= -16.56 \\ f(5) &= -37.2. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 5]$ is $\boxed{0}$ (and this occurs at the endpoint $x = 0$), while the absolute minimum value of f is $\boxed{-37.2}$ (and this occurs at the endpoint $x = 5$).

(c) The figure below displays the graph of f, which $\boxed{\text{does support the conclusions from part (b)}}$: the absolute maximum occurs at 0 and the absolute minimum occurs at 5.



71. (a) Let $C(x) = 3.60\left(\dfrac{2500}{x} + x\right)$ and consider the interval $[10, 75]$. Because C is continuous on this closed interval, the Extreme Value Theorem guarantees that C has an absolute maximum and an absolute minimum on $[10, 75]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. The critical numbers of C occur where $C'(x) = 0$ or where $C'(x)$ does not exist. Now,

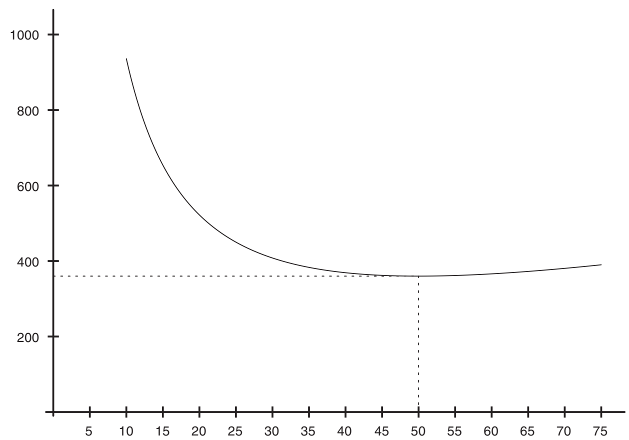
$$C'(x) = 3.60\left(-\frac{2500}{x^2} + 1\right),$$

so $C'(x) = 0$ when $x = \pm 50$ and $C'(x)$ does not exist when $x = 0$. Because $x = 0$ is not in the domain of C, 0 is not a critical number. Therefore, ± 50 are the critical numbers of C; however, $x = -50$ is not inside the open interval $(10, 75)$, so this number can be excluded. Evaluating C at the endpoints of the interval $[10, 75]$ and at the critical number 50 yields

$$\begin{aligned} C(10) &= 3.60\left(\frac{2500}{10} + 10\right) = 3.60(260) = 936 \\ C(50) &= 3.60\left(\frac{2500}{50} + 50\right) = 3.60(100) = 360 \\ C(75) &= 3.60\left(\frac{2500}{75} + 75\right) = 3.60 \cdot \frac{325}{3} = 390. \end{aligned}$$

Therefore, the absolute minimum value of C on the interval $[10, 75]$ is \$360 and this occurs at $x = 50$. The most economical speed for the truck to travel is $\boxed{\text{50 miles per hour}}$.

(b) The figure below displays the graph of C.



73. Let $R(\theta) = \dfrac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta - \cos\theta)$.

(a) Consider the closed interval $[45°, 90°]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[45°, 90°]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, R is differentiable everywhere, which means that the critical numbers of R occur where $R'(\theta) = 0$. Moreover,

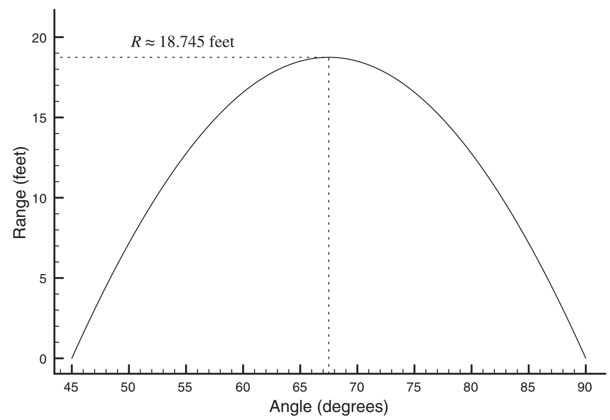
$$\begin{aligned} R'(\theta) &= \frac{v_0^2\sqrt{2}}{16}\left[\cos\theta(\cos\theta + \sin\theta) - (\sin\theta - \cos\theta)\sin\theta\right] \\ &= \frac{v_0^2\sqrt{2}}{16}\left(\cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta\right) \\ &= \frac{v_0^2\sqrt{2}}{16}(\cos 2\theta + \sin 2\theta). \end{aligned}$$

Therefore, $R'(\theta) = 0$ when $\cos 2\theta + \sin 2\theta = 0$. On the interval $45° \leq \theta \leq 90°$, $\cos 2\theta + \sin 2\theta = 0$ when $\theta = 67.5°$. Evaluating f at the endpoints of the interval $[45°, 90°]$ and the critical number $67.5°$ yields

$$\begin{aligned} R(45°) &= \frac{v_0^2\sqrt{2}}{16}\cos 45°(\sin 45° - \cos 45°) = 0 \\ R(67.5°) &= \frac{v_0^2\sqrt{2}}{16}\cos 67.5°(\sin 67.5° - \cos 67.5°) \approx 0.0183v_0^2 \\ R(90°) &= \frac{v_0^2\sqrt{2}}{16}\cos 90°(\sin 90° - \cos 90°) = 0. \end{aligned}$$

Therefore, the absolute maximum value of R on the interval $[45°, 90°]$ occurs when $\boxed{\theta = 67.5°}$. The maximum value of R is approximately $\boxed{0.0183v_0^2}$.

(b) The figure below displays the graph of R using $v_0 = 32$ ft/s. The dashed vertical line is drawn at $\theta = 67.5°$.



(c) Based on the graph in part (b), the angle that maximizes R is $\theta = 67.5°$, and the maximum range is approximately

$$0.0183(32)^2 \approx 18.745 \text{ feet}.$$

75. (a) Let $R = \dfrac{2v_0^2}{g}\sin\theta\cos\theta = \dfrac{v_0^2}{g}\sin 2\theta$, and consider the interval $[0°, 90°]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[0°, 90°]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, R is differentiable everywhere, which means that the critical numbers of R occur where $R'(\theta) = 0$. Moreover,

$$R'(\theta) = \frac{2v_0^2}{g}\cos 2\theta = 0$$

when $\cos 2\theta = 0$. On the interval $[0°, 90°]$, $\cos 2\theta = 0$ when $\theta = 45°$. Evaluating f at the endpoints of the interval $[0°, 90°]$ and the critical number $45°$ yields

$$\begin{aligned} R(0°) &= \frac{v_0^2}{g}\sin 0° = 0 \\ R(45°) &= \frac{v_0^2}{g}\sin 90° = \frac{v_0^2}{g} \\ R(90°) &= \frac{v_0^2}{g}\sin 180° = 0 = 0. \end{aligned}$$

Therefore, the absolute maximum value of R is achieved when $\theta = 45°$; that is, the golf ball achieves its maximum range when the golfer hits the ball at an angle of $45°$.

(b) With $v_0 = 91.1$ m/s and $g = 9.8$ m/s^2, the maximum range is

$$\frac{v_0^2}{g} = \frac{91.1^2}{9.8} \approx \boxed{846.858 \text{ m}}.$$

77. With $t = \sqrt{27 - 3q^2}$, the tax revenue R is given by

$$R = tq = q\sqrt{27 - 3q^2}.$$

The domain of this function is the set $\{q | -3 \le q \le 3\}$; however, from an economics standpoint, the quantity consumed must be greater than or equal to 0, so consider the function R over the interval $[0, 3]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[0, 3]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of R occur where $R'(q) = 0$ or where $R'(q)$ does not exist. Moreover,

$$\begin{aligned} R'(q) &= q \cdot \frac{1}{2}(27 - 3q^2)^{-1/2}(-6q) + \sqrt{27 - 3q^2} \\ &= -\frac{3q^2}{\sqrt{27 - 3q^2}} + \sqrt{27 - 3q^2} = \frac{-3q^2 + 27 - 3q^2}{\sqrt{27 - 3q^2}} = \frac{27 - 6q^2}{\sqrt{27 - 3q^2}}, \end{aligned}$$

so $R'(q) = 0$ when $27 - 6q^2 = 0$, or when $q = \pm\frac{3\sqrt{2}}{2}$, and $R'(q)$ does not exist when $27 - 3q^2 = 0$, or when $q = \pm 3$. Of these numbers, only $\frac{3\sqrt{2}}{2}$ lies inside the open interval $(0, 3)$. Evaluating R at the endpoints of the interval $[0, 3]$ and the critical number $\frac{3\sqrt{2}}{2}$ yields

$$\begin{aligned} R(0) &= 0\sqrt{27} = 0 \\ R\left(\frac{3\sqrt{2}}{2}\right) &= \frac{3\sqrt{2}}{2}\sqrt{27 - 3\left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{9\sqrt{3}}{2} \approx 7.794 \\ R(3) &= 3\sqrt{27 - 3(3)^2} = 0. \end{aligned}$$

Therefore, the absolute maximum value of R on the interval $[0, 3]$ is $\frac{9\sqrt{3}}{2}$ (and this occurs at the critical number $q = \frac{3\sqrt{2}}{2}$). Substituting this value for q into the formula for t gives

$$t = \sqrt{27 - 3\left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{3\sqrt{6}}{2} \approx 3.674.$$

The $\boxed{\text{maximum tax revenue is approximately 7.794}}$, and this is produced with a $\boxed{\text{tax rate of approximately 3.674}}$.

79. Let $y = 15\cosh\frac{x}{15} - 10$ and consider the interval $[-6, 6]$, thus assuming that the x-axis is placed along the ground with the two poles at $(-6, 0)$ and $(6, 0)$. Because y is continuous on this closed interval, the Extreme Value Theorem guarantees that y has an absolute

maximum and an absolute minimum on $[-6, 6]$. The absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, y is differentiable everywhere, which means that the critical numbers of y occur where $y'(x) = 0$. Moreover,

$$y'(x) = \sinh \frac{x}{15} = 0$$

when $x = 0$. Evaluating y at the endpoints of the interval $[-6, 6]$ and the critical number 0 yields

$$\begin{aligned} y(-6) &= 15 \cosh\left(-\frac{6}{15}\right) - 10 \approx 6.216 \\ y(0) &= 15 \cosh 0 - 10 = 5 \\ y(6) &= 15 \cosh\left(\frac{6}{15}\right) - 10 \approx 6.216. \end{aligned}$$

Therefore, the absolute minimum value of y on the interval $[-6, 6]$ is 5; that is, the height of the cable at its lowest point is $\boxed{5 \text{ m}}$.

81. (a) Let $f(x) = \dfrac{x^3}{3} - 0.055x^2 + 0.0028x - 4$. Because f is a polynomial function, it is differentiable everywhere. Therefore, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = x^2 - 0.11x + 0.0028 = 0$$

when

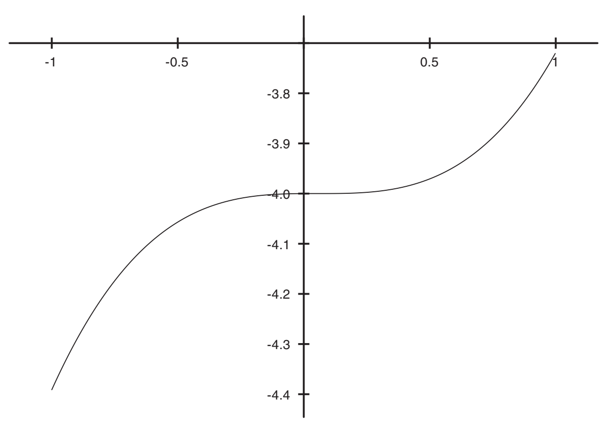
$$x = \frac{0.11 \pm \sqrt{0.11^2 - 4(0.0028)}}{2} = \frac{0.11 \pm \sqrt{0.0009}}{2} = \frac{0.11 \pm 0.03}{2} = \boxed{0.07, 0.04}.$$

(b) Evaluating f at the endpoints of the closed interval $[-1, 1]$ and at the critical numbers 0.04 and 0.07 yields

$$\begin{aligned} f(-1) &\approx -4.391133333 \\ f(0.04) &\approx -3.999954667 \\ f(0.07) &\approx -3.999959167 \\ f(1) &\approx -3.718866667. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is approximately $\boxed{-3.719}$ (which occurs at the endpoint $x = 1$), and the absolute minimum value is approximately $\boxed{-4.391}$ (which occurs at the endpoint $x = -1$).

(c) The figure below displays the graph of f on the interval $[-1, 1]$.



83. Let $f(x) = \sqrt{1+x^2}+|x-2|$, and consider the interval $[0,3]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0,3]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. On the interval $[0,2)$, $x-2<0$, so $f(x) = \sqrt{1+x^2} - x + 2$ and

$$f'(x) = \frac{1}{2}(1+x^2)^{-1/2}(2x) - 1 = \frac{x}{\sqrt{1+x^2}} - 1.$$

This derivative exists and is never equal to zero for $0 \le x < 2$, so there are no critical numbers on this interval. On the interval $(2,3]$, $x-2>0$, so $f(x) = \sqrt{1+x^2} + x - 2$ and

$$f'(x) = \frac{1}{2}(1+x^2)^{-1/2}(2x) + 1 = \frac{x}{\sqrt{1+x^2}} + 1.$$

This derivative exists and is never equal to zero for $2 < x \le 3$, so there are no critical numbers on this interval. At $x = 2$, $f'(x)$ does not exist because of the $|x-2|$ term, so 2 is a critical number of f. Evaluating f at the endpoints of the interval $[0,3]$ and at the critical number 2 yields

$$\begin{aligned} f(0) &= \sqrt{1} + |0-2| = 1+2 = 3 \\ f(2) &= \sqrt{1+4} + 0 = \sqrt{5} \\ f(3) &= \sqrt{1+9} + |3-2| = \sqrt{10} + 1. \end{aligned}$$

Therefore, the absolute maximum value of the function f on the interval $[0,3]$ is $\boxed{\sqrt{10}+1}$ (and this occurs at the endpoint $\boxed{x=3}$), while the absolute minimum value is $\boxed{\sqrt{5}}$ (and this occurs at the critical point $\boxed{x=2}$).

85. Let $f(x) = [(16-x^2)(x^2-9)]^{1/2}$.

(a) The domain of f is determined by the inequality $(16-x^2)(x^2-9) \ge 0$. Now, $16-x^2$ is positive for $-4 < x < 4$ and is negative for $|x| > 4$; on the other hand, x^2-9 is positive for $|x| > 3$ and is negative for $-3 < x < 3$. It follows that $16-x^2$ and x^2-9 have the same sign for $-4 < x < -3$ and $3 < x < 4$. Additionally, $(16-x^2)(x^2-9) = 0$ for $x = \pm 3$ and $x \pm 4$. Therefore, the domain of f is the set

$$\boxed{\{x \mid -4 \le x \le -3 \text{ or } 3 \le x \le 4\}},$$

or in interval notation $\boxed{[-4,-3] \cup [3,4]}$.

(b) Note that

$$f(-x) = [(16-(-x)^2)((-x)^2-9)]^{1/2} = [(16-x^2)(x^2-9)]^{1/2} = f(x),$$

so f is an even function. To determine the absolute maximum value of f on its domain, it is therefore sufficient to consider the closed interval $[3,4]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{1}{2}[(16-x^2)(x^2-9)]^{-1/2}[(16-x^2)(2x) + (x^2-9)(-2x)] \\ &= \frac{1}{[(16-x^2)(x^2-9)]^{1/2}} \cdot (16x - x^3 - x^3 + 9x) \\ &= \frac{x(25-2x^2)}{[(16-x^2)(x^2-9)]^{1/2}}, \end{aligned}$$

so $f'(x) = 0$ when $25 - 2x^2 = 0$ or $x = \pm\frac{5\sqrt{2}}{2}$ and $f'(x)$ does not exist when $x = \pm 3$ and when $x = \pm 4$. Of these numbers, only $\frac{5\sqrt{2}}{2}$ is inside the open interval $(3, 4)$. Evaluating f at the endpoints of the closed interval $[3, 4]$ and the critical number $\frac{5\sqrt{2}}{2}$ yields

$$\begin{aligned} f(3) &= 0 \\ f\left(\frac{5\sqrt{2}}{2}\right) &= \sqrt{\left(16 - \frac{25}{2}\right)\left(\frac{25}{2} - 9\right)} = \frac{7}{2} \\ f(4) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[3, 4]$ is $\frac{7}{2}$. Because f is an even function, it follows that the absolute maximum value on the interval $[-4, -3]$ is also $\frac{7}{2}$, so the absolute maximum value of f on its domain is $\boxed{\frac{7}{2}}$.

87. (a) $\boxed{\text{Not necessarily true}}$. Just because a function is continuous on a closed interval $[a, b]$ is no guarantee that the function is differentiable on (a, b).

(b) $\boxed{\text{Not necessarily true}}$. An absolute maximum can occur at an endpoint of the interval $[a, b]$; even if the absolute maximum occurs at a critical number, the derivative might not exist at that critical number.

(c) $\boxed{\text{True}}$. This is the definition of continuity on the open interval (a, b).

(d) $\boxed{\text{Not necessarily true}}$. Just because a function is continuous on a closed interval $[a, b]$ is no guarantee that the function is differentiable on (a, b); even if the function is differentiable, the derivative does not need to equal 0 for some x, $a \le x \le b$.

(e) $\boxed{\text{True}}$. This follows from the Extreme Value Theorem.

89. The absolute extreme values of a continuous function defined on a closed interval $[a, b]$ can only occur at the endpoints of the interval or at the critical numbers inside the open interval (a, b). Therefore, first determine the critical numbers of the function: determine where inside the open interval that the derivative is equal to zero and where the derivative does not exist. Next, evaluate the function at the endpoints of the interval and at the critical numbers inside the interval. Finally, identify the absolute maximum value (the largest of the function values obtained in the previous step) and the absolute minimum value (the smallest of the function values obtained in the previous step).

91. Suppose f has a local minimum at c. Then, for all x in an open interval containing c, $f(c) \le f(x)$. Multiplying this inequality by -1 yields $-f(c) \ge -f(x)$. Now, let $g(x) = -f(x)$. Then the last inequality becomes $g(c) \ge g(x)$ for all x in an open interval containing c. Therefore, g has a local maximum at c.

Challenge Problems

93. (a) Let a, b, c, and d be real numbers, $n \ge 1$ an integer, and $f(x) = \frac{ax^{2n} + b}{cx^n + d}$. If $a = 0$, then

$$f(x) = \frac{b}{cx^n + d} = b(cx^n + d)^{-1} \quad \text{and} \quad f'(x) = -bcnx^{n-1}(cx^n + d)^{-2}.$$

Now, $f'(x)$ does not exist when $cx^n + d = 0$, but any x that satisfies this equation is not in the domain of f, so it cannot be a critical number. Therefore, the only critical numbers of f occur where $f'(x) = 0$, which is at $x = 0$. So, if $a = 0$, then f has only one critical number.

Next, consider the case $c = 0$ but $d \neq 0$ (c and d cannot be simultaneously equal to zero, as that would cause division by zero for all x). Then

$$f(x) = \frac{a}{d}x^{2n} + \frac{b}{d} \quad \text{and} \quad f'(x) = \frac{2an}{d}x^{2n-1}.$$

Now, $f'(x)$ exists for all x and is equal to zero only at $x = 0$. When $c = 0$ but $d \neq 0$, f has only one critical number.

Finally, suppose $a \neq 0$ and $c \neq 0$. Then

$$\begin{aligned} f'(x) &= \frac{(cx^n + d)(2anx^{2n-1}) - (ax^{2n} + b)(cnx^{n-1})}{(cx^n + d)^2} \\ &= \frac{2acnx^{3n-1} + 2adnx^{2n-1} - acnx^{3n-1} - bcnx^{n-1}}{(cx^n + d)^2} \\ &= \frac{acnx^{n-1}\left(x^{2n} + 2\frac{d}{c}x^n - \frac{b}{a}\right)}{(cx^n + d)^2}. \end{aligned}$$

Now, $f'(x)$ does not exist when $cx^n + d = 0$, but any x that satisfies this equation is not in the domain of f, so it cannot be a critical number. Therefore, the only critical numbers of f occur where $f'(x) = 0$. Clearly, $f'(x) = 0$ when $x = 0$. Any other critical numbers are the zeros of the function

$$g(x) = x^{2n} + 2\frac{d}{c}x^n - \frac{b}{a} = \left(x^n + \frac{d}{c}\right)^2 - \left(\frac{b}{a} + \frac{d^2}{c^2}\right).$$

The number of zeros of g depends on the values of a, b, c, and d:

i. If $\frac{b}{a} + \frac{d^2}{c^2} < 0$, g has no zeros.

ii. If $\frac{b}{a} + \frac{d^2}{c^2} = 0$, then g has one zero if n is odd (which will not be 0 provided $d \neq 0$) but 2 zeroes if n is even and $d/c < 0$.

iii. If $\frac{b}{a} + \frac{d^2}{c^2} > 0$, then g has two zeros if n is odd (neither of which will be 0 provided $b \neq 0$) but can have 0, 2, or 4 zeros if n is even depending on the signs of

$$-\frac{d}{c} + \sqrt{\frac{b}{a} + \frac{d^2}{c^2}} \quad \text{and} \quad -\frac{d}{c} - \sqrt{\frac{b}{a} + \frac{d^2}{c^2}}.$$

It follows that f can have at most five critical numbers.

(b) Based on part (a), in order for f to have five critical numbers, n must be even and

$$\frac{b}{a} + \frac{d^2}{c^2}, \quad -\frac{d}{c} + \sqrt{\frac{b}{a} + \frac{d^2}{c^2}}, \quad \text{and} \quad -\frac{d}{c} - \sqrt{\frac{b}{a} + \frac{d^2}{c^2}}$$

must all be positive. Answers will vary, but one choice of parameter values is $n = 2$, $a = 1$, $b = -1$, $c = 1$, and $d = -2$. Note that

$$\begin{aligned} \frac{b}{a} + \frac{d^2}{c^2} &= -1 + 4 = 3 > 0 \\ -\frac{d}{c} + \sqrt{\frac{b}{a} + \frac{d^2}{c^2}} &= 2 + \sqrt{3} > 0 \\ -\frac{d}{c} - \sqrt{\frac{b}{a} + \frac{d^2}{c^2}} &= 2 - \sqrt{3} > 0. \end{aligned}$$

The function

$$\boxed{f(x) = \frac{x^4 - 1}{x^2 - 2}}$$

has exactly five critical numbers:

$$0, \quad \pm\sqrt{2+\sqrt{3}}, \quad \text{and} \quad \pm\sqrt{2-\sqrt{3}}.$$

AP® Practice Problems

1. $g(x) = \sin x + \cos x$. The domain is $(0, 2\pi)$

   The critical numbers of g occur where $g'(x) = 0$ or where $g'(x)$ does not exist. This happens when

$$\begin{aligned} g'(x) &= 0 \\ \cos x - \sin x &= 0 \\ \cos x &= \sin x \\ \tan x &= 1 \\ x &= \frac{\pi}{4} \text{ and } x = \frac{5\pi}{4}, \text{ on the domain } (0, 2\pi) \end{aligned}$$

   Since both of these numbers are in the domain they are both critical numbers. And $g'(x)$ is defined everywhere, so there are no other critical numbers on the domain. Therefore $\boxed{\frac{\pi}{4} \text{ and } \frac{5\pi}{4}}$ are the critical numbers.

   CHOICE C

3. $f(x) = 2x^3 - 15x^2 + 36x$. The domain is $[0, 4]$.

   The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist.

   This happens when

$$\begin{aligned} f'(x) &= 0 \\ 6x^2 - 30x + 36 &= 0 \\ 6(x-3)(x-2) &= 0 \\ x &= 3 \text{ or } x = 2 \end{aligned}$$

   Since both $x = 3$ and $x = 2$ are in the domain, they are critical numbers. Since $f(x)$ is defined everywhere, there are no other critical values. Evaluate f at the critical numbers, 3 and 2, and at the endpoints 0 and 4.

x	$f(x) = 2x^3 - 15x^2 + 36x$	Conclusion
0	$2(0)^3 - 15(0)^2 + 36(0) = 0$	
2	$2(2)^3 - 15(2)^2 + 36(2) = 28$	
3	$2(3)^3 - 15(3)^2 + 36(3) = 27$	
4	$2(4)^3 - 15(4)^2 + 36(4) = \boxed{32}$	Absolute Maximum

   CHOICE C

5. $f(x) = \begin{cases} x^2 + 1 & \text{if } -2 \le x \le 1 \\ 3x^2 - 4x + 3 & \text{if } 1 < x \le 3 \end{cases}$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist.

$$f'(x) = \begin{cases} 2x & \text{if } -2 \leq x \leq 1 \\ 6x - 4 & \text{if } 1 < x \leq 3 \end{cases}$$

For $-2 \leq x \leq 1$,

$$\begin{aligned} f'(x) &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

which *is* a critical number, since $-2 \leq 0 \leq 1$.

For $1 < x \leq 3$,

$$\begin{aligned} f'(x) &= 0 \\ 6x - 4 &= 0 \\ x &= \frac{3}{2} \end{aligned}$$

which is *not* a critical number, since $\frac{3}{2} \not\leq 1$, so $\frac{3}{2}$ is not in the domain of this part of the definitions of $f(x)$ and $f'(x)$.

Therefore, the sole critical number is $\boxed{0}$.

CHOICE D

4.3 The Mean Value Theorem

Concepts and Vocabulary

1. $\boxed{\text{False}}$. If a function f is defined and continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then Rolle's Theorem guarantees that there is at least one number c in the interval (a, b) for which $f'(c) = 0$. The conclusion of Rolle's Theorem involves the derivative of the function, not the function itself.

3. $\boxed{\text{True}}$. If two functions f and g are differentiable on an open interval (a, b) and if $f'(x) = g'(x)$ for all numbers x in (a, b), then f and g differ by a constant on (a, b).

Skill Building

5. Let $f(x) = x^2 - 3x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 3]$ and differentiable on the open interval $(0, 3)$. Additionally,

$$f(0) = 0 \quad \text{and} \quad f(3) = 3^2 - 3(3) = 0,$$

so $f(0) = f(3)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, 3]$. Now, $f'(x) = 2x - 3$, so $f'(c) = 0$ when $\boxed{c = \frac{3}{2}}$.

7. Let $g(x) = x^2 - 2x - 2$. The polynomial function g is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$. Additionally,

$$g(0) = -2 \quad \text{and} \quad g(2) = 2^2 - 2(2) - 2 = -2,$$

so $g(0) = g(2)$. The function g therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, 2]$. Now, $g'(x) = 2x - 2$, so $g'(c) = 0$ when $\boxed{c = 1}$.

9. Let $f(x) = x^3 - x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-1, 0]$ and differentiable on the open interval $(-1, 0)$. Additionally,

$$f(-1) = (-1)^3 - (-1) = 0 \quad \text{and} \quad f(0) = 0,$$

so $f(-1) = f(0)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[-1, 0]$. Now, $f'(x) = 3x^2 - 1$, so $f'(c) = 0$ when $c = \pm\frac{\sqrt{3}}{3}$. Of these two numbers, only $\boxed{c = -\frac{\sqrt{3}}{3}}$ is in the interval $[-1, 0]$.

11. Let $f(t) = t^3 - t + 2$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-1, 1]$ and differentiable on the open interval $(-1, 1)$. Additionally,

$$f(-1) = (-1)^3 - (-1) + 2 = 2 \quad \text{and} \quad f(1) = 1^3 - 1 + 2 = 2,$$

so $f(-1) = f(1)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[-1, 1]$. Now, $f'(t) = 3t^2 - 1$, so $f'(c) = 0$ when $\boxed{c = \pm\frac{\sqrt{3}}{3}}$.

13. Let $s(t) = t^4 - 2t^2 + 1$. The polynomial function s is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. Additionally,

$$s(-2) = (-2)^4 - 2(-2)^2 + 1 = 9 \quad \text{and} \quad s(2) = 2^4 - 2(2)^2 + 1 = 9,$$

so $s(-2) = s(2)$. The function s therefore satisfies all three conditions of Rolle's Theorem on the interval $[-2, 2]$. Now, $s'(t) = 4t^3 - 4t = 4t(t^2 - 1)$, so $s'(c) = 0$ when $\boxed{c = 0 \text{ or } c = \pm 1}$.

15. Let $f(x) = \sin(2x)$. The trigonometric function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, \pi]$ and differentiable on the open interval $(0, \pi)$. Additionally,

$$f(0) = \sin 0 = 0 \quad \text{and} \quad f(\pi) = \sin(2\pi) = 0,$$

so $f(0) = f(\pi)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, \pi]$. Now, $f'(x) = 2\cos(2x)$, so $f'(c) = 0$ on the interval $(0, \pi)$ when $\boxed{c = \frac{\pi}{4} \text{ or } c = \frac{3\pi}{4}}$.

17. Let $f(x) = x^2 - 2x + 1$. Though the polynomial function f is continuous on the closed interval $[-2, 1]$ and differentiable on the open interval $(-2, 1)$,

$$f(-2) = (-2)^2 - 2(-2) + 1 = 9 \quad \text{and} \quad f(1) = 1 - 2 + 1 = 0.$$

Therefore, $\boxed{f(-2) \neq f(1)}$, so Rolle's Theorem does not apply for the function f on the interval $[-2, 1]$.

19. Let $f(x) = x^{1/3} - x$. Though f is continuous on the closed interval $[-1, 1]$,

$$f'(x) = \frac{1}{3}x^{-2/3} - 1,$$

so f is not differentiable at 0 and therefore is $\boxed{\text{not differentiable on the open interval } (-1, 1)}$. Consequently, Rolle's Theorem cannot be applied to the function f on the interval $[-1, 1]$.

21. Let $f(x) = x^2 + 1$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[0, 2]$. Now,

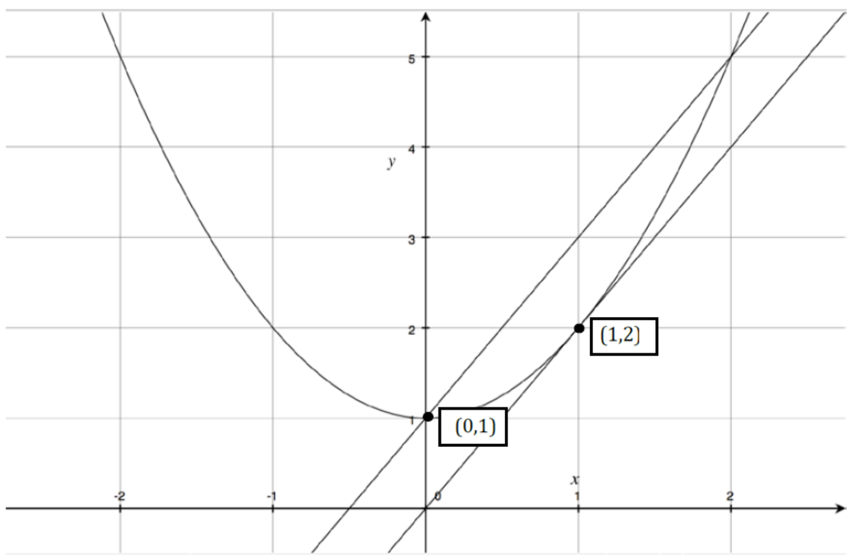
$$f'(x) = 2x \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(0)}{2 - 0} = \frac{5 - 1}{2} = 2,$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 2c = 2,$$

or when $\boxed{c = 1}$.

At $x = c = 1$, the slope of the tangent line to the graph of f is the same as the slope of the secant line connecting the points $(0, 1)$ and $(2, 5)$. That is, $m_{\tan} = f'(c) = f'(1) = 2$ and $m_{\sec} = \frac{f(2)-f(0)}{2-0} = \frac{5-1}{2} = 2$. The tangent line and the secant line are parallel.



23. Let $f(x) = \ln \sqrt{x} = \dfrac{1}{2} \ln x$. The function f is continuous and differentiable on the set $\{x | x > 0\}$, so it is continuous on the closed interval $[1, e]$ and differentiable on the open interval $(1, e)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, e]$. Now,

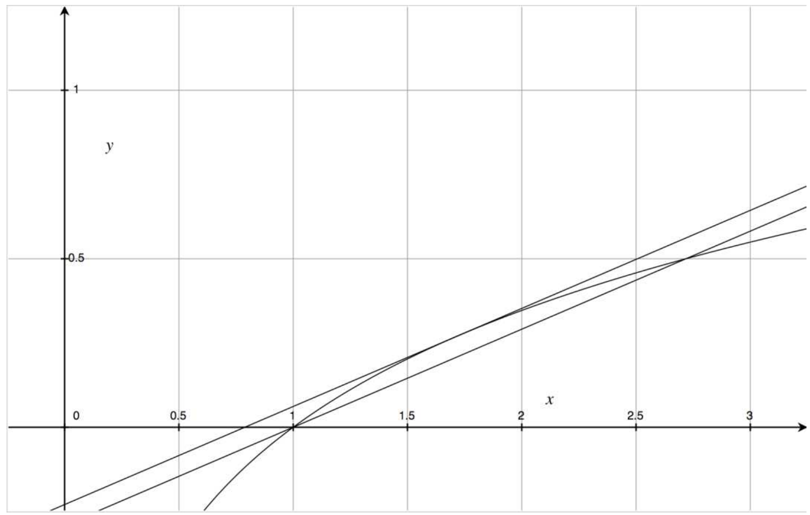
$$f'(x) = \frac{1}{2x} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(e) - f(1)}{e - 1} = \frac{\frac{1}{2} - 0}{e - 1} = \frac{1}{2(e - 1)},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{1}{2c} = \frac{1}{2(e - 1)},$$

or when $\boxed{c = e - 1}$. Note that $e - 1 \approx 1.718 > 1$.

At $x = c = e - 1$, the slope of the tangent line to the graph of f is the same as the slope of the secant line connecting the points $(1, 0)$ and $\left(e, \frac{1}{2}\right)$. That is, $m_{\tan} = f'(c) = f'(e - 1) = \frac{1}{2(e-1)}$ and $m_{\sec} = \frac{f(e)-f(1)}{e-1} = \frac{\frac{1}{2}-0}{e-1} = \frac{1}{2(e-1)}$. The tangent line and the secant line are parallel.



25. Let $f(x) = x^3 - 5x^2 + 4x - 2$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[1, 3]$ and differentiable on the open interval $(1, 3)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 3]$. Now,

$$f'(x) = 3x^2 - 10x + 4 \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-8 - (-2)}{2} = -3,$$

so

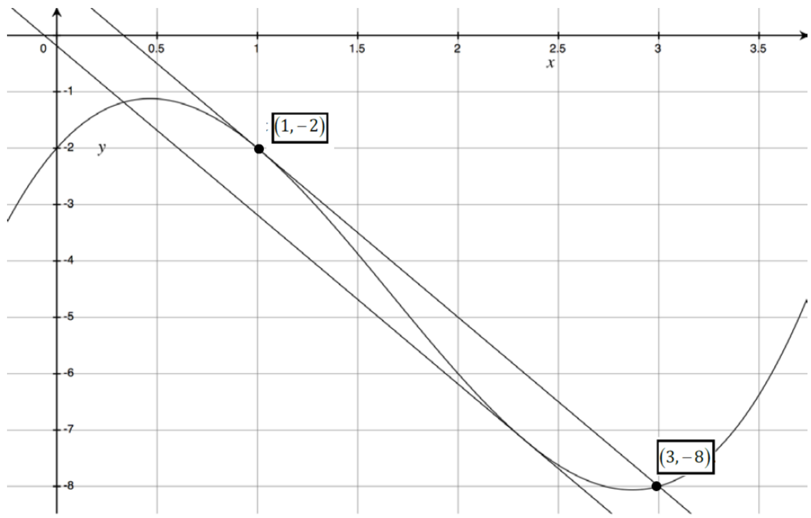
$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 3c^2 - 10c + 4 = -3.$$

Therefore,

$$\begin{aligned} 3c^2 - 10c + 7 &= 0 \\ (3c - 7)(c - 1) &= 0 \\ c &= \frac{7}{3}, 1. \end{aligned}$$

Of these numbers, only $\boxed{c = \frac{7}{3}}$ is in the interval $(1, 3)$.

At $x = c = \frac{7}{3}$, the slope of the tangent line to the graph of f is the same as the slope of the secant line connecting the points $(1, -2)$ and $(3, -8)$. That is, $m_{\tan} = f'(c) = f'\left(\frac{7}{3}\right) = -3$ and $m_{\sec} = \frac{f(3)-f(1)}{3-1} = \frac{-8-(-2)}{2} = -3$. The tangent line and the secant line are parallel.



27. Let $f(x) = \dfrac{x+1}{x} = 1 + \dfrac{1}{x}$. The function f is continuous and differentiable on the set $\{x|x \neq 0\}$, so it is continuous on the closed interval $[1,3]$ and differentiable on the open interval $(1,3)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1,3]$. Now,

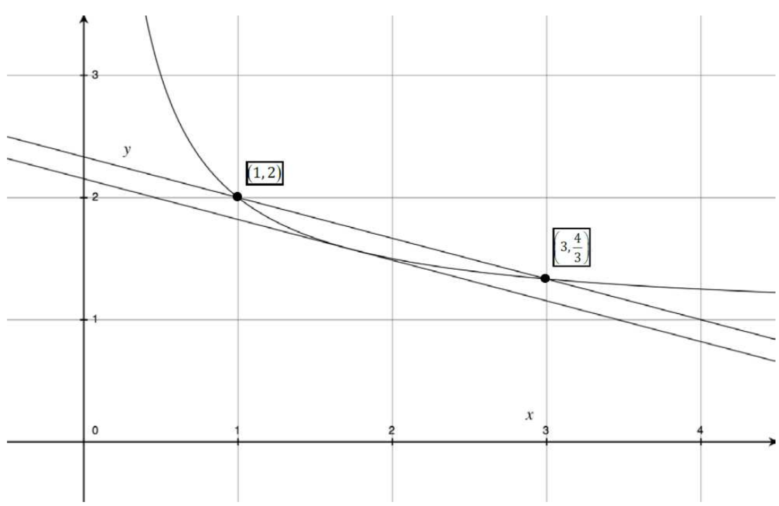
$$f'(x) = -\frac{1}{x^2} \quad \text{and} \quad \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{\frac{4}{3}-2}{2} = -\frac{1}{3},$$

so

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{when} \quad -\frac{1}{c^2} = -\frac{1}{3}.$$

Therefore, $c^2 = 3$ and $c = \pm\sqrt{3}$. Of these numbers, only $\boxed{c = \sqrt{3}}$ is in the interval $(1,3)$.

At $x = c = \sqrt{3}$, the slope of the tangent line to the graph of f is the same as the slope of the secant line connecting the points $(1,2)$ and $\left(3, \frac{4}{3}\right)$. That is, $m_{\tan} = f'(c) = f'\left(\sqrt{3}\right) = -\frac{1}{3}$ and $m_{\sec} = \frac{f(3)-f(1)}{3-1} = \frac{\frac{4}{3}-2}{2} = -\frac{1}{3}$. The tangent line and the secant line are parallel.



29. Let $f(x) = \sqrt[3]{x^2}$. The function f is continuous on the set of all real numbers and differentiable on the set $\{x|x \neq 0\}$, so it is continuous on the closed interval $[1, 8]$ and differentiable on the open interval $(1, 8)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 8]$. Now,

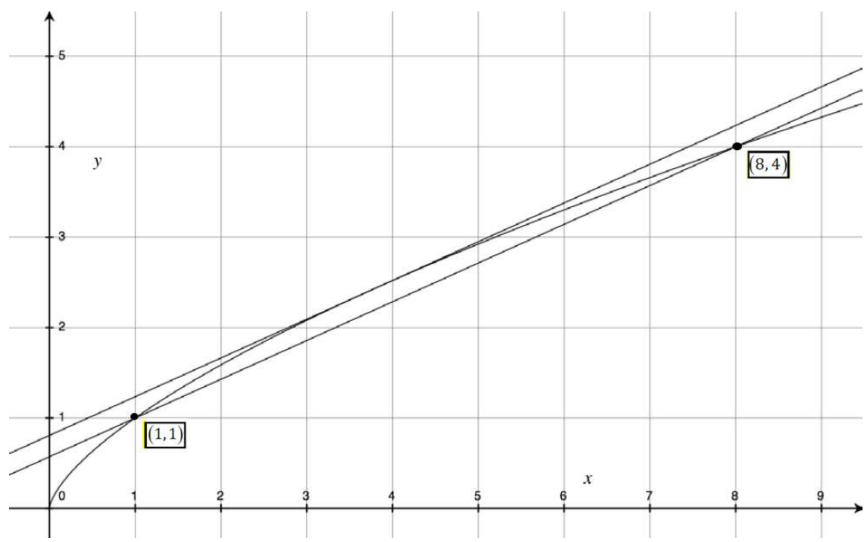
$$f'(x) = \frac{2}{3}x^{-1/3} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(8) - f(1)}{8 - 1} = \frac{4 - 1}{7} = \frac{3}{7},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{2}{3}c^{-1/3} = \frac{3}{7}.$$

Therefore, $c^{1/3} = \dfrac{14}{9}$ and $\boxed{c = \left(\dfrac{14}{9}\right)^3 = \dfrac{2744}{729}}$.

At $x = c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729}$, the slope of the tangent line to the graph of f is the same as the slope of the secant line connecting the points $(1, 1)$ and $(8, 4)$. That is, $m_{\tan} = f'(c) = f'\left[\left(\frac{14}{9}\right)^3\right] = f'\left(\frac{2744}{729}\right) = \frac{3}{7}$ and $m_{\sec} = \frac{f(8)-f(1)}{8-1} = \frac{4-1}{7} = \frac{3}{7}$. The tangent line and the secant line are parallel.



31. Let $f(x) = x^3 + 6x^2 + 12x + 1$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 + 12x + 12 = 3(x^2 + 4x + 4) = 3(x + 2)^2,$$

so -2 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Because $f'(x) > 0$ for all $x \neq -2$, it follows that f is increasing on the intervals $(-\infty, -2)$ and $(-2, \infty)$. Since f is continuous on its domain, we can say that f is $\boxed{\text{increasing on } (-\infty, \infty)}$.

33. Let $f(x) = x^3 - 3x + 1$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 - 3 = 3\left(x^2 - 1\right) = 3(x + 1)(x - 1)$$

so -1 and 1 are critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers -1 and 1 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x+1$	Sign of $x-1$	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	$-$	$-$	$+$	f is increasing
$(-1, 1)$	$+$	$-$	$-$	f is decreasing
$(1, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$ and decreasing on the interval $(-1, 1)$. Since f is continuous on its domain, we can say that f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$ and decreasing on the interval $[-1, 1]$.

35. Let $f(x) = x^4 - 4x^2 + 1$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 4x^3 - 8x = 4x\left(x^2 - 2\right) = 4x\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$$

so $-\sqrt{2}$, 0, and $\sqrt{2}$ are critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $-\sqrt{2}$, 0, and $\sqrt{2}$ to form four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $4x$	Sign of $x+\sqrt{2}$	Sign of $x-\sqrt{2}$	Sign of $f'(x)$	Conclusion
$\left(-\infty, -\sqrt{2}\right)$	$-$	$-$	$-$	$-$	f is decreasing
$\left(-\sqrt{2}, 0\right)$	$-$	$+$	$-$	$+$	f is increasing
$\left(0, \sqrt{2}\right)$	$+$	$+$	$-$	$+$	f is decreasing
$\left(\sqrt{2}, \infty\right)$	$+$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right)$ and decreasing on the intervals $\left(-\infty, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$. Since f is continuous on its domain, we can say that f is increasing on the intervals $\left[-\sqrt{2}, 0\right]$ and $\left[\sqrt{2}, \infty\right)$ and decreasing on the intervals $\left(-\infty, -\sqrt{2}\right]$ and $\left[0, \sqrt{2}\right]$.

37. Let $f(x) = x^{2/3}(x^2 - 4)$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{2/3}(2x) + (x^2 - 4) \cdot \frac{2}{3}x^{-1/3} = \frac{3x(2x) + 2(x^2 - 4)}{3x^{1/3}} = \frac{8x^2 - 8}{3x^{1/3}} = \frac{8(x-1)(x+1)}{3x^{1/3}},$$

so -1, 0, and 1 are the critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers -1, 0, and 1 to form four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8(x-1)$	Sign of $x+1$	Sign of $3x^{1/3}$	Sign of $f'(x)$	Conclusion
$(-\infty,-1)$	$-$	$-$	$-$	$-$	f is decreasing
$(-1,0)$	$-$	$+$	$-$	$+$	f is increasing
$(0,1)$	$-$	$+$	$+$	$-$	f is decreasing
$(1,\infty)$	$+$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-1,0)$ and $(1,\infty)$ and decreasing on the intervals $(-\infty,-1)$ and $(0,1)$. Since f is continuous on its domain, we can say that f is $\boxed{\text{increasing on the intervals } [-1,0] \text{ and } [1,\infty)}$ and $\boxed{\text{decreasing on the intervals } (-\infty,-1] \text{ and } [0,1]}$.

39. Let

$$f(x) = |x^3+3| = \begin{cases} -(x^3+3), & x < -\sqrt[3]{3} \\ x^3+3, & x \geq -\sqrt[3]{3}. \end{cases}$$

On the interval $(-\infty,-\sqrt[3]{3})$, $f'(x) = -3x^2$ exists and is never equal to 0; f has no critical numbers on this interval. On the interval $(-\sqrt[3]{3},\infty)$, $f'(x) = 3x^2$ exists and is equal to 0 when $x = 0$. It follows that 0 is a critical number. At $x = -\sqrt[3]{3}$, the rule for f changes, so it is necessary to investigate the existence of $f'(-\sqrt[3]{3})$. Now,

$$\begin{aligned} \lim_{x\to-\sqrt[3]{3}^-} \frac{f(x)-f(-\sqrt[3]{3})}{x-(-\sqrt[3]{3})} &= \lim_{x\to-\sqrt[3]{3}^-} \frac{-(x^3+3)-0}{x+\sqrt[3]{3}} \\ &= \lim_{x\to-\sqrt[3]{3}^-} \frac{-(x+\sqrt[3]{3})(x^2-\sqrt[3]{3}x+\sqrt[3]{9})}{x+\sqrt[3]{3}} \\ &= \lim_{x\to-\sqrt[3]{3}^-} -(x^2-\sqrt[3]{3}x+\sqrt[3]{9}) = -3\sqrt[3]{9}, \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to-\sqrt[3]{3}^+} \frac{f(x)-f(-\sqrt[3]{3})}{x-(-\sqrt[3]{3})} &= \lim_{x\to-\sqrt[3]{3}^+} \frac{(x^3+3)-0}{x+\sqrt[3]{3}} \\ &= \lim_{x\to-\sqrt[3]{3}^+} \frac{(x+\sqrt[3]{3})(x^2-\sqrt[3]{3}x+\sqrt[3]{9})}{x+\sqrt[3]{3}} \\ &= \lim_{x\to-\sqrt[3]{3}^+} (x^2-\sqrt[3]{3}x+\sqrt[3]{9}) = 3\sqrt[3]{9}. \end{aligned}$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = -\sqrt[3]{3}$. Therefore, $-\sqrt[3]{3}$ is also a critical number of f.

The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $(-\infty,-\sqrt[3]{3})$, $f'(x) = -3x^2 < 0$, while on the intervals $(-\sqrt[3]{3},0)$ and $(0,\infty)$, $f'(x) = 3x^2 > 0$. Therefore, f is decreasing on the interval $(-\infty,-\sqrt[3]{3})$ and increasing on the intervals $(-\sqrt[3]{3},0)$ and $(0,\infty)$. Since f is continuous on its domain, we can say that f is $\boxed{\text{decreasing on the interval } (-\infty,-\sqrt[3]{3}]}$ and is $\boxed{\text{increasing on the interval } [-\sqrt[3]{3},\infty)}$.

41. Let $f(x) = 3\sin x$. The function f is a constant multiple of the trigonometric function $\sin x$, which is differentiable everywhere; therefore, f is differentiable everywhere. The critical numbers of f therefore occur where $f'(x) = 0$. Now,

$$f'(x) = 3\cos x,$$

so the critical numbers on the interval $(0, 2\pi)$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $\left(0, \frac{\pi}{2}\right)$, $f'(x) > 0$; on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $f'(x) < 0$; finally, on the interval $\left(\frac{3\pi}{2}, 2\pi\right)$, $f'(x) > 0$. Therefore, f is increasing on the intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ and decreasing on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Since f is continuous on its domain, we can say that f is $\boxed{\text{increasing on the intervals } \left[0, \frac{\pi}{2}\right] \text{ and } \left[\frac{3\pi}{2}, 2\pi\right]}$ and $\boxed{\text{decreasing on the interval } \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]}$.

43. Let $f(x) = xe^x$. The function g is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = xe^x + e^x = (x+1)e^x,$$

so -1 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number -1 to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x+1$	Sign of e^x	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	$-$	$+$	$-$	f is decreasing
$(-1, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the interval $(-1, \infty)$ and decreasing on the interval $(-\infty, -1)$. Since f is continuous on its domain, we can say that f is $\boxed{\text{increasing on the interval } [-1, \infty)}$ and $\boxed{\text{decreasing on the interval } (-\infty, -1]}$.

45. Let $f(x) = e^x \sin x$. The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = e^x \cos x + e^x \sin x = e^x(\sin x + \cos x),$$

so the critical numbers on the interval $(0, 2\pi)$ are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $\sin x + \cos x$	Sign of $f'(x)$	Conclusion
$\left(0, \frac{3\pi}{4}\right)$	$+$	$+$	$+$	f is increasing
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$+$	$-$	$-$	f is decreasing
$\left(\frac{7\pi}{4}, 2\pi\right)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and decreasing on the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. Since f is continuous on its domain, we can say that f is $\boxed{\text{increasing on the intervals } \left[0, \frac{3\pi}{4}\right] \text{ and } \left[\frac{7\pi}{4}, 2\pi\right]}$ and $\boxed{\text{decreasing on the interval } \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]}$.

47. (a) $\{x | x \neq -2, x \neq 2\}$
   (b) -2, 0, 2, and 4
   (c) 0 and 4
   (d) 2
   (e) -2
   (f) $(-\infty, 0]$ and $[2, 4]$
   (g) $[0, 2]$ and $[4, \infty)$

49. (a) $\{x | x \neq -1, x \neq 0\}$
   (b) -2, -1, 0, 1, and 2
   (c) -2, 1, and 2
   (d) -1
   (e) 0
   (f) $(-\infty, 1]$ and $[2, \infty)$
   (g) $[1, 2]$

Applications and Extensions

51. Let $f(x) = 2x^3 - 6x^2 + 6x - 5$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 6x^2 - 12x + 6 = 6(x^2 - 2x + 1) = 6(x - 1)^2,$$

so 1 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Because $f'(x) > 0$ for all $x \neq 1$, it follows that f is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$. Since f is continuous on its domain, we can say that f is increasing for all x.

53. Let $f(x) = \dfrac{x}{x+1}$. The rational function f is differentiable on its domain, the set $\{x | x \neq -1\}$, so the critical numbers of f occur where $f'(x) = 0$. Now,

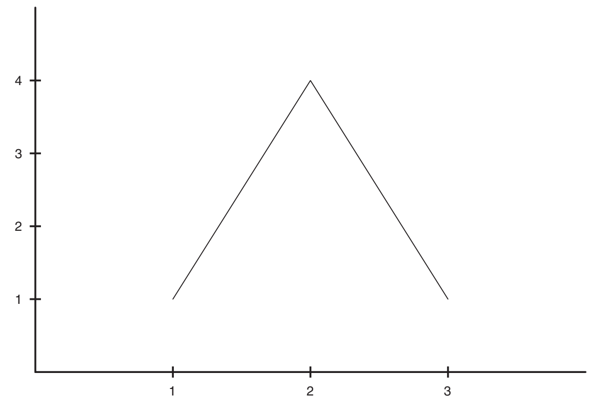
$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x + 1 - x}{(x+1)^2} = \frac{1}{(x+1)^2},$$

so f has no critical numbers. For $x \neq -1$, $f'(x) > 0$, so the Increasing/Decreasing Function Test indicates that f is increasing on the intervals $(-\infty, -1)$ and $(-1, \infty)$; that is, f is increasing on any interval that does not contain $x = -1$.

55. Answers will vary. The figure below displays the graph of a function that is continuous on the closed interval $[1, 3]$ but not differentiable on the open interval $(1, 3)$ – the function is not differentiable at $x = 2$ – and for which the conclusion of the Mean Value Theorem does not hold. Note that

$$\frac{f(3) - f(1)}{3 - 1} = \frac{1 - 1}{2} = 0,$$

but the graph of the function does not have a horizontal tangent line anywhere on the open interval $(1, 3)$.



57. Let $s(t)$ denote the distance function of the automobile, and assume that this function is differentiable. If the automobile traveled 20 miles at an average speed of 40 mph, the trip took one half hour to complete. Let $t = 0$ denote the beginning of the trip and $t = 0.5$ denote the end of the trip. Because s is continuous on the closed interval $[0, 0.5]$ and differentiable on the open interval $(0, 0.5)$, the conditions of the Mean Value Theorem are satisfied. Therefore, there exists at least one c in $(0, 0.5)$ such that

$$s'(c) = \frac{s(0.5) - s(0)}{0.5 - 0} = \frac{20 - 0}{0.5} = 40 \text{ mph}.$$

Now, s' is the speed of the automobile, so the speed was exactly 40 mph at some time during the trip.

59. Let $f(t) = f_2(t) - f_1(t)$, and let $t = 0$ denote the start of the race and $t = T$ denote the end of the race for these two cars. Because the cars start the race together and finish in a tie,

$$f(0) = f_2(0) - f_1(0) = 0 \quad \text{and} \quad f(T) = f_2(T) - f_1(T) = 0.$$

Assuming that f_1 and f_2 are continuous on the closed interval $[0, T]$ and differentiable on the open interval $(0, T)$, the function f will also be continuous on the closed interval $[0, T]$ and differentiable on the open interval $(0, T)$. The Mean Value Theorem therefore applies to f over $[0, T]$, so there is at least one c in $(0, T)$ such that

$$f'(c) = \frac{f(T) - f(0)}{T - 0} = \frac{0 - 0}{T - 0} = 0.$$

Recalling the definition of f, this last statement is equivalent to $f_2'(c) - f_1'(c) = 0$, or $f_2'(c) = f_1'(c)$. Therefore, at some time during the race, the two cars are traveling at the same speed.

61. Let $d = -\frac{1}{192}x^4 + \frac{25}{384}x^3 - \frac{25}{128}x^2$ on the closed interval $[0, 5]$.

(a) The polynomial function d is continuous on the closed interval $[0, 5]$ and differentiable on the open interval $(0, 5)$. Moreover, $d(0) = 0$ and

$$d(5) = -\frac{1}{192}5^4 + \frac{25}{384}5^3 - \frac{25}{128}5^2 = \frac{5^4}{384}(-2 + 5 - 3) = 0.$$

Therefore, d satisfies the conditions of Rolle's Theorem on the interval $[0, 5]$.

(b) According to part (a), $\boxed{d(0) = d(5) = 0}$, so the deflection at the ends of the beam is always zero; therefore, the ends of the beam must be held fixed in place.

(c) Differentiating d yields

$$d'(x) = -\frac{1}{48}x^3 + \frac{25}{128}x^2 - \frac{25}{64}x = -\frac{x}{384}(8x^2 - 75x + 150).$$

Therefore, $d'(c) = 0$ when $c = 0$ and when $8c^2 - 75c + 150 = 0$. Applying the quadratic formula to this last equation gives

$$c = \frac{75 \pm \sqrt{(-75)^2 - 4(8)(150)}}{16} = \frac{75 \pm \sqrt{825}}{16} = \frac{75 \pm 5\sqrt{33}}{16}.$$

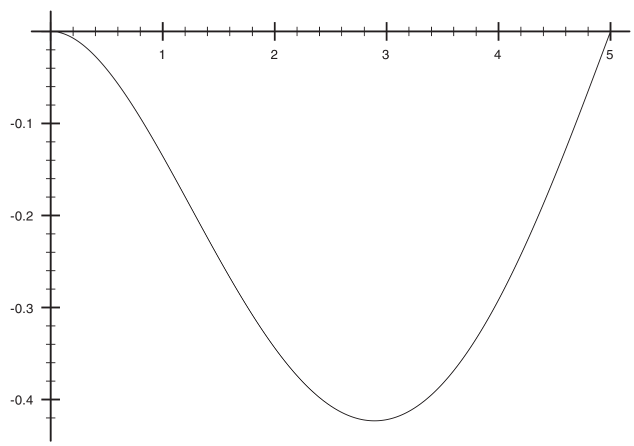
Now,

$$c = \frac{75 + 5\sqrt{33}}{16} \approx 6.483 \quad \text{and} \quad c = \frac{75 - 5\sqrt{33}}{16} \approx 2.892,$$

so the only c in $(0, 5)$ that satisfies the conclusion of Rolle's Theorem is $\boxed{c = \frac{75 - 5\sqrt{33}}{16} \approx 2.892 \text{ ft}}$. Using the computer algebra system *Maple*,

$$d\left(\frac{75 - 5\sqrt{33}}{16}\right) = \boxed{-\frac{24375 + 34375\sqrt{33}}{524288} \text{ ft} \approx -0.423 \text{ ft}}.$$

(d) The figure below displays the graph of d on the interval $[0, 5]$.



63. Let $f(x) = (x - 1)\sin x$. The function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. Additionally, $f(0) = (-1)\sin 0 = 0$, and $f(1) = (0)\sin 1 = 0$, so $f(0) = f(1)$, and all three conditions of Rolle's Theorem are satisfied. Therefore, there exists at least one c in $(0, 1)$ such that $f'(c) = 0$. Now,

$$f'(x) = (x - 1)\cos x + \sin x,$$

so

$$(c - 1)\cos c + \sin c = 0.$$

Because $c \in (0, 1)$, $\cos c \neq 0$, so the previous equation can be divided by $\cos c$ to yield

$$c - 1 + \tan c = 0 \quad \text{or} \quad \tan c + c = 1.$$

In conclusion, the equation $\tan x + x = 1$ has a solution in the interval $(0, 1)$.

65. Let $f(x) = (x-8)^3$. Then $f(8) = (8-8)^3 = 0$, so 8 is a real zero of f. Suppose, for the sake of contradiction, that f does have another real zero at z. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval between z and 8 and differentiable on the open interval between z and 8. Additionally, $f(z) = f(8) = 0$, so Rolle's Theorem applies and there must exist a c between z and 8 such that $f'(c) = 0$. However, $f'(x) = 3(x-8)^2 = 0$ only for $x = 8$ which is not between the positive numbers z and 8. Therefore, f has exactly one real zero.

67. Though the function $f(x) = |x|$ is continuous on the closed interval $[-1, 1]$, it is $\boxed{\text{not differentiable on the open interval } (-1,1)}$ because $f'(x)$ does not exist at $x = 0$. Therefore, Rolle's Theorem does not apply to f on the interval $[-1, 1]$.

69. Let $f(x) = \sin^{-1} x$. Then

$$f(1) - f(0) = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

and $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$. Therefore,

$$f'(N) = \frac{1}{\sqrt{1-N^2}} = \frac{\pi}{2}$$

when

$$\begin{aligned} \sqrt{1-N^2} &= \frac{2}{\pi} \\ 1 - N^2 &= \frac{4}{\pi^2} \\ N^2 &= 1 - \frac{4}{\pi^2} \\ N &= \pm\sqrt{1 - \frac{4}{\pi^2}}. \end{aligned}$$

Of these numbers, $\boxed{N = \sqrt{1 - \frac{4}{\pi^2}}}$ is contained in the interval $(0, 1)$.

71. (a) Let $f(x) = \sqrt{x}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Now,

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} > 0$$

for $x > 0$, so $\boxed{f \text{ is increasing on the interval } (0,\infty)}$.

(b) Because f is increasing on the interval $(0, \infty)$ and is continuous for $x \geq 0$, $\boxed{f \text{ is increasing for all } x \geq 0}$; that is, f is increasing on its domain.

73. (a) $\boxed{\text{Not necessarily true}}$. Let $f(x) = e^x$ and $g(x) = -1$. Then $f'(x) = e^x$ and $g'(x) = 0$ exist and $f'(x) > g'(x)$ for all real x; however, the graphs of $y = f(x)$ and $y = g(x)$ never intersect because $f(x) > 0$ for all real x.

(b) $\boxed{\text{True}}$. Suppose for sake of contradiction that the graphs of $y = f(x)$ and $y = g(x)$ intersect more than once. In particular, suppose the graphs intersect at a and b with $a < b$. Consider the function $h(x) = f(x) - g(x)$. The functions f and g are continuous and differentiable everywhere, so the function h is also continuous and differentiable everywhere. Additionally,

$$h(a) = f(a) - g(a) = 0 \quad \text{and} \quad h(b) = f(b) - g(b) = 0,$$

so $h(a) = h(b)$, and Rolle's Theorem applies. It follows that there exists at least one c in (a, b) such that $h'(c) = f'(c) - g'(c) = 0$; equivalently, $f'(c) = g'(c)$, in violation of the condition that $f'(x) > g'(x)$ for all real x. Therefore, the graphs of $y = f(x)$ and $y = g(x)$ can intersect no more than once.

(c) $\boxed{\text{Not necessarily true}}$. Let $f(x) = 2x$ and $g(x) = x$. Then $f'(x) = 2$ and $g'(x) = 1$ exist and $f'(x) > g'(x)$ for all real x; however, the graphs of $y = f(x)$ and $y = g(x)$ intersect at the origin.

(d) $\boxed{\text{False}}$. See the proof for part (b).

(e) $\boxed{\text{False}}$. Even if the graphs intersect, the tangent lines will be different because the condition $f'(x) > g'(x)$ guarantees that the tangent lines will have different slopes.

75. Consider the function $f(x) = \dfrac{e^x}{x^2}$ for $x > 0$. Now,

$$f'(x) = \frac{x^2 e^x - e^x(2x)}{x^4} = \frac{e^x(x-2)}{x^3},$$

so the only critical number on $x > 0$ is 2. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number 2 to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $x-2$	Sign of x^3	Sign of $f'(x)$	Conclusion
$(0,2)$	$+$	$-$	$+$	$-$	f is decreasing
$(2,\infty)$	$+$	$+$	$+$	$+$	f is increasing

Because f is decreasing on the interval $(0, 2)$ and increasing on the interval $(2, \infty)$, f must have an absolute minimum at 2. Therefore,

$$f(x) = \frac{e^x}{x^2} \geq f(2) = \frac{e^2}{4} \approx 1.847 > 1,$$

so that, for $x > 0$

$$\frac{e^x}{x^2} > 1 \quad \text{or} \quad e^x > x^2.$$

77. For $x > 1$, $\ln x > 0$. Next, consider the function $f(x) = x - \ln x$ for $x > 1$. Now, $f'(x) = 1 - \dfrac{1}{x}$. For $x > 1$

$$\frac{1}{x} < 1 \quad \text{so that} \quad 1 - \frac{1}{x} > 0;$$

that is, $f'(x) > 0$. Therefore, by the Increasing/Decreasing Function Test, f is increasing for $x > 1$, so that $f(x) > f(1) = 1 - \ln 1 = 1 > 0$. Therefore, for $x > 1$

$$f(x) = x - \ln x > 0 \quad \text{or} \quad x > \ln x.$$

79.

$$\begin{aligned} y &= \sin^{-1} x + \cos^{-1} x \\ y' &= \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} \\ &= 0. \end{aligned}$$

Since $y' = 0$ for all x, therefore y is a constant function, meaning that substituting any x from the domain of all real numbers will generate the same value of y. That is, substituting an arbitrary value such as $x = 0$ will generate a y-value that will be the same for any other values of x. For $x = 0$,

$$\begin{aligned} y &= \sin^{-1} 0 + \cos^{-1} 0 \\ &= 0 + \frac{\pi}{2} \\ &= \frac{\pi}{2}. \end{aligned}$$

81. Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Further, suppose that $f(x) = 0$ for three different numbers in (a, b). Let x_1, x_2, and x_3 denote the three zeros with $x_1 < x_2 < x_3$. Consider the interval $[x_1, x_2]$. Because $[x_1, x_2]$ is contained in $[a, b]$, f is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2). Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Next, consider the interval $[x_2, x_3]$. Because $[x_2, x_3]$ is contained in $[a, b]$, f is continuous on the closed interval $[x_2, x_3]$ and differentiable on the open interval (x_2, x_3). Additionally, $f(x_2) = f(x_3) = 0$, so Rolle's Theorem applies and there exists at least one number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. By construction, $c_1 < x_2$ and $c_2 > x_2$, so c_1 cannot be equal to c_2. Therefore, there are at least two numbers in (a, b) at which $f'(x) = 0$.

83. Suppose, for sake of contradiction, that f has an absolute extreme value on (a, b), say at c. Because c is in (a, b) – and, in particular, not at an endpoint of the interval – $f(c)$ must also be a local extreme value. Now, local extreme values can only occur at critical numbers. It follows that $f'(c)$ must be equal to zero or not exist. However, $f'(x)$ exists and is never equal to 0 for x in (a, b). Therefore, f cannot have an extreme value on (a, b).

Challenge Problems

85. Let a, b, c, and d be real numbers and consider the function $f(x) = ax^3 + bx^2 + cx + d$. This polynomial function is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3ax^2 + 2bx + c.$$

The discriminant of this quadratic function is $(2b)^2 - 4(3a)(c) = 4(b^2 - 3ac)$. Consider the following cases:

- Case I: $b^2 - 3ac < 0$: In this case, $f'(x)$ is never equal to zero and is always positive when $a > 0$ and always negative when $a < 0$. Therefore, f is increasing for all x when $a > 0$ and decreasing for all x when $a < 0$.
- Case II: $b^2 - 3ac = 0$: In this case, $f'(x) = 0$ when $x = -\dfrac{b}{3a}$. For all other x, $f'(x) > 0$ when $a > 0$ and $f'(x) < 0$ when $a < 0$. Therefore, f is increasing for all $x \neq -\dfrac{b}{3a}$ when $a > 0$ and decreasing for all $x \neq -\dfrac{b}{3a}$ when $a < 0$. Because f is continuous on its domain, we can say f is increasing for all x when $a > 0$ and decreasing for all x when $a < 0$.
- Case III: $b^2 - 3ac > 0$: In this case, $f'(x) = 0$ when

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

For notation, let

$$x_1 = \min\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) \quad \text{and} \quad x_2 = \max\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right).$$

If $a > 0$, then $f'(x) > 0$ and f is increasing on the intervals $(-\infty, x_1)$ and (x_2, ∞) and $f'(x) < 0$ and f is decreasing on the interval (x_1, x_2). If $a < 0$, then $f'(x) > 0$ and f is increasing on the interval (x_1, x_2) and $f'(x) < 0$ and f is decreasing on the intervals $(-\infty, x_1)$ and (x_2, ∞).

87. Let $f(x) = x^n + ax + b$, where n is a positive odd integer. Suppose, for sake of contradiction, that f has more than three distinct real zeros. Let x_1, x_2, x_3, and x_4, with $x_1 < x_2 < x_3 < x_4$, denote four of the distinct real zeros of f. Consider the interval $[x_1, x_2]$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2). Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Next, consider the interval $[x_2, x_3]$. Following the same reasoning as above, f is continuous on the closed interval $[x_2, x_3]$ and differentiable on the open interval (x_2, x_3). Additionally, $f(x_2) = f(x_3) = 0$, so Rolle's Theorem applies and there exists at least one number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. Finally, consider the interval $[x_3, x_4]$. Following the same reasoning as above, f is continuous on the closed interval $[x_3, x_4]$ and differentiable on the open interval (x_3, x_4). Additionally, $f(x_3) = f(x_4) = 0$, so Rolle's Theorem applies and there exists at least one number c_3 in (x_3, x_4) such that $f'(c_3) = 0$. It follows that there are at least three distinct numbers in (a, b) at which $f'(x) = 0$. However, $f'(x) = nx^{n-1} + a$, which has only at most two distinct real zeros because n is an odd integer, so $n - 1$ is an even integer. Therefore, the function $f(x) = x^n + ax + b$, where n is a positive odd integer, has at most three distinct real zeros.

89. Let $f(x) = x^n + ax^2 + b$, where n is a positive even integer. Suppose, for sake of contradiction, that f has more than four distinct real zeros. Let x_1, x_2, x_3, x_4, and x_5, with $x_1 < x_2 < x_3 < x_4 < x_5$, denote five of the distinct real zeros of f. Consider the interval $[x_1, x_2]$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2). Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. By similar reasoning, there exists at least one number c_2 in (x_2, x_3), at least one number c_3 in (x_3, x_4), and at least one number c_4 in (x_4, x_5) such that $f'(c_2) = f'(c_3) = f'(c_4) = 0$. However, $f'(x) = nx^{n-1} + 2ax = x(nx^{n-2} + 2a)$, which has at most three distinct real zeros because n is an even integer, so $n - 2$ is also an even integer. Therefore, the function $f(x) = x^n + ax^2 + b$, where n is a positive even integer, has at most four distinct real zeros.

91. Let a, b, c, and d be real numbers with $ad - bc \neq 0$. Additionally, let $n \geq 2$ be an integer. Consider the function $f(x) = \dfrac{ax^n + b}{cx^n + d}$. The rational function f is differentiable on its domain, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$\begin{aligned} f'(x) &= \frac{(cx^n + d)(anx^{n-1}) - (ax^n + b)(cnx^{n-1})}{(cx^n + d)^2} \\ &= \frac{acnx^{2n-1} + adnx^{n-1} - acnx^{2n-1} - bcnx^{n-1}}{(cx^n + d)^2} = \frac{(ad - bc)nx^{n-1}}{(cx^n + d)^2}, \end{aligned}$$

so $\boxed{\text{0 is the only critical number}}$. Suppose that n is odd, so that $n-1$ is even. If $ad-bc > 0$, then $f'(x) > 0$ and f is increasing everywhere on the domain of f except at $x = 0$. Because f is continuous on its domain, we can say that f is increasing on its domain. If $ad-bc < 0$, then $f'(x) < 0$ and f is decreasing everywhere on the domain of f except at $x = 0$. Because f is continuous on its domain, we can say that f is decreasing on its domain. On the other hand, suppose that n is even, so that $n - 1$ is odd. If $ad - bc > 0$, then $f'(x) > 0$ and f is increasing on that portion of the domain of f where $x > 0$, while $f'(x) < 0$ and f is decreasing on that portion of the domain of f where $x < 0$. If $ad - bc < 0$, then $f'(x) > 0$ and f is increasing on that portion of the domain of f where $x < 0$, while $f'(x) < 0$ and f is decreasing on that portion of the domain of f where $x > 0$.

93. Let a and b be real numbers with $0 < a < b$ and consider the function $f(x) = \ln x$ on the interval $[a, b]$. The natural logarithm function is continuous and differentiable on its

domain, so it is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). By the Mean Value Theorem, there exists a number c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln b - \ln a}{b - a} = \frac{1}{b - a} \ln \frac{b}{a}.$$

Now,

$$f'(x) = \frac{1}{x} \quad \text{so} \quad f'(c) = \frac{1}{c}.$$

With $a < c < b$, it follows that

$$\frac{1}{b} < \frac{1}{c} = f'(c) < \frac{1}{a},$$

so, substituting for $f'(c)$ from the Mean Value Theorem,

$$\frac{1}{b} < \frac{1}{b - a} \ln \frac{b}{a} < \frac{1}{a}.$$

Taking the reciprocal of each component of this compound inequality and then multiplying by $\ln(b/a)$, which is positive because $b > a$ so that $b/a > 1$, yields

$$a \ln \frac{b}{a} < b - a < b \ln \frac{b}{a}.$$

95. This is because the ranges of $\cot^{-1}$ and $\tan^{-1}$ are the same on the domain $x > 0$ but different on the domain $x < 0$.

For $x > 0$ (which also means that $\frac{1}{x} > 0$), both $\cot^{-1} x$ and $\tan^{-1} \frac{1}{x}$ have the range $\left(0, \frac{\pi}{2}\right)$. Then, as shown in Section 3.3 Problem 58, they are equal.

But for $x < 0$ (which also means that $\frac{1}{x} < 0$), $\cot^{-1} x$ has the range $\left(\frac{\pi}{2}, \pi\right)$ but $\tan^{-1} \frac{1}{x}$ has the range $\left(-\frac{\pi}{2}, 0\right)$. However, $\tan^{-1} \frac{1}{x} + \pi$ has the range $\left(-\frac{\pi}{2} + \pi, 0 + \pi\right) = \left(\frac{\pi}{2}, \pi\right)$. So $\cot^{-1} x$ and $\tan^{-1} \frac{1}{x} + \pi$ have the same range and, as above, are equal.

AP® Practice Problems

1. f is continuous on the closed interval $[-2, 5]$ and differentiable on the open interval $(-2, 5)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[-2, 5]$, which guarantees that there is at least one number c in $(-2, 5)$, for which $f'(c) = \frac{f(5) - f(-2)}{5 - (-2)} = \frac{3 - 3}{7} = 0$.

Since f is continuous and differentiable it could be a horizontal line from the point $(-2, 3)$ to $(5, 3)$, in which case $f'(c) = 0$ for all c in $(-2, 5)$ so B and C are false, and $f(c) = 3$ for all c in $(-2, 5)$ so A is false.

A second option would be that f increases from $(-2, 3)$, but then it would have to decrease in some interval in order to pass through $(5, 3)$.

A third option would be that f decreases from $(-2, 3)$, but then it would have to increase in some interval in order to pass through $(5, 3)$.

For either the second or third option there *could* be a number c in the interval $(-2, 5)$ for which $f(c) = 0$ and A is true, but it is not *necessary.*

Because f must both increase and decrease in the second or third options, choices B and C are not possible.

CHOICE D

3. $f(x) = \sqrt{x} = x^{1/2}$, which is defined for $x \geq 0$.

   $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$, so $f(x)$ is differentiable for $x \geq 0$.

   Since $f(x)$ is differentiable on $x > 0$, it is continuous on $(0, 4)$. Also, f is continuous at its endpoints $(0, 0)$ and $(4, 2)$ since $\lim_{x \to 0^+} f(x) = f(0) = 0$ and $\lim_{x \to 4^-} f(x) = f(4) = 2$.

   Therefore, the conditions for the Mean Value Theorem are satisfied, so there is at least one number c in $(0, 4)$, such that $f'(c) =$ the slope of the line between $(0, 0)$ and $(4, 2)$.

$$\begin{aligned} f'(c) &= \frac{f(4) - f(0)}{4 - 0} \\ \frac{1}{2c^{1/2}} &= \frac{2 - 0}{4} \\ &= \frac{1}{2} \\ c &= \boxed{1}. \end{aligned}$$

   CHOICE C

5. $f(x) = x^4 - 4x^3 + 4x^2 + 1$ is decreasing on the interval(s) where $f'(x)$ is negative.

   We determine the possible interval(s) where $f(x)$ is decreasing by setting $f'(x) = 0$ and solving for x, the endpoints of the intervals to be evaluated.

$$\begin{aligned} f'(x) &= 0 \\ 4x\left(x^2 - 3x + 2\right) &= 0 \\ 4x(x - 1)(x - 2) &= 0 \\ x &= 0,\ x = 1,\ \text{or } x = 2 \end{aligned}$$

Interval	Sign of x	Sign of $x - 1$	Sign of $x - 2$	Sign of $f'(x) = 4x^3 - 12x^2 + 8x$	Conclusion
$(-\infty, 0)$	$-$	$-$	$-$	$-$	Decreasing
$(0, 1)$	$+$	$-$	$-$	$+$	Increasing
$(1, 2)$	$+$	$+$	$-$	$-$	Decreasing
$(2, \infty)$	$+$	$+$	$+$	$+$	Increasing

   Therefore, f is decreasing on the intervals $\boxed{(-\infty, 0] \text{ and } [1, 2]}$.

   CHOICE A

7. $h(x) = f(x)g(x)$

   By the Product Rule, $h'(x) = f'(x)g(x) + g'(x)f(x)$

   But, by the given, $h'(x) = f'(x)g(x)$

   Therefore

$$\begin{aligned} f'(x)g(x) + g'(x)f(x) &= f'(x)g(x) \\ g'(x)f(x) &= 0 \\ g'(x) &= 0 \text{ since } f'(x) \neq 0 \text{ since } f'(x) > 0, \text{ by the given.} \end{aligned}$$

   Since $g'(x) = 0$ for all real numbers x, therefore $g(x)$ is a constant function (a horizontal line).

Since $g(0) = 4$, therefore $g(x) = \boxed{4}$ for all real numbers x.

CHOICE D

9. An object in rectilinear motion is at rest when $v(t) = x'(t) = 0$.

For $x(t) = t^3 + \frac{3}{2}t^2 - 18t + 4$, $x'(t) = 3t^2 + 3t - 18$.

So

$$\begin{aligned} x'(t) &= 0 \\ 3t^2 + 3t - 18 &= 0 \\ 3(t+3)(t-2) &= 0 \\ t &= -3 \text{ or } t = 2 \end{aligned}$$

The only one of those which is in the domain, $t \geq 0$, is $\boxed{t = 2}$.

CHOICE B

4.4 Local Extrema and Concavity

Concepts and Vocabulary

1. $\boxed{\text{False}}$. If a function f is continuous on the interval $[a, b]$, differentiable on the interval (a, b), and changes from an increasing function to a decreasing function at the point $(c, f(c))$, then $f(c)$ **is a local maximum value of** f.

3. Suppose a function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). If the graph of f lies above each of its tangent lines on the interval (a, b), then f is $\boxed{\textbf{(a) concave up}}$ on (a, b).

5. Suppose f is a function that is differentiable on an open interval containing c and the concavity of f changes at the point $(c, f(c))$. Then $(c, f(c))$ is an $\boxed{\textbf{(a) inflection point}}$ of f.

7. $\boxed{\text{True}}$. Suppose f is a function for which f' and f'' exist on an open interval (a, b) and suppose c, $a < c < b$, is a critical number of f. If $f''(c) = 0$, then the Second Derivative Test cannot be used to determine if there is a local extremum at c.

Skill Building

9. (a) Based on the graph of f, f has
   - a $\boxed{\text{local maximum value at the point } (-1, 0)}$,
   - a $\boxed{\text{local minimum value at the points } (-2.5, -4) \text{ and } (0.5, -4)}$, and
   - a $\boxed{\text{point of inflection at the points } (-1.8, -2) \text{ and } (-0.2, -2)}$.

   (b) Based on the graph of f, f is increasing on the intervals $(-2.5, -1)$ and $(0.5, \infty)$ and decreasing on the intervals $(-\infty, -2.5)$ and $(-1, 0.5)$. Since f is continuous on its domain, we can say that f is
   - $\boxed{\text{increasing on the intervals } [-2.5, -1] \text{ and } [0.5, \infty)}$ and
   - $\boxed{\text{decreasing on the intervals } (-\infty, -2.5] \text{ and } [-1, 0.5]}$.

Additionally, f is

- concave up on the intervals $(-\infty, -1.8)$ and $(-0.2, \infty)$, and
- concave down on the interval $(-1.8, -0.2)$.

11. (a) Based on the graph of f, f has

   - a local maximum value at the points $(-2, 3)$ and $(12, 10)$,
   - a local minimum value at the point $(0, 0)$, and
   - no points of inflection.

   (b) Based on the graph of f, f is increasing on the intervals $(-\infty, -2)$ and $(0, 12)$ and decreasing on the intervals $(-2, 0)$ and $(12, \infty)$. Since f is continuous on its domain, we can say that f is

   - increasing on the intervals $(-\infty, -2]$ and $[0, 12]$ and
   - decreasing on the intervals $[-2, 0]$ and $[12, \infty)$.

   Additionally, f is concave down on the intervals $(-\infty, 0)$ and $(0, \infty)$. The function is never concave up.

13. Let $f(x) = x^3 - 6x^2 + 2$.

   (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

   $$f'(x) = 3x^2 - 12x = 3x(x - 4),$$

   so 0 and 4 are critical numbers.

   (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers 0 and 4 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3x$	Sign of $x - 4$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$-$	$-$	$+$	f is increasing
$(0, 4)$	$+$	$-$	$-$	f is decreasing
$(4, \infty)$	$+$	$+$	$+$	f is increasing

   Therefore, by the First Derivative Test, f has a local maximum value at 0 and a local minimum value at 4. The local maximum value is $f(0) = 2$; the local minimum value is $f(4) = -30$.

15. Let $f(x) = 3x^4 - 4x^3$.

   (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

   $$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1),$$

   so 0 and 1 are critical numbers.

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers 0 and 1 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $12x^2$	Sign of $x-1$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$+$	$-$	$-$	f is decreasing
$(0, 1)$	$+$	$-$	$-$	f is decreasing
$(1, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, by the First Derivative Test, f has neither a local maximum value nor a local minimum value at 0 and a local minimum value at 1. The $\boxed{\text{local minimum value is } f(1) = -1}$.

17. Let $f(x) = (5 - 2x)e^x$.

(a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = (5 - 2x)e^x - 2e^x = (3 - 2x)e^x,$$

so $\boxed{\frac{3}{2}}$ is the only critical number.

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number $\frac{3}{2}$ to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3-2x$	Sign of e^x	Sign of $f'(x)$	Conclusion
$\left(-\infty, \frac{3}{2}\right)$	$+$	$+$	$+$	f is increasing
$\left(\frac{3}{2}, \infty\right)$	$-$	$+$	$-$	f is decreasing

Therefore, by the First Derivative Test, f has a local maximum value at $\frac{3}{2}$. The $\boxed{\text{local maximum value is } f\left(\frac{3}{2}\right) = 2e^{3/2}}$.

19. Let $f(x) = x^{2/3} + x^{1/3}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3} = \frac{2x^{1/3} + 1}{3x^{2/3}},$$

so $f'(x) = 0$ when $x = -\frac{1}{8}$ and $f'(x)$ does not exist when $x = 0$. Therefore, $\boxed{-\frac{1}{8} \text{ and } 0}$ are the critical numbers of f.

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $-\frac{1}{8}$ and 0 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2x^{1/3}+1$	Sign of $3x^{2/3}$	Sign of $f'(x)$	Conclusion
$\left(-\infty,-\frac{1}{8}\right)$	$-$	$+$	$-$	f is decreasing
$\left(-\frac{1}{8},0\right)$	$+$	$+$	$+$	f is increasing
$(0,\infty)$	$+$	$+$	$+$	f is increasing

Therefore, by the First Derivative Test, f has a local minimum value at $-\dfrac{1}{8}$ but has neither a local maximum value nor a local minimum value at 0. The $\boxed{\text{local minimum value is } f\left(-\frac{1}{8}\right)=-\frac{1}{4}}$.

21. Let $g(x)=x^{2/3}(x^2-4)$.

(a) The critical numbers of g occur where $g'(x)=0$ or where $g'(x)$ does not exist. Now,

$$g'(x)=x^{2/3}(2x)+\frac{2}{3}x^{-1/3}(x^2-4)=\frac{6x^2+2(x^2-4)}{3x^{1/3}}=\frac{8(x-1)(x+1)}{3x^{1/3}},$$

so $g'(x)=0$ when $x=\pm1$ and $g'(x)$ does not exist when $x=0$. Therefore, $\boxed{-1,\ 0,\text{ and }1}$ are the critical numbers of g.

(b) The Increasing/Decreasing Function Test states that g is increasing on intervals where $g'(x)>0$ and that g is decreasing on intervals where $g'(x)<0$. These inequalities are solved by using the critical numbers -1, 0, and 1 to form four intervals. The sign of $g'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8(x-1)$	Sign of $x+1$	Sign of $3x^{1/3}$	Sign of $g'(x)$	Conclusion
$(-\infty,-1)$	$-$	$-$	$-$	$-$	g is decreasing
$(-1,0)$	$-$	$+$	$-$	$+$	g is increasing
$(0,1)$	$-$	$+$	$+$	$-$	g is decreasing
$(1,\infty)$	$+$	$+$	$+$	$+$	g is increasing

Therefore, by the First Derivative Test, g has a local minimum value at -1, a local maximum value at 0, and a local minimum value at 1. The $\boxed{\text{local minimum values are } g(-1)=-3 \text{ and } g(1)=-3}$, and the $\boxed{\text{local maximum value is } g(0)=0}$.

23. Let $f(x)=\dfrac{\ln x}{x^3}$.

(a) The function f is differentiable on its domain, the set $\{x|x>0\}$, so the critical numbers of f occur where $f'(x)=0$. Now,

$$f'(x)=\frac{x^3\cdot\frac{1}{x}-3x^2\ln x}{x^6}=\frac{1-3\ln x}{x^4},$$

so $f'(x)=0$ when $x=\sqrt[3]{e}$. Therefore, $\boxed{\sqrt[3]{e}}$ is the only critical number of f.

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x)>0$ and that f is decreasing on intervals where $f'(x)<0$. These inequalities are solved by using the critical number $\sqrt[3]{e}$ to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $1-3\ln x$	Sign of x^4	Sign of $f'(x)$	Conclusion
$(0,\sqrt[3]{e})$	+	+	+	f is increasing
$(\sqrt[3]{e},\infty)$	−	+	−	f is decreasing

Therefore, by the First Derivative Test, f has a local maximum value at $\sqrt[3]{e}$. The $\boxed{\text{local maximum value is } f(\sqrt[3]{e}) = \dfrac{1}{3e}}$.

25. Let $f(\theta) = \sin\theta - 2\cos\theta$.

(a) The function f is the difference of the trigonometric function $\sin x$ and a constant multiple of the trigonometric function $\cos x$, so it is differentiable everywhere. The critical numbers of f therefore occur where $f'(\theta) = 0$. Now,

$$f'(\theta) = \cos\theta + 2\sin\theta,$$

so $f'(\theta) = 0$ when $\cos\theta = -2\sin\theta$ or $\tan\theta = -\dfrac{1}{2}$. It follows that the critical numbers of f are $\boxed{-\tan^{-1}\dfrac{1}{2} + k\pi}$, where k is any integer. Remember that the tangent function is periodic with period π.

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(\theta) > 0$ and that f is decreasing on intervals where $f'(\theta) < 0$. The critical numbers of f divide the number line into intervals of the form

$$\left(-\tan^{-1}\frac{1}{2} + 2k\pi, -\tan^{-1}\frac{1}{2} + (2k+1)\pi\right)$$

and

$$\left(-\tan^{-1}\frac{1}{2} + (2k+1)\pi, -\tan^{-1}\frac{1}{2} + (2k+2)\pi\right)$$

for each integer k. Select a test number in each interval and evaluate f' at that number. If $f'(\theta) < 0$ at the test number, then $f'(\theta) < 0$ throughout the interval; if $f'(\theta) > 0$ at the test number, then $f'(\theta) > 0$ throughout the interval. The following table summarizes the results:

Interval	Test Number	Value of $f'(\theta)$ at Test Number	Sign of $f'(\theta)$	Conclusion
$\left(-\tan^{-1}\frac{1}{2} + 2k\pi, -\tan^{-1}\frac{1}{2} + (2k+1)\pi\right)$	$2k\pi$	1	+	f is increasing
$\left(-\tan^{-1}\frac{1}{2} + (2k+1)\pi, -\tan^{-1}\frac{1}{2} + (2k+2)\pi\right)$	$(2k+1)\pi$	−1	−	f is decreasing

Therefore, f is increasing on the intervals $\left(-\tan^{-1}\dfrac{1}{2} + 2k\pi, -\tan^{-1}\dfrac{1}{2} + (2k+1)\pi\right)$ and decreasing on the intervals $\left(-\tan^{-1}\dfrac{1}{2} + (2k+1)\pi, -\tan^{-1}\dfrac{1}{2} + (2k+2)\pi\right)$. By the First Derivative Test, it follows that f has a local maximum value at all points of the form $-\tan^{-1}\dfrac{1}{2} + (2k+1)\pi$ and a local minimum value at all points of the form $-\tan^{-1}\dfrac{1}{2} + 2k\pi$. The local maximum values are

$$\begin{aligned} f\left(-\tan^{-1}\frac{1}{2} + (2k+1)\pi\right) &= \sin\left(-\tan^{-1}\frac{1}{2} + (2k+1)\pi\right) - 2\cos\left(-\tan^{-1}\frac{1}{2} + (2k+1)\pi\right) \\ &= \sin\left(\tan^{-1}\frac{1}{2}\right) + 2\cos\left(\tan^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{5}} + 2\frac{2}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \boxed{\sqrt{5}}, \end{aligned}$$

and the local minimum values are

$$\begin{aligned} f\left(-\tan^{-1}\frac{1}{2}+2k\pi\right) &= \sin\left(-\tan^{-1}\frac{1}{2}+2k\pi\right)-2\cos\left(-\tan^{-1}\frac{1}{2}+2k\pi\right) \\ &= -\sin\left(\tan^{-1}\frac{1}{2}\right)-2\cos\left(\tan^{-1}\frac{1}{2}\right)=-\frac{1}{\sqrt{5}}-2\frac{2}{\sqrt{5}}=-\frac{5}{\sqrt{5}}=\boxed{-\sqrt{5}}. \end{aligned}$$

27. Let $s = t^2 - 2t + 3$.

(a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 2t - 2 = 2(t-1),$$

so 1 is the only critical number of s. When $t < 1$, $v(t) < 0$ and when $t > 1$, $v(t) > 0$. Therefore, the object moves to the $\boxed{\text{right on the interval } (1,\infty)}$ and moves to the $\boxed{\text{left on the interval } (0,1)}$.

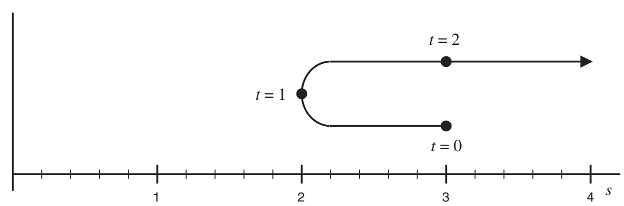
(b) The object reverses direction at $\boxed{t = 1}$.

(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

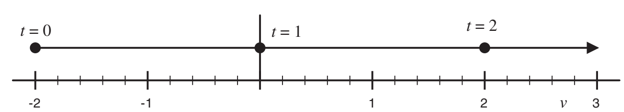
$$a(t) = v'(t) = 2 > 0$$

for all $t \geq 0$. Therefore, the velocity is $\boxed{\text{increasing on the interval } (0,\infty)}$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



29. Let $s = 2t^3 + 6t^2 - 18t + 1$.

(a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 6t^2 + 12t - 18 = 6(t^2 + 2t - 3) = 6(t+3)(t-1).$$

As the domain of s is the set $\{t | t \geq 0\}$ and -3 is not in this domain, -3 is not a critical number. When $t < 1$, $v(t) < 0$ and when $t > 1$, $v(t) > 0$. Therefore, the object moves to the $\boxed{\text{right on the interval } (1,\infty)}$ and moves to the $\boxed{\text{left on the interval } (0,1)}$.

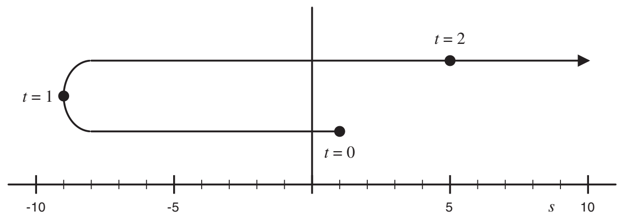
(b) The object reverses direction at $\boxed{t = 1}$.

(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

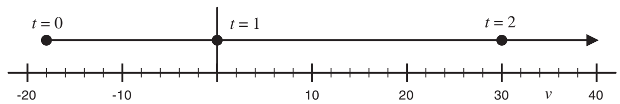
$$a(t) = v'(t) = 12t + 12 = 12(t+1) > 0$$

for all $t \geq 0$. Therefore, the velocity is $\boxed{\text{increasing on the interval } (0,\infty)}$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



31. Let $s = 2t - \dfrac{6}{t}$.

(a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 2 + \frac{6}{t^2} > 0$$

for all $t > 0$. Therefore, the object is moving to the $\boxed{\text{right on the interval } (0, \infty)}$.

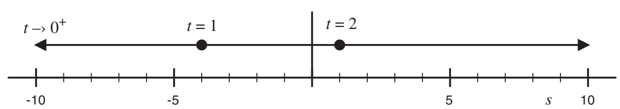
(b) The object $\boxed{\text{does not reverse direction}}$.

(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

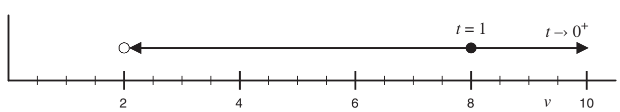
$$a(t) = v'(t) = -\frac{12}{t^3} < 0$$

for all $t > 0$. Therefore, the velocity of the object is $\boxed{\text{decreasing on the interval } (0, \infty)}$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



33. Let $s = 2\sin(3t)$ for $0 \le t \le \dfrac{2\pi}{3}$.

(a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 6\cos(3t).$$

When $0 < t < \dfrac{\pi}{6}$, $v(t) > 0$; when $\dfrac{\pi}{6} < t < \dfrac{\pi}{2}$, $v(t) < 0$; and when $\dfrac{\pi}{2} < t < \dfrac{2\pi}{3}$, $v(t) > 0$. Therefore, the object is moving to the $\boxed{\text{right on the intervals } \left(0, \frac{\pi}{6}\right) \text{ and } \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)}$ and is moving to the $\boxed{\text{left on the interval } \left(\frac{\pi}{6}, \frac{\pi}{2}\right)}$.

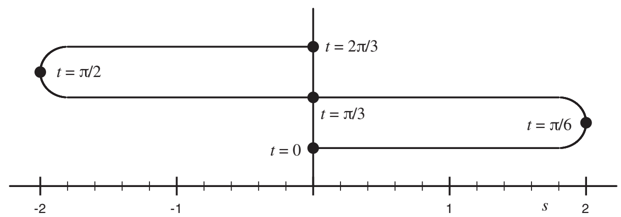
(b) The object reverses direction at $\boxed{t = \frac{\pi}{6} \text{ and } t = \frac{\pi}{2}}$.

(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now,

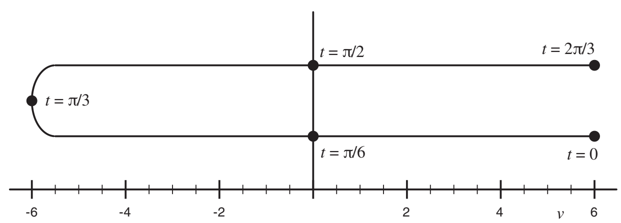
$$a(t) = v'(t) = -18\sin(3t).$$

When $0 < t < \frac{\pi}{3}$, $a(t) < 0$, and when $\frac{\pi}{3} < t < \frac{2\pi}{3}$, $a(t) > 0$. Therefore, the velocity of the object is $\boxed{\text{decreasing on the interval } \left(0, \frac{\pi}{3}\right)}$ and is $\boxed{\text{increasing on the interval } \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)}$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



35. (a) $0, 1$
(b) $[0, \infty)$
(c) $(-\infty, 0]$
(d) 0
(e) none

37. (a) $-3, 0, 1$
(b) $(-\infty, -3]$, $[1, \infty)$
(c) $[-3, 1]$
(d) 1
(e) -3

39. Let $f(x) = 2x^3 - 6x^2 + 6x - 3$.

(a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 6x^2 - 12x + 6 = 6(x^2 - 2x + 1) = 6(x - 1)^2,$$

so 1 is a critical number of f. For $x \neq 1$, $f'(x) > 0$, so that f is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$. By the First Derivative Test, it follows that f has neither a local maximum value nor a local minimum value at 1. Hence, f has $\boxed{\text{no local extrema}}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 12x - 12 = 12(x - 1).$$

For $x < 1$, $f''(x) < 0$, and for $x > 1$, $f''(x) > 0$. Therefore, f is $\boxed{\text{concave down on the interval } (-\infty, 1)}$ and $\boxed{\text{concave up on the interval } (1, \infty)}$.

(c) Because the concavity of f changes at 1, the point $\boxed{(1, f(1)) = (1, -1)}$ is a point of inflection of f.

41. Let $f(x) = x^4 - 4x$.

(a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1),$$

so 1 is a critical number of f. For $x < 1$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, 1)$; for $x > 1$, $f'(x) > 0$, so f is increasing on the interval $(1, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at 1. The $\boxed{\text{local minimum value is } f(1) = -3}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 12x^2,$$

so $f''(x) > 0$ for $x \neq 0$. Therefore, f is $\boxed{\text{concave up on the intervals } (-\infty, 0) \text{ and } (0, \infty)}$.

(c) Because the concavity of f never changes, f has $\boxed{\text{no points of inflection}}$.

43. Let $f(x) = 5x^4 - x^5$.

(a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 20x^3 - 5x^4 = 5x^3(4 - x),$$

so 0 and 4 are critical numbers of f. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and 4 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $5x^3$	Sign of $4 - x$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$-$	$+$	$-$	f is decreasing
$(0, 4)$	$+$	$+$	$+$	f is increasing
$(4, \infty)$	$+$	$-$	$-$	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(4, \infty)$ and increasing on the interval $(0, 4)$. By the First Derivative Test, it follows that f has a local minimum value at 0 and a local maximum value at 4. The $\boxed{\text{local minimum value is } f(0) = 0}$, and the $\boxed{\text{local maximum value is } f(4) = 256}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 60x^2 - 20x^3 = 20x^2(3 - x),$$

so $f''(x) = 0$ when $x = 0$ and when $x = 3$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and 3 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $20x^2$	Sign of $3-x$	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	$+$	$+$	$+$	f is concave up
$(0, 3)$	$+$	$+$	$+$	f is concave up
$(3, \infty)$	$+$	$-$	$-$	f is concave down

Therefore, f is $\boxed{\text{concave up on the intervals } (-\infty, 0) \text{ and } (0, 3)}$ and $\boxed{\text{concave down on the interval } (3, \infty)}$.

(c) Because the concavity of f changes at 3, the point $\boxed{(3, f(3)) = (3, 162)}$ is a point of inflection of f.

45. Let $f(x) = 3x^5 - 20x^3$.

(a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x - 2)(x + 2),$$

so -2, 0, and 2 are critical numbers of f. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -2, 0, and 2 to divide the number line into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $15x^2$	Sign of $x-2$	Sign of $x+2$	Sign of $f'(x)$	Conclusion
$(-\infty, -2)$	$+$	$-$	$-$	$+$	f is increasing
$(-2, 0)$	$+$	$-$	$+$	$-$	f is decreasing
$(0, 2)$	$+$	$-$	$+$	$-$	f is decreasing
$(2, \infty)$	$+$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and decreasing on the intervals $(-2, 0)$ and $(0, 2)$. By the First Derivative Test, it follows that f has a local maximum value at -2, neither a local maximum value nor a local minimum value at 0, and a local minimum value at 2. The $\boxed{\text{local maximum value is } f(-2) = 64}$, and the $\boxed{\text{local minimum value is } f(2) = -64}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 60x^3 - 120x = 60x(x^2 - 2) = 60x(x - \sqrt{2})(x + \sqrt{2}),$$

so $f''(x) = 0$ when $x = \pm\sqrt{2}$ and when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-\sqrt{2}$, 0, and $\sqrt{2}$ to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $60x$	Sign of $x-\sqrt{2}$	Sign of $x+\sqrt{2}$	Sign of $f''(x)$	Conclusion
$(-\infty, -\sqrt{2})$	$-$	$-$	$-$	$-$	f is concave down
$(-\sqrt{2}, 0)$	$-$	$-$	$+$	$+$	f is concave up
$(0, \sqrt{2})$	$+$	$-$	$+$	$-$	f is concave down
$(\sqrt{2}, \infty)$	$+$	$+$	$+$	$+$	f is concave up

Therefore, f is $\boxed{\text{concave up on the intervals } (-\sqrt{2}, 0) \text{ and } (\sqrt{2}, \infty)}$ and $\boxed{\text{concave down on the intervals } (-\infty, -\sqrt{2}) \text{ and } (0, \sqrt{2})}$.

(c) Because the concavity of f changes at $-\sqrt{2}$, 0, and $\sqrt{2}$, the points $\boxed{(-\sqrt{2}, f(-\sqrt{2})) = (-\sqrt{2}, 28\sqrt{2})}$, $\boxed{(0, f(0)) = (0, 0)}$, and $\boxed{(\sqrt{2}, f(\sqrt{2})) = (\sqrt{2}, -28\sqrt{2})}$ are points of inflection of f.

47. Let $f(x) = x^2 e^x$.

(a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = x^2 e^x + 2xe^x = xe^x(x + 2),$$

so -2 and 0 are critical numbers of f. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -2 and 0 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of xe^x	Sign of $x + 2$	Sign of $f'(x)$	Conclusion
$(-\infty, -2)$	$-$	$-$	$+$	f is increasing
$(-2, 0)$	$-$	$+$	$-$	f is decreasing
$(0, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$ and decreasing on the interval $(-2, 0)$. By the First Derivative Test, it follows that f has a local maximum value at -2 and a local minimum value at 0. The $\boxed{\text{local maximum value is } f(-2) = 4e^{-2}}$, and the $\boxed{\text{local minimum value is } f(0) = 0}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = (x^2 + 2x)e^x + (2x + 2)e^x = e^x(x^2 + 4x + 2) = e^x[(x + 2)^2 - 2],$$

so $f''(x) = 0$ when $x = -2 \pm \sqrt{2}$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-2-\sqrt{2}$ and $-2+\sqrt{2}$ to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $x^2 + 4x + 2$	Sign of $f''(x)$	Conclusion
$(-\infty, -2 - \sqrt{2})$	$+$	$+$	$+$	f is concave up
$(-2 - \sqrt{2}, -2 + \sqrt{2})$	$+$	$-$	$-$	f is concave down
$(-2 + \sqrt{2}, \infty)$	$+$	$+$	$+$	f is concave up

Therefore, f is $\boxed{\text{concave up on the intervals } (-\infty, -2 - \sqrt{2}) \text{ and } (-2 + \sqrt{2}, \infty)}$ and $\boxed{\text{concave down on the interval } (-2 - \sqrt{2}, -2 + \sqrt{2})}$.

(c) Because the concavity of f changes at $-2 - \sqrt{2}$ and $-2 + \sqrt{2}$, the points

$$\boxed{(-2 - \sqrt{2}, f(-2 - \sqrt{2})) = (-2 - \sqrt{2}, (6 + 4\sqrt{2})e^{-2-\sqrt{2}})}$$

and

$$\boxed{(-2 + \sqrt{2}, f(-2 + \sqrt{2})) = (-2 + \sqrt{2}, (6 - 4\sqrt{2})e^{-2+\sqrt{2}})}$$

are points of inflection of f.

49. Let $f(x) = 6x^{4/3} - 3x^{1/3}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = 8x^{1/3} - x^{-2/3} = \frac{8x-1}{x^{2/3}},$$

so 0 and $\frac{1}{8}$ are critical numbers. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\frac{1}{8}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8x-1$	Sign of $x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$-$	$+$	$-$	f is decreasing
$\left(0, \frac{1}{8}\right)$	$-$	$+$	$-$	f is decreasing
$\left(\frac{1}{8}, \infty\right)$	$+$	$+$	$+$	f is increasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $\left(0, \frac{1}{8}\right)$ and increasing on the interval $\left(\frac{1}{8}, \infty\right)$. By the First Derivative Test, f has neither a local maximum value nor a local minimum value at 0 and a local minimum value at $\frac{1}{8}$. The $\boxed{\text{local minimum value is } f\left(\frac{1}{8}\right) = -\frac{9}{8}}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \frac{8}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{8x+2}{3x^{5/3}},$$

so $f''(x) = 0$ when $x = -\frac{1}{4}$ and $f''(x)$ does not exist when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-\frac{1}{4}$ and 0 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8x+2$	Sign of $3x^{5/3}$	Sign of $f''(x)$	Conclusion
$\left(-\infty, -\frac{1}{4}\right)$	$-$	$-$	$+$	f is concave up
$\left(-\frac{1}{4}, 0\right)$	$+$	$-$	$-$	f is concave down
$(0, \infty)$	$+$	$+$	$+$	f is concave up

Therefore, f is $\boxed{\text{concave up on the intervals } \left(-\infty, -\frac{1}{4}\right) \text{ and } (0, \infty)}$ and $\boxed{\text{concave down on the interval } \left(-\frac{1}{4}, 0\right)}$.

(c) Because the concavity of f changes at $-\frac{1}{4}$ and 0, the points $\boxed{\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right) = \left(-\frac{1}{4}, \frac{9\sqrt[3]{2}}{4}\right)}$ and $\boxed{(0, f(0)) = (0, 0)}$ are points of inflection of f.

51. Let $f(x) = x^{2/3}(x^2 - 8)$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{2/3}(2x) + (x^2 - 8) \cdot \frac{2}{3}x^{-1/3} = \frac{6x^2 + 2(x^2 - 8)}{3x^{1/3}} = \frac{8(x - \sqrt{2})(x + \sqrt{2})}{3x^{1/3}},$$

so 0 and $\pm\sqrt{2}$ are critical numbers of f. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\pm\sqrt{2}$ to divide the number line into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8(x - \sqrt{2})$	Sign of $x + \sqrt{2}$	Sign of $3x^{1/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -\sqrt{2})$	$-$	$-$	$-$	$-$	f is decreasing
$(-\sqrt{2}, 0)$	$-$	$+$	$-$	$+$	f is increasing
$(0, \sqrt{2})$	$-$	$+$	$+$	$-$	f is decreasing
$(\sqrt{2}, \infty)$	$+$	$+$	$+$	$+$	f is increasing

Therefore, f is decreasing on the intervals $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$ and increasing on the intervals $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$. By the First Derivative Test, it follows that f has local minimum values at $-\sqrt{2}$ and $\sqrt{2}$ and a local maximum value at 0. The

$$\boxed{\text{local minimum values are } f(-\sqrt{2}) = -6\sqrt[3]{2} \text{ and } f(\sqrt{2}) = -6\sqrt[3]{2}},$$

and the $\boxed{\text{local maximum value is } f(0) = 0}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \frac{3x^{1/3}(16x) - 8(x^2 - 2) \cdot x^{-2/3}}{9x^{2/3}} = \frac{40x^2 + 16}{9x^{4/3}},$$

so $f''(x)$ does not exist when $x = 0$ and is never equal to zero. For $x \neq 0$, $f''(x) > 0$, so f is $\boxed{\text{concave up on the intervals } (-\infty, 0) \text{ and } (0, \infty)}$.

(c) Because the concavity of f does not change, f has $\boxed{\text{no points of inflection}}$.

53. Let $f(x) = x^2 - \ln x$. Note that the domain of f is the set $\{x | x > 0\}$.

(a) The function f is differentiable on its domain, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 2x - \frac{1}{x} = \frac{2x^2 - 1}{x},$$

so $\frac{\sqrt{2}}{2}$ is a critical number of f. For $0 < x < \frac{\sqrt{2}}{2}$, $f'(x) < 0$, so f is decreasing on the interval $\left(0, \frac{\sqrt{2}}{2}\right)$; for $x > \frac{\sqrt{2}}{2}$, $f'(x) > 0$, so f is increasing on the interval $\left(\frac{\sqrt{2}}{2}, \infty\right)$. By the First Derivative Test, it follows that f has a local minimum value at $\frac{\sqrt{2}}{2}$. The

$$\boxed{\text{local minimum value is } f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2}\ln 2}.$$

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 2 + \frac{1}{x^2} > 2 > 0$$

for all $x > 0$. Therefore, f is $\boxed{\text{concave up on the interval } (0, \infty)}$.

(c) Because the concavity of f does not change, f has $\boxed{\text{no points of inflection}}$.

55. Let $f(x) = \dfrac{x}{(1+x^2)^{5/2}}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{(1+x^2)^{5/2} - x \cdot \frac{5}{2}(1+x^2)^{3/2}(2x)}{(1+x^2)^5} = \frac{1-4x^2}{(1+x^2)^{7/2}} = \frac{(1-2x)(1+2x)}{(1+x^2)^{7/2}},$$

so $\pm\dfrac{1}{2}$ are critical numbers of f. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers $\pm\dfrac{1}{2}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $1-2x$	Sign of $1+2x$	Sign of $(1+x^2)^{7/2}$	Sign of $f'(x)$	Conclusion
$\left(-\infty, -\frac{1}{2}\right)$	$+$	$-$	$+$	$-$	f is decreasing
$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$+$	$+$	$+$	$+$	f is increasing
$\left(\frac{1}{2}, \infty\right)$	$-$	$+$	$+$	$-$	f is decreasing

Therefore, f is decreasing on the intervals $\left(-\infty, -\dfrac{1}{2}\right)$ and $\left(\dfrac{1}{2}, \infty\right)$ and increasing on the interval $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. By the First Derivative Test, it follows that f has a local minimum value at $-\dfrac{1}{2}$ and a local maximum value at $\dfrac{1}{2}$. The $\boxed{\text{local minimum value is } f\left(-\frac{1}{2}\right) = -\frac{16}{5^{5/2}} = -\frac{16\sqrt{5}}{125}}$, and the $\boxed{\text{local maximum value is } f\left(\frac{1}{2}\right) = \frac{16}{5^{5/2}} = \frac{16\sqrt{5}}{125}}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$\begin{aligned} f''(x) &= \frac{(1+x^2)^{7/2}(-8x) - (1-4x^2) \cdot \frac{7}{2}(1+x^2)^{5/2}(2x)}{(1+x^2)^7} \\ &= \frac{-8x - 8x^3 - 7x + 28x^3}{(1+x^2)^{9/2}} = \frac{5x(4x^2-3)}{(1+x^2)^{9/2}}, \end{aligned}$$

so $f''(x)$ exists for all x and is equal to zero when $x = 0$ and when $x = \pm\dfrac{\sqrt{3}}{2}$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and $\pm\dfrac{\sqrt{3}}{2}$ to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $5x$	Sign of $4x^2-3$	Sign of $(1+x^2)^{9/2}$	Sign of $f''(x)$	Conclusion
$\left(-\infty,-\frac{\sqrt{3}}{2}\right)$	−	+	+	−	f is concave down
$\left(-\frac{\sqrt{3}}{2},0\right)$	−	−	+	+	f is concave up
$\left(0,\frac{\sqrt{3}}{2}\right)$	+	−	+	−	f is concave down
$\left(\frac{\sqrt{3}}{2},\infty\right)$	+	+	+	+	f is concave up

Therefore, f is $\boxed{\text{concave down on the intervals } \left(-\infty,-\frac{\sqrt{3}}{2}\right) \text{ and } \left(0,\frac{\sqrt{3}}{2}\right)}$ and $\boxed{\text{concave up on the intervals } \left(-\frac{\sqrt{3}}{2},0\right) \text{ and } \left(\frac{\sqrt{3}}{2},\infty\right)}$.

(c) Because the concavity of f changes at $\pm\frac{\sqrt{3}}{2}$ and 0, the points

$$\boxed{\left(-\frac{\sqrt{3}}{2}, f\left(-\frac{\sqrt{3}}{2}\right)\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{16\sqrt{3}}{7^{5/2}}\right)},$$

$$\boxed{\left(\frac{\sqrt{3}}{2}, f\left(\frac{\sqrt{3}}{2}\right)\right) = \left(\frac{\sqrt{3}}{2}, \frac{16\sqrt{3}}{7^{5/2}}\right)},$$

and $\boxed{(0, f(0)) = (0,0)}$ are points of inflection of f.

57. Let $f(x) = x^2\sqrt{1-x^2}$. Note that the domain of f is the set $\{x| -1 \le x \le 1\}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^2 \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) + 2x\sqrt{1-x^2} = \frac{-x^3+2x-2x^3}{\sqrt{1-x^2}} = \frac{x(2-3x^2)}{\sqrt{1-x^2}},$$

so ± 1, 0, and $\pm\frac{\sqrt{6}}{3}$ are critical numbers of f. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers ± 1, 0, and $\pm\frac{\sqrt{6}}{3}$ to divide $[-1,1]$ into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of x	Sign of $2-3x^2$	Sign of $\sqrt{1-x^2}$	Sign of $f'(x)$	Conclusion
$\left(-1,-\frac{\sqrt{6}}{3}\right)$	−	−	+	+	f is increasing
$\left(-\frac{\sqrt{6}}{3},0\right)$	−	+	+	−	f is decreasing
$\left(0,\frac{\sqrt{6}}{3}\right)$	+	+	+	+	f is increasing
$\left(\frac{\sqrt{6}}{3},1\right)$	+	−	+	−	f is decreasing

Therefore, f is increasing on the intervals $\left(-1, -\frac{\sqrt{6}}{3}\right)$ and $\left(0, \frac{\sqrt{6}}{3}\right)$ and decreasing on the intervals $\left(-\frac{\sqrt{6}}{3}, 0\right)$ and $\left(\frac{\sqrt{6}}{3}, 1\right)$. By the First Derivative Test, it follows that f has local maximum values at $\pm\frac{\sqrt{6}}{3}$ and a local minimum value at 0. The

$$\boxed{\text{local maximum values are } f\left(\pm\frac{\sqrt{6}}{3}\right) = \frac{2\sqrt{3}}{9}, \text{ and the local minimum value is } f(0) = 0}.$$

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$\begin{aligned} f''(x) &= \frac{\sqrt{1-x^2}(2-9x^2) - (2x-3x^3)\cdot\frac{1}{2}(1-x^2)^{-1/2}(-2x)}{1-x^2} \\ &= \frac{(1-x^2)(2-9x^2) + x(2x-3x^3)}{(1-x^2)^{3/2}} = \frac{6x^4-9x^2+2}{(1-x^2)^{3/2}}, \end{aligned}$$

so $f''(x)$ does not exist at $x = \pm 1$ and is equal to zero when

$$x^2 = \frac{9 \pm \sqrt{81 - 4(6)(2)}}{12} = \frac{9 \pm \sqrt{33}}{12},$$

or when

$$x = \pm\sqrt{\frac{9 \pm \sqrt{33}}{12}} = \pm\frac{1}{6}\sqrt{27 \pm 3\sqrt{33}}.$$

Note that

$$\pm\frac{1}{6}\sqrt{27 + 3\sqrt{33}} \approx \pm 1.108$$

are not in the domain of f and can be excluded. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $\pm\frac{1}{6}\sqrt{27 - 3\sqrt{33}}$ to divide $[-1, 1]$ into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $6x^4 - 9x^2 + 2$	Sign of $(1-x^2)^{3/2}$	Sign of $f''(x)$	Conclusion
$\left(-1, -\frac{1}{6}\sqrt{27-3\sqrt{33}}\right)$	$-$	$+$	$-$	f is concave down
$\left(-\frac{1}{6}\sqrt{27-3\sqrt{33}}, \frac{1}{6}\sqrt{27-3\sqrt{33}}\right)$	$+$	$+$	$+$	f is concave up
$\left(\frac{1}{6}\sqrt{27-3\sqrt{33}}, 1\right)$	$-$	$+$	$-$	f is concave down

Therefore, f is concave down on the intervals

$$\boxed{\left(-1, -\frac{1}{6}\sqrt{27-3\sqrt{33}}\right) \text{ and } \left(\frac{1}{6}\sqrt{27-3\sqrt{33}}, 1\right)}$$

and concave up on the interval

$$\boxed{\left(-\frac{1}{6}\sqrt{27-3\sqrt{33}}, \frac{1}{6}\sqrt{27-3\sqrt{33}}\right)}.$$

(c) Because the concavity of f changes at $\pm\frac{1}{6}\sqrt{27-3\sqrt{33}}$, the points

$$\boxed{\left(\pm\frac{1}{6}\sqrt{27-3\sqrt{33}}, f\left(\pm\frac{1}{6}\sqrt{27-3\sqrt{33}}\right)\right) = \left(\pm\frac{1}{6}\sqrt{27-3\sqrt{33}}, \frac{9-\sqrt{33}}{12}\sqrt{\frac{3+\sqrt{33}}{12}}\right)}$$

are points of inflection of f.

59. Let $f(x) = x - 2\sin x$, and consider the interval $[0, 2\pi]$.

(a) The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 1 - 2\cos x,$$

so $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ are critical numbers. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ to divide $[0, 2\pi]$ into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $f'(x)$	Conclusion
$(0, \frac{\pi}{3})$	$-$	f is decreasing
$(\frac{\pi}{3}, \frac{5\pi}{3})$	$+$	f is increasing
$(\frac{5\pi}{3}, 2\pi)$	$-$	f is decreasing

Therefore, f is decreasing on the intervals $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$ and increasing on the interval $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$. By the First Derivative Test, it follows that f has a local minimum value at $\frac{\pi}{3}$ and a local maximum value at $\frac{5\pi}{3}$. The $\boxed{\text{local minimum value is } f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}}$, and the $\boxed{\text{local maximum value is } f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 2\sin x,$$

so $f''(x) = 0$ when $x = 0, \pi$, and 2π. For $0 < x < \pi$, $f''(x) > 0$, so f is $\boxed{\text{concave up on the interval } (0, \pi)}$; for $\pi < x < 2\pi$, $f''(x) < 0$, so f is $\boxed{\text{concave down on the interval } (\pi, 2\pi)}$.

(c) Because the concavity of f changes at π, the point $\boxed{(\pi, f(\pi)) = (\pi, \pi)}$ is a point of inflection of f.

61. Let $f(x) = \dfrac{e^x + e^{-x}}{2} = \cosh x$.

(a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = \sinh x,$$

so 0 is a critical number of f. For $x < 0$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, 0)$; for $x > 0$, $f'(x) > 0$, so f is increasing on the interval $(0, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at 0. The $\boxed{\text{local minimum value is } f(0) = \cosh 0 = 1}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \cosh x \geq 1 > 0$$

for all x. Therefore, f is $\boxed{\text{concave up on the interval } (-\infty, \infty)}$.

(c) Because the concavity of f does not change, f has $\boxed{\text{no points of inflection}}$.

63. (a) $-\sqrt{3}$, 0, $\sqrt{3}$
(b) $\left[-\sqrt{3}, 0\right]$, $\left[\sqrt{3}, \infty\right)$
(c) $\left(-\infty, -\sqrt{3}\right]$, $\left[0, \sqrt{3}\right]$
(d) $-\sqrt{3}$, $\sqrt{3}$
(e) 0
(f) $(-\infty, -1)$, $(1, \infty)$
(g) $(-1, 1)$
(h) $-1, 1$

65. (a) 0, 1
(b) $(-\infty, 0]$, $[1, \infty)$
(c) $[0, 1]$
(d) 1
(e) 0
(f) $\left(\sqrt[3]{\frac{1}{4}}, \infty\right)$
(g) $\left(-\infty, \sqrt[3]{\frac{1}{4}}\right)$
(h) $\sqrt[3]{\frac{1}{4}}$

67. Let $f(x) = -2x^3 + 15x^2 - 36x + 7$. The polynomial function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -6x^2 + 30x - 36 = -6(x^2 - 5x + 6) = -6(x-3)(x-2),$$

so 2 and 3 are the critical numbers of f.

(a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 2 and 3 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $-6(x-3)$	Sign of $x-2$	Sign of $f'(x)$	Conclusion
$(-\infty, 2)$	+	−	−	f is decreasing
$(2, 3)$	+	+	+	f is increasing
$(3, \infty)$	−	+	−	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, 2)$ and $(3, \infty)$ and increasing on the interval $(2, 3)$. By the First Derivative Test, it follows that f has a local minimum value at 2 and a local maximum value at 3. The

$$\boxed{\text{local minimum value is } f(2) = -21, \text{ and the local maximum value is } f(3) = -20}.$$

(b) The second derivative of f is

$$f''(x) = -12x + 30.$$

Evaluating the second derivative at the critical numbers yields

$$f''(2) = 6 > 0 \quad \text{and} \quad f''(3) = -6 < 0.$$

By the Second Derivative Test, it follows that f has a $\boxed{\text{local minimum value at 2}}$ and a $\boxed{\text{local maximum value at 3}}$.

(c) Answers will vary, but here is one possible response. Neither test is particularly difficult to use. The First Derivative Test requires the calculation of only one derivative, but the Second Derivative Test requires fewer steps.

69. (a) $f(x)$ is defined for all x, so find the critical points by setting the derivative equal to 0:

$$\begin{aligned} f'(x) &= 0 \\ \frac{d}{dx}\left(x^4 - 8x^2 - 5\right) &= 0 \\ 4x^3 + 16x &= 0 \\ 4x\left(x^2 + 4\right) &= 0 \\ 4x &= 0, \quad \text{since } x^2 + 4 > 0 \text{ for all } x \\ x &= 0 \end{aligned}$$

The critical point, and hence the boundary of the regions to be analyzed, is $x = 0$.

For $x < 0$, $f'(x) = 4(\text{negative})(\text{positive}) = \text{negative}$, and for $x > 0$, $f'(x) = 4(\text{positive})(\text{positive}) = \text{positive}$.

Therefore no local maximum; local minimum $f(0) = (0)^4 - 8(0)^2 - 5 = -5$ at $x = 0$.

(b) Analyze the second derivative at the critical point:

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) \\ &= \frac{d}{dx}\left(4x^3 + 16x\right) \\ &= 12x^2 + 16 \\ f''(0) &= 12(0)^2 + 16 = 16 > 0 \end{aligned}$$

The graph is concave up at $x = 0$, so $(0, -5)$ is a local minimum.

(c) Answers will vary, including, but not limited to, the statement that with the Second Derivative Test one only has to substitute a single value into a relatively easy second-derivative formula to determine that f is concave up at the single critical number and therefore has a local minimum there.

71. (a) $f(x)$ is defined for all x, so find the critical points by setting the derivative equal to 0:

$$\begin{aligned} f'(x) &= 0 \\ \frac{d}{dx}\left(3x^5 + 5x^4 + 1\right) &= 0 \\ 15x^4 + 20x^3 &= 0 \\ 5x^3(3x + 4) &= 0 \\ x &= 0 \quad \text{or} \quad x = -\frac{4}{3} \end{aligned}$$

The critical points, and hence the boundaries of the regions to be analyzed, are $x = -\frac{4}{3}$ and $x = 0$.

For $x < -\frac{4}{3}$, $f'(x) = 5(\text{negative})^3(\text{negative}) = \text{positive}$.

For $-\frac{4}{3} < x < 0$, $f'(x) = 5(\text{negative})^3(\text{positive}) = \text{negative}$.

For $x > 0$, $f'(x) = 5(\text{positive})^3(\text{positive}) = \text{positive}$

Therefore local maximum $f\left(-\frac{4}{3}\right) = 3\left(-\frac{4}{3}\right)^5 + 5\left(-\frac{4}{3}\right)^4 + 1 = \frac{337}{81}$ at $x = -\frac{4}{3}$; local minimum $f(0) = 3(0)^5 + 5(0)^4 + 1 = 1$ at $x = 0$.

(b) Analyze the second derivative at the critical points:

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) \\ &= \frac{d}{dx}\left(15x^4 + 20x^3\right) \\ &= 60x^3 + 60x^2 \\ f''\left(-\frac{4}{3}\right) &= 60\left(-\frac{4}{3}\right)^3 + 60\left(-\frac{4}{3}\right)^2 = -\frac{320}{9} < 0 \\ f''(0) &= 60(0)^3 + 60(0)^2 = 0 \end{aligned}$$

The graph is concave down at $x = -\frac{4}{3}$, so $\left(-\frac{4}{3}, \frac{337}{81}\right)$ is a local maximum.

The graph is (momentarily) neither concave up nor concave down at $x = 0$, so the Second Derivative Test (by itself) provides no information about whether $(0, 1)$ is a local maximum, a local minimum, or an inflection point.

(c) Answers will vary, including, but not limited to, the statement that the Second Derivative test does not provide enough information about one of the points.

73. Let $f(x) = (x-3)^2 e^x$. The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = (x-3)^2 e^x + 2(x-3)e^x = (x^2 - 4x + 3)e^x = (x-3)(x-1)e^x,$$

so 1 and 3 are the critical numbers of f

(a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 1 and 3 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x-3$	Sign of $(x-1)e^x$	Sign of $f'(x)$	Conclusion
$(-\infty, 1)$	$-$	$-$	$+$	f is increasing
$(1, 3)$	$-$	$+$	$-$	f is decreasing
$(3, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-\infty, 1)$ and $(3, \infty)$ and decreasing on the interval $(1, 3)$. By the First Derivative Test, it follows that f has a local maximum value at 1 and a local minimum value at 3. The

$$\boxed{\text{local maximum value is } f(1) = 4e, \text{ and the local minimum value is } f(3) = 0}.$$

(b) The second derivative of f is

$$f''(x) = (x^2 - 4x + 3)e^x + (2x - 4)e^x = (x^2 - 2x - 1)e^x.$$

Evaluating the second derivative at the critical numbers yields

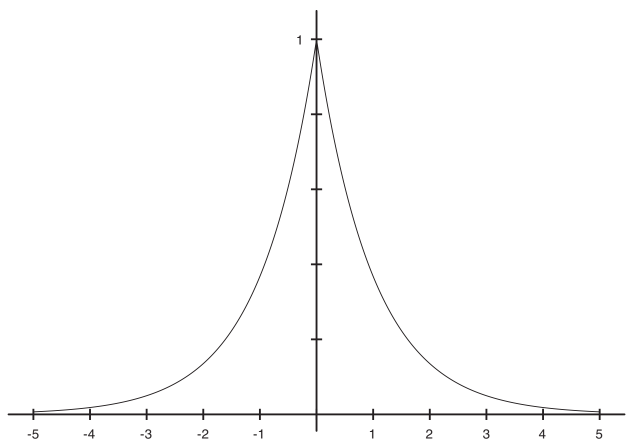
$$f''(1) = -2e < 0 \quad \text{and} \quad f''(3) = 2e > 0.$$

By the Second Derivative Test, it follows that f has a $\boxed{\text{local maximum value at 1}}$ and a $\boxed{\text{local minimum value at 3}}$.

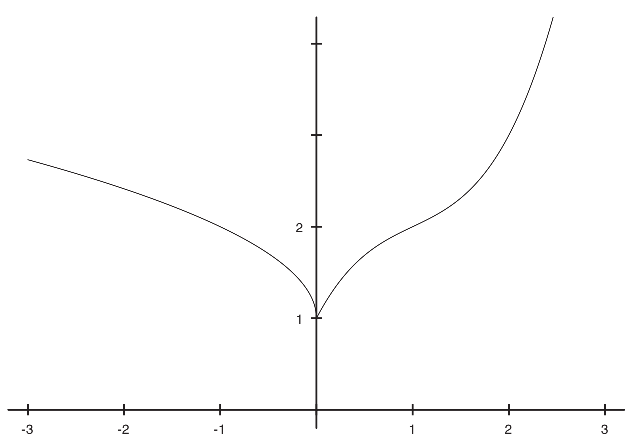
(c) Answers will vary, but here is one possible response. Neither test is particularly difficult to use. The First Derivative Test requires the calculation of only one derivative, but the Second Derivative Test requires fewer steps.

Applications and Extensions

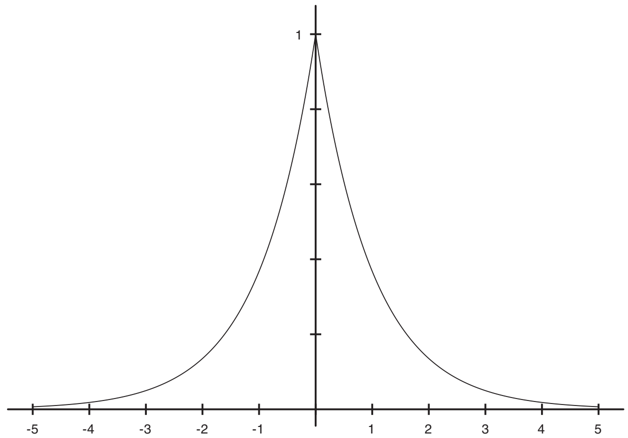
75. Answers will vary. The figure below displays the graph of a function f with the properties: f is concave up on $(-\infty, \infty)$, increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$, and $f(0) = 1$.



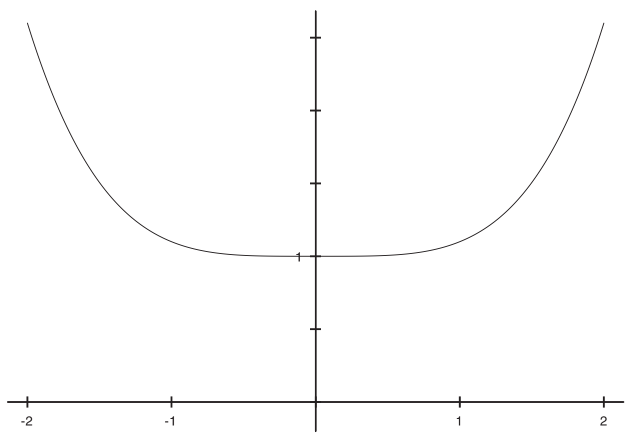
77. Answers will vary. The figure below displays the graph of a function f with the properties: f is concave down on $(-\infty, 1)$, concave up on $(1, \infty)$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, and $f(0) = 1$ and $f(1) = 2$.



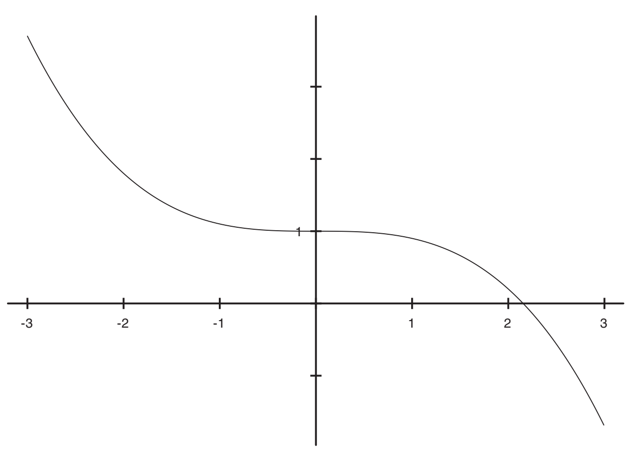
79. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(x) > 0$ if $x < 0$, $f'(x) < 0$ if $x > 0$, $f''(x) > 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



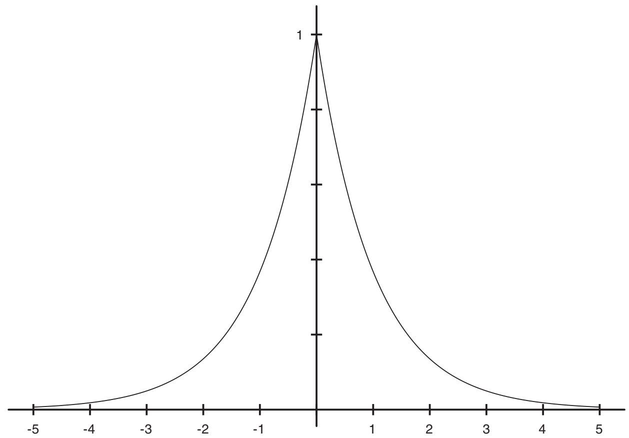
81. Answers will vary. The figure below displays the graph of a function f with the properties: $f''(0) = 0$, $f'(0) = 0$, $f''(x) > 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



83. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(0) = 0$, $f'(x) < 0$ if $x \neq 0$, $f''(x) > 0$ if $x < 0$, $f''(x) < 0$ if $x > 0$, and $f(0) = 1$.



85. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(0)$ does not exist, $f''(x) > 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



87. Let $f(x) = e^{-(x-2)^2}$.

(a) Using the command

```
Reduce [ D [ Exp[-(x-2)²], {x,2} ] > 0, x ]
```

in *Mathematica* yields the solution

$$x < 2 - \frac{\sqrt{2}}{2} \quad \text{or} \quad x > 2 + \frac{\sqrt{2}}{2}$$

to the inequality $f''(x) > 0$; the command

```
Reduce [ D [ Exp[-(x-2)²], {x,2} ] < 0, x ]
```

yields the solution

$$2 - \frac{\sqrt{2}}{2} < x < 2 + \frac{\sqrt{2}}{2}$$

to the inequality $f''(x) < 0$. Therefore, f is

$$\boxed{\text{concave up on the intervals } \left(-\infty, 2 - \frac{\sqrt{2}}{2}\right) \text{ and } \left(2 + \frac{\sqrt{2}}{2}, \infty\right)}$$

and $\boxed{\text{concave down on the interval } \left(2 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right)}$.

(b) Because the concavity of f changes at $2 \pm \frac{\sqrt{2}}{2}$, the points

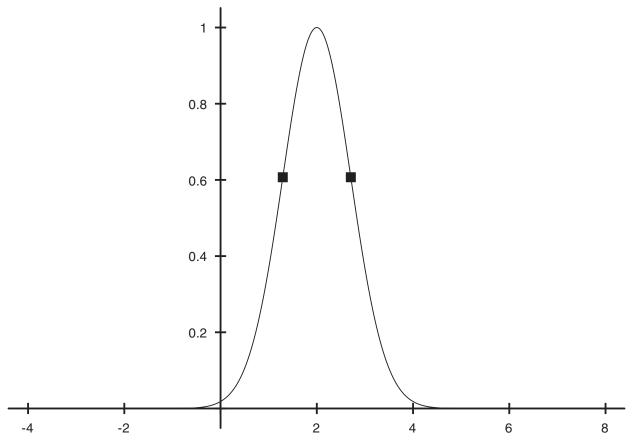
$$\boxed{\left(2 - \frac{\sqrt{2}}{2}, f\left(2 - \frac{\sqrt{2}}{2}\right)\right) = \left(2 - \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{e}}\right)}$$

and

$$\boxed{\left(2 + \frac{\sqrt{2}}{2}, f\left(2 + \frac{\sqrt{2}}{2}\right)\right) = \left(2 + \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{e}}\right)}$$

are points of inflection of f.

(c) The figure below displays the graph of f with the points of inflection marked with black squares. At the point on the left, the concavity changes from up to down, while at the point on the right, the concavity changes from down to up.



89. Let $f(x) = \dfrac{2-x}{2x^2-2x+1}$.

(a) Using the command

```
N [ Reduce [ D [ (2-x)/(2x^2 - 2x + 1), {x,2} ] > 0, x ] ]
```

in *Mathematica* yields the approximate solution

$$x < 0.134792 \quad \text{or} \quad 0.7211 < x < 5.14411$$

to the inequality $f''(x) > 0$; the command

```
N [ Reduce [ D [ (2-x)/(2x^2 - 2x + 1), {x,2} ] < 0, x ] ]
```

yields the approximate solution

$$0.134792 < x < 0.7211 \quad \text{or} \quad x > 5.14411$$

to the inequality $f''(x) < 0$. Therefore, f is

$$\boxed{\text{concave up on the intervals } (-\infty, 0.134792) \text{ and } (0.7211, 5.14411)}$$

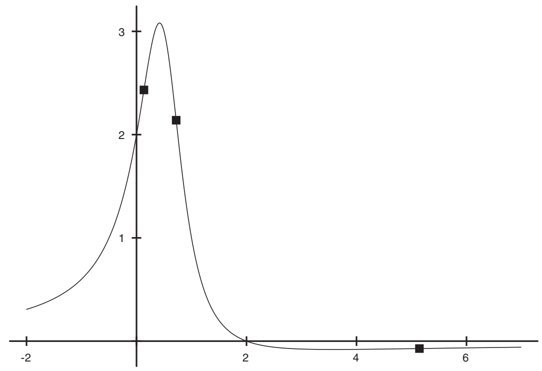
and $\boxed{\text{concave down on the intervals } (0.134792, 0.7211) \text{ and } (5.14411, \infty)}$.

(b) Because the concavity of f changes at approximately 0.134792, 0.7211, and 5.14411, the points

$$\begin{aligned}
(0.134792, f(0.134792)) &= \boxed{(0.134792, 2.43260)},\\
(0.7211, f(0.7211)) &= \boxed{(0.7211, 2.13945)}, \text{ and}\\
(5.14411, f(5.14411)) &= \boxed{(5.14411, -0.07205)}
\end{aligned}$$

are approximately the points of inflection of f.

(c) The figure below displays the graph of f with the points of inflection marked with black squares. From left to right, the concavity changes from up to down, down to up, and up to down at the inflection points.



91. Let $f(x) = ax^3 + bx^2$. For the point $(1, 6)$ to be on the graph of f, $f(1)$ must be equal to 6. As $f(1) = a + b$, the numbers a and b must satisfy the equation $a + b = 6$. Because the polynomial function f is twice differentiable everywhere, if f is to have an inflection point at the point $(1, 6)$, $f''(1)$ must be equal to 0. Now,

$$f'(x) = 3ax^2 + 2bx \quad \text{and} \quad f''(x) = 6ax + 2b,$$

so $f''(1) = 6a + 2b$ and another equation that a and b must satisfy is $6a + 2b = 0$. Solving $a + b = 6$ and $6a + 2b = 0$ yields $\boxed{a = -3 \text{ and } b = 9}$.

93. Let $N(t) = \dfrac{10,000}{1 + 9999e^{-t}}$.

(a) The rate of change of the infection is

$$\boxed{N'(t) = \frac{(1 + 9999e^{-t}) \cdot 0 - 10,000 \cdot -9999e^{-t}}{(1 + 9999e^{-t})^2} = \frac{99,990,000e^{-t}}{(1 + 9999e^{-t})^2}}.$$

(b) $N'(t)$ is increasing when $N''(t) > 0$ and decreasing when $N''(t) < 0$. Now,

$$\begin{aligned} N''(t) &= \frac{(1 + 9999e^{-t})^2 \cdot -99,990,000e^{-t} - 99,990,000e^{-t} \cdot 2(1 + 9999e^{-t})(-9999e^{-t})}{(1 + 9999e^{-t})^4} \\ &= \frac{99,990,000e^{-t}(19998e^{-t} - 1 - 9999e^{-t})}{(1 + 9999e^{-t})^3} = \frac{99,990,000e^{-t}(9999e^{-t} - 1)}{(1 + 9999e^{-t})^3}, \end{aligned}$$

so $N''(t) = 0$ when

$$9999e^{-t} - 1 = 0 \quad \text{or} \quad t = \ln 9999.$$

For $0 < t < \ln 9999$, $N''(t) > 0$, so $\boxed{N'(t) \text{ is increasing on the interval } (0, \ln 9999)}$.

For $t > \ln 9999$, $N''(t) < 0$, so $\boxed{N'(t) \text{ is decreasing on the interval } (\ln 9999, \infty)}$.

(c) Based on the results of part (b) and the First Derivative Test, the rate of change of the infection is a maximum when $\boxed{t = \ln 9999}$.

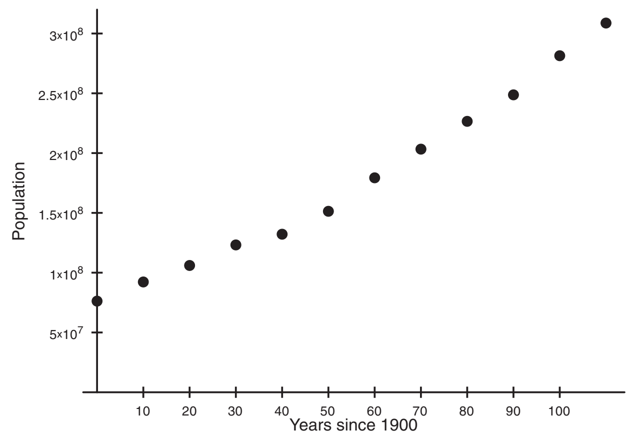
(d) Based on the results of part (b), N is concave up on the interval $(0, \ln 9999)$ and concave down on the interval $(\ln 9999, \infty)$. Therefore, the concavity of N changes at $\ln 9999$, and the point

$$\boxed{(\ln 9999, N(\ln 9999)) = (\ln 9999, 5000)}$$

is an inflection point of N.

(e) Comparing the results of parts (c) and (d), we see that the point of inflection of N is the point at which $N'(t)$ is maximum.

95. (a) The figure below displays a scatterplot of the population data.



(b) Using the `Logistic` regression function on a TI-84 Plus calculator, the logistic function that best fits the given data is

$$P(t) = \frac{762,176,717.8}{1 + 8.743e^{-0.0162t}}.$$

(c) The rate of change in population is

$$P'(t) = \frac{(1 + 8.743e^{-0.0162t}) \cdot 0 - 762,176,717.8 \cdot -0.1416366e^{-0.0162t}}{(1 + 8.743e^{-0.0162t})^2} = \boxed{\frac{107,952,118.9e^{-0.0162t}}{(1 + 8.743e^{-0.0162t})^2}}.$$

(d) $P'(t)$ is increasing when $P''(t) > 0$ and decreasing when $P''(t) < 0$. Now,

$$\begin{aligned} P''(t) &= \frac{(1 + 8.743e^{-0.0162t})^2 \cdot -1,748,824.326e^{-0.0162t}}{(1 + 8.743e^{-0.162t})^4} - \\ &\qquad \frac{107,952,118.9e^{-0.0162t} \cdot 2(1 + 8.743e^{-0.0162t})(-0.1416366e^{-0.0162t})}{(1 + 8.743e^{-0.162t})^4} \\ &= \frac{1,748,824.326e^{-0.0162t}(17.486e^{-0.0162t} - 1 - 8.743e^{-0.0162t})}{(1 + 8.743e^{-0.0162t})^3} \\ &= \frac{1,748,824.326e^{-0.0162t}(8.743e^{-0.0162t} - 1)}{(1 + 8.743e^{-0.0162t})^3}, \end{aligned}$$

so $P''(t) = 0$ when

$$8.743e^{-0.0162t} - 1 = 0 \quad \text{or} \quad t = \frac{1}{0.0162} \ln 8.743 \approx 133.843.$$

For $0 < t < 133.843$, $P''(t) > 0$, so $\boxed{P'(t) \text{ is increasing on the interval } (0, 133.843)}$.
For $t > 133.843$, $P''(t) < 0$, so $\boxed{P'(t) \text{ is decreasing on the interval } (133.843, \infty)}$.

(e) Based on the results of part (d) and the First Derivative Test, the rate of change in population is a maximum when $\boxed{t \approx 133.843}$.

(f) Based on the results of part (d), P is concave up on the interval $(0, 133.843)$ and concave down on the interval $(133.843, \infty)$. Therefore, the concavity of P changes at 133.843, and the point

$$\boxed{(133.843, P(133.843)) \approx (133.843, 381088358.9)}$$

is an inflection point of P.

(g) Comparing the results of parts (e) and (f), we see that the point of inflection of P is the point at which $P'(t)$ is maximum.

97. Let $B(t) = -12.8t^3 + 163.4t^2 - 614.0t + 390.6$.

(a) The polynomial function B is differentiable everywhere, so the critical numbers of B occur where $B'(t) = 0$. Now,

$$B'(t) = -38.4t^2 + 326.8t - 614.0,$$

so the critical numbers of B are

$$t = \frac{-326.8 \pm \sqrt{326.8^2 - 4(-38.4)(-614.0)}}{-76.8} = \frac{-326.8 \pm \sqrt{12487.84}}{-76.8} \approx 2.80, 5.71.$$

Evaluating $B''(t) = -76.8t + 326.8$ at the critical numbers yields

$$B''(2.80) = 111.76 > 0 \quad \text{and} \quad B''(5.71) = -111.728 < 0,$$

so B has a local minimum value at 2.80 and a local maximum value at 5.71. The

$$\boxed{\text{local minimum value is } B(2.80) \approx -328.53 \text{ billion dollars}},$$

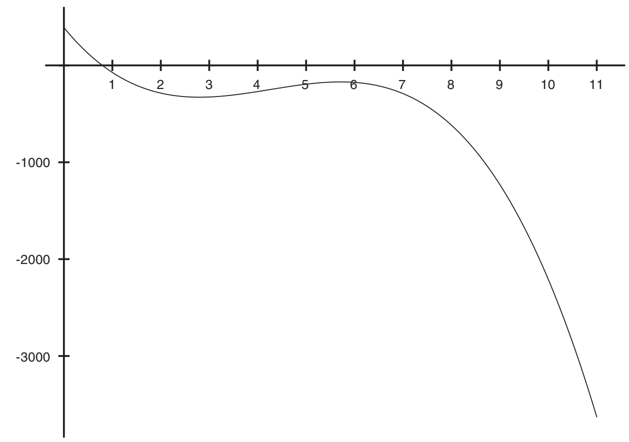
and the $\boxed{\text{local maximum value is } B(5.71) \approx -170.80 \text{ billion dollars}}$.

(b) Because both local extreme values are negative, both represent a $\boxed{\text{budget deficit}}$.

(c) From part (a), $B''(t) = -76.8t + 326.8$, so $B''(t) = 0$ when $t \approx 4.26$. For $0 < t < 4.26$, $B''(t) > 0$, so B is $\boxed{\text{concave up on the interval } (0, 4.26)}$; for $4.26 < t < 9$, $B''(t) < 0$, so B is $\boxed{\text{concave down on the interval } (4.26, 9)}$. Because the concavity of B changes at 4.26, the point $\boxed{(4.26, B(4.26)) \approx (4.26, -249.27)}$ is a point of inflection of B.

(d) Because $B''(t) > 0$ for $t < 4.26$, to the left of the point of inflection, the rate of change of the budget is increasing at an increasing rate; because $B''(t) < 0$ for $t > 4.26$, to the right of the point of inflection, the rate of change of the budget is increasing at a decreasing rate.

(e) The figure below displays the graph of B. Given that B predicts a 3.6 trillion dollar budget deficit in 2011, which would indicate that the government took in nothing in 2011, B does not seem to be an accurate predictor for the budget for the years 2010 and beyond.



99. Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. In order for the graph of f to contain the points $(0, 5)$ and $(4, 33)$, $f(0)$ must be equal to 5 and $f(4)$ must be equal to 33. These conditions yield the equations $d = 5$ and $64a + 16b + 4c + d = 33$. The polynomial function f is differentiable everywhere, so for f to have a local minimum at 0 and a local maximum at 4, both $f'(0)$ and $f'(4)$ must be equal to 0. Now,

$$f'(x) = 3ax^2 + 2bx + c,$$

so the derivative conditions yield the equations $c = 0$ and $48a + 8b + c = 0$. The solution of the equations $d = 5$, $64a + 16b + 4c + d = 33$, $c = 0$, and $48a + 8b + c = 0$ is $\boxed{a = -\frac{7}{8},\ b = \frac{21}{4},\ c = 0, \text{ and } d = 5}$.

101. Let $y = \sqrt{3}\sin x + \cos x$, and consider the interval $0 \le x \le 2\pi$. Because the function y is differentiable everywhere, local extrema can only occur where $y'(x) = 0$. Now,

$$y'(x) = \sqrt{3}\cos x - \sin x,$$

so $y'(x) = 0$ when $\tan x = \sqrt{3}$, which is when $x = \frac{\pi}{3}$ and when $x = \frac{4\pi}{3}$. For $0 < x < \frac{\pi}{3}$, $y'(x) > 0$, so y is increasing on the interval $\left(0, \frac{\pi}{3}\right)$; for $\frac{\pi}{3} < x < \frac{4\pi}{3}$, $y'(x) < 0$, so y is decreasing on the interval $\left(\frac{\pi}{3}, \frac{4\pi}{3}\right)$; and for $\frac{4\pi}{3} < x < 2\pi$, $y'(x) > 0$, so y is increasing on the interval $\left(\frac{4\pi}{3}, 2\pi\right)$. By the First Derivative Test, it follows that y has a local maximum value at $\frac{\pi}{3}$ and a local minimum value at $\frac{4\pi}{3}$. The $\boxed{\text{local maximum value is } f\left(\frac{\pi}{3}\right) = 2}$, and the $\boxed{\text{local minimum value is } f\left(\frac{4\pi}{3}\right) = -2}$.

Next,

$$y''(x) = -\sqrt{3}\sin x - \cos x,$$

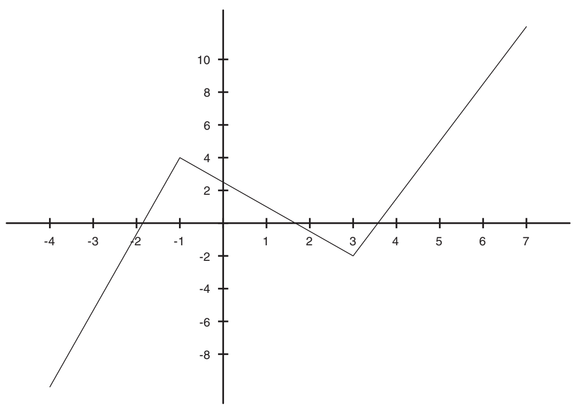
so $y''(x) = 0$ when $\tan x = -\frac{\sqrt{3}}{3}$, which is when $x = \frac{5\pi}{6}$ and when $x = \frac{11\pi}{6}$. For $0 < x < \frac{5\pi}{6}$, $y''(x) < 0$, so y is concave down on the interval $\left(0, \frac{5\pi}{6}\right)$; for $\frac{5\pi}{6} < x < \frac{11\pi}{6}$, $y''(x) > 0$, so y is concave up on the interval $\left(\frac{5\pi}{6}, \frac{11\pi}{6}\right)$; and for $\frac{11\pi}{6} < x < 2\pi$, $y''(x) < 0$, so y is concave down on the interval $\left(\frac{11\pi}{6}, 2\pi\right)$. Because the concavity of y changes at $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$, the points $\boxed{\left(\frac{5\pi}{6}, f\left(\frac{5\pi}{6}\right)\right) = \left(\frac{5\pi}{6}, 0\right)}$ and $\boxed{\left(\frac{11\pi}{6}, f\left(\frac{11\pi}{6}\right)\right) = \left(\frac{11\pi}{6}, 0\right)}$ are points of inflection of y.

103. Let $f(x) = x^{2/3}$. Then

$$f'(x) = \frac{2}{3}x^{-1/3} \quad \text{and} \quad f''(x) = -\frac{2}{9}x^{-4/3}.$$

It follows that 0 is the only critical number of f, but $\boxed{f''(x) \text{ does not exist at } 0}$; therefore, the Second Derivative Test cannot be applied to identify the extreme value of $f(x) = x^{2/3}$.

105. (a) $\boxed{\text{Not necessarily true}}$. The figure below displays a function f that is continuous for all x and has a local maximum at $(-1, 4)$ and a local minimum at $(3, -2)$, but does not have a point of inflection somewhere between $x = -1$ and $x = 3$.



(b) $\boxed{\text{Not necessarily true}}$. See the graph in part (a) for which $f'(-1)$ does not exist.

(c) $\boxed{\text{Not necessarily true}}$. See the graph in part (a) which does not have a horizontal asymptote.

(d) $\boxed{\text{Not necessarily true}}$. See the graph in part (a) which does not have a tangent line at $x = 3$.

(e) $\boxed{\text{True}}$. The function is continuous for all x, so $f(0)$ is defined, meaning the graph of f has a y-intercept, and the graph of f intersects the y-axis. Moreover, because f is continuous on the closed interval $[-1, 3]$ with $f(-1) = 4 > 0$ and $f(3) = -2 < 0$, the Intermediate Value Theorem guarantees there exists a c in $(-1, 3)$ such that $f(c) = 0$. Therefore, the graph of f intersects the x-axis as well.

107. Consider the function $f(x) = x - \sin x$. Now,

$$f'(x) = 1 + \cos x \geq 0$$

on the interval $0 \leq x \leq 2\pi$. Therefore, f is increasing on the interval $0 \leq x \leq 2\pi$, so $f(x) \geq f(0) = 0$ on the interval $0 \leq x \leq 2\pi$. That is, $x - \sin x \geq 0$, or $x \geq \sin x$ on the interval $0 \leq x \leq 2\pi$.

109. Let $f(x) = 2\sqrt{x} - 3 + \frac{1}{x}$. Now,

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}.$$

For $x > 1$,

$$x^2 > \sqrt{x} \quad \text{so that} \quad \frac{1}{x^2} < \frac{1}{\sqrt{x}}.$$

It follows that $f'(x) > 0$ and f is increasing for $x > 1$. Therefore, $f(x) > f(1) = 0$ for $x > 1$; that is,

$$2\sqrt{x} - 3 + \frac{1}{x} > 0 \quad \text{or} \quad 2\sqrt{x} > 3 - \frac{1}{x}$$

for $x > 1$.

111. Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 40$. Then

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2),$$

so -1, 0, and 2 are critical numbers of f. Consider the closed interval $[-1, 2]$. Evaluating f at -1, 0, and 2 yields

$$f(-1) = 35, \quad f(0) = 40, \quad \text{and} \quad f(2) = 8,$$

so the absolute minimum value of f on the interval $[-1, 2]$ is 8. For $x < -1$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, -1)$. This means that $f(x) \geq f(-1) = 35$ for $x < -1$. Moreover, for $x > 2$, $f'(x) > 0$, so f is increasing on the interval $(2, \infty)$. This means that $f(x) > f(2) = 8$ for $x > 2$. Bringing all of this information together, it follows that f has an absolute minimum value of 8 at 2. Therefore, $f(x) \geq 8 > 0$ for all x.

113. Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$, so $f''(x) = 0$ when $x = -\dfrac{b}{3a}$. If $a < 0$, then

$$f''(x) > 0 \text{ when } x < -\frac{b}{3a} \quad \text{and} \quad f''(x) < 0 \text{ when } x > -\frac{b}{3a},$$

so the concavity of f changes and there is a point of inflection at $-\dfrac{b}{3a}$; if $a > 0$, then

$$f''(x) < 0 \text{ when } x < -\frac{b}{3a} \quad \text{and} \quad f''(x) > 0 \text{ when } x > -\frac{b}{3a},$$

so, again, the concavity of f changes and there is a point of inflection at $-\dfrac{b}{3a}$. Therefore, every polynomial of degree 3 has exactly one point of inflection.

115. Let $f(x) = (x-a)^n$, where a is a constant and $n \geq 3$ is an odd integer. Then,

$$f'(x) = n(x-a)^{n-1} \quad \text{and} \quad f''(x) = n(n-1)(x-a)^{n-2},$$

so $f''(x)$ exists everywhere and is equal to zero only for $x = a$. Because n is an odd integer, $n - 2$ is also an odd integer, so $f''(x) < 0$ for $x < a$ and $f''(x) > 0$ for $x > a$. It follows that the concavity of f changes and f has a point of inflection at a. There are no other candidates for points of inflection, so the function f has exactly one point of inflection.

117. Consider the function $f(x) = \dfrac{ax+b}{ax+d}$. The rational function f is differentiable on its domain, so critical numbers occur only when $f'(x) = 0$ and points of inflection can only occur when $f''(x) = 0$. Now,

$$f'(x) = \frac{(ax+d)(a) - (ax+b)(a)}{(ax+d)^2} = \frac{a(d-b)}{(ax+d)^2},$$

and

$$f''(x) = -\frac{2a^2(d-b)}{(ax+d)^3}.$$

Provided $a \neq 0$ and $d \neq b$, $f'(x)$ and $f''(x)$ are never equal to zero, so the function f has no critical numbers and no points of inflection. If $a = 0$ or $d = b$, then f is a constant function so that all real numbers x are critical numbers but f still has no points of inflection.

119. Let f be a continuous function on some interval I. Suppose c is a critical number of f and (a, b) is some open interval in I containing c with $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$. Then the function f is decreasing on the interval (a, c) and increasing on the interval (c, b); in other words, for all x in (a, b), $f(x) \geq f(c)$. Therefore, $f(c)$ is a local minimum value.

121. Let f be a function that is continuous on the closed interval $[a, b]$. Moreover, suppose that f' and f'' exist on the open interval (a, b) with $f''(x) < 0$ on (a, b). Let c be any fixed number in (a, b). An equation of the tangent line to f at the point $(c, f(c))$ is

$$y = f(c) + f'(c)(x - c).$$

To show that f is concave down on (a, b), it must be established that the graph of f lies below each of its tangent lines for all x in (a, b); that is, that

$$f(x) \leq f(c) + f'(c)(x - c)$$

for all x in (a, b). If $x = c$, then $f(x) = f(c)$ and there is nothing more to prove. If $x \neq c$, then by applying the Mean Value Theorem to the function f, there is a number x_1 between c and x for which

$$f'(x_1) = \frac{f(x) - f(c)}{x - c}.$$

Solve this equation for $f(x)$ to obtain $f(x) = f(c) + f'(x_1)(x - c)$. There are two cases to consider:

- Case I: $c < x_1 < x$: Because $f''(x) < 0$ on the interval (a, b), it follows that f' is decreasing on (a, b). For $x_1 > c$, this means that $f'(x_1) < f'(c)$. Multiplying by $x - c$, which is positive, and adding $f(c)$ then yields

$$f(x) = f(c) + f'(x_1)(x - c) < f(c) + f'(c)(x - c);$$

that is, the graph of f lies below each of its tangent lines to the right of c in (a, b).

- Case II: $x < x_1 < c$: Because $f''(x) < 0$ on the interval (a, b), it follows that f' is decreasing on (a, b). For $x_1 < c$, this means that $f'(x_1) > f'(c)$. Multiplying by $x - c$, which is negative, and adding $f(c)$ then yields

$$f(x) = f(c) + f'(x_1)(x - c) < f(c) + f'(c)(x - c);$$

that is, the graph of f lies below each of its tangent lines to the left of c in (a, b).

Therefore, f is concave down.

Challenge Problems

123. Let $f(x) = (x + 1)\tan^{-1} x$. Then

$$f'(x) = (x + 1) \cdot \frac{1}{1 + x^2} + \tan^{-1} x = \frac{x + 1}{1 + x^2} + \tan^{-1} x,$$

and

$$f''(x) = \frac{(1 + x^2) \cdot 1 - (x + 1)(2x)}{(1 + x^2)^2} + \frac{1}{1 + x^2} = \frac{1 + x^2 - 2x^2 - 2x}{(1 + x^2)^2} + \frac{1 + x^2}{(1 + x^2)^2} = \frac{2 - 2x}{(1 + x^2)^2},$$

so $f''(x)$ exists everywhere and is equal to zero when $x = 1$. For $x < 1$, $f''(x) > 0$, so f is concave up on the interval $(-\infty, 1)$; for $x > 1$, $f''(x) < 0$, so f is concave down on the interval $(1, \infty)$. Because the concavity of f changes at 1, the point $\boxed{(1, f(1)) = \left(1, \frac{\pi}{2}\right)}$ is a point of inflection of f.

AP® Practice Problems

1. Given that $f''(x) = x(x+1)^2(x-2)$, determine when $f''(x) = 0$ to determine the intervals on which we should test the concavity of f. Set $f''(x) = x(x+1)^2(x-2) = 0$ and solve, to find $x = 0, x = -1$ and $x = 2$.

   The function f is concave up in the interval(s) where $f''(x) > 0$ and concave down in the interval(s) where $f''(x) < 0$. The sign of $f''(x)$ with the determination of whether the graph is concave up or concave down in the specified interval is shown in the following table.

Interval	Sign of x	Sign of $(x+1)^2$	Sign of $x-2$	Sign of $f''(x)$	Conclusion
$(-\infty, -1)$	$-$	$+$	$-$	$+$	Concave Up
$(-1, 0)$	$-$	$+$	$-$	$+$	Concave Up
$(0, 2)$	$+$	$+$	$-$	$-$	Concave Down
$(2, \infty)$	$+$	$+$	$+$	$+$	Concave Up

   From the chart, since a Point of Inflection is at the point where the concavity changes, the Inflection Points are at $\boxed{\text{0 and 2 only}}$.

   CHOICE D

3. $g(x) = x^5 + x^3 - 2x - 1$. The polynomial function f is differentiable everywhere, so the critical numbers for g occur where $g'(x) = 0$.

   $g'(x) = 5x^4 + 3x^2 - 2$

   Determine the critical numbers by setting $g'(x) = 0$ and solving for x.

$$\begin{aligned} g'(x) &= 0 \\ 5x^4 + 3x^2 - 2 &= 0 \\ \left(5x^2 - 2\right)\left(x^2 + 1\right) &= 0 \\ 5x^2 - 2 &= 0, \quad \text{since } x^2 + 1 > 0 \\ x &= \pm\sqrt{\frac{2}{5}} = \pm\frac{\sqrt{10}}{5} \end{aligned}$$

   By the First Derivative Test $g(x)$ has a local minimum at the critical number(s) where $g'(x)$ changes from negative to positive.

   To determine where $g'(x) > 0$ and $g'(x) < 0$ use the numbers $x = \frac{\sqrt{10}}{5}$ and $x = -\frac{\sqrt{10}}{5}$ to determine the intervals to test for the function increasing or decreasing. The sign of $g'(x)$ with the determination of whether the graph is increasing or decreasing in the specified interval is shown in the following table.

Interval	Sign of $5x^2 - 2$	Sign of $\left(x^2 + 1\right)$	Sign of $f'(x)$	Conclusion
$\left(-\infty, -\frac{\sqrt{10}}{5}\right)$	$+$	$+$	$+$	Increasing
$\left(\frac{-\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right)$	$-$	$+$	$-$	Decreasing
$\left(\frac{\sqrt{10}}{5}, \infty\right)$	$+$	$+$	$+$	Increasing

   By the First Derivative Test, g has a local minimum at $\boxed{x = \frac{\sqrt{10}}{5}}$

   CHOICE B

5. $f(2) = 0$. $f'(2)$ is the slope of the line tangent to f at $(2, f(2))$, which appears to be negative. $f''(2)$ is the concavity of f at $(2, f(2))$, which appears to be positive, as the graph is concave up for the entire domain. Therefore $\boxed{f'(2) < f(2) < f''(2)}$.

   CHOICE B

7. $v(t) = s'(t)$. A value of $v(t)$ in the chart represents the $s'(t)$ value, which is the slope of the line tangent to the graph of $s(t)$ at the specified value of t. Numerous values in the chart could be examined, but note that $v(3) = 0$ and $v(5) = 0$, indicating a horizontal tangent line to $s(t)$ at $(3, s(3))$ and $(5, s(5))$. Reviewing the given graphs shows that only Choice $\boxed{\text{B}}$ satisfies these conditions.

   CHOICE B

9. $$f(x) = \frac{x^4}{2} - 2x^3 - 9x^2 - 12x + 5$$
   $$f'(x) = 2x^3 - 6x^2 - 18x - 12$$
   $$f''(x) = 6x^2 - 12x - 18$$

   Set $f''(x) = 0$ and solve for x.

   $$6x^2 - 12x - 18 = 0$$
   $$6(x-3)(x+1) = 0$$
   $$x = 3 \quad \text{or} \quad x = -1$$

   The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$.

   To determine where $f''(x) > 0$ and $f''(x) < 0$ use the numbers $x = -1$ and $x = 3$ to determine the intervals to test for concavity. The sign of $f''(x)$ with the determination of whether the graph is concave up or concave down in the specified interval is shown in the following table.

Interval	Sign of $x-3$	Sign of $x+1$	Sign of $f''(x)$	Conclusion
$(-\infty, -1)$	$-$	$-$	$+$	Concave Up
$(-1, 3)$	$-$	$+$	$-$	Concave Down
$(3, \infty)$	$+$	$+$	$+$	Concave Up

   f is concave down on the interval $\boxed{(-1, 3)}$

   CHOICE A

11. $$f(x) = (x-1)^{4/5} - 2$$
    $$f'(x) = \frac{4}{5}(x-1)^{-1/5}$$
    $$= \frac{4}{5(x-1)^{1/5}}$$
    $$f''(x) = \frac{4}{5}\left(-\frac{1}{5}\right)(x-1)^{-6/5}$$
    $$= -\frac{4}{25(x-1)^{6/5}}$$

    f has a critical number at $x = 1$, since $f'(x)$ does not exist at $x = 1$. There are no other critical numbers, since $f'(x)$ never equals 0.

Interval	Sign of $f'(x)$	Conclusion	Sign of $f''(x)$	Conclusion
$(-\infty, 1)$	$-$	Decreasing	$-$	Concave Down
$(1, \infty)$	$+$	Increasing	$-$	Concave Down

By the First Derivative Test, $(1, -2)$ is a local minimum. Choice A is true.

f is concave down on $(-\infty, 1)$ and $(1, \infty)$. Choice B is true.

The point $(1, -2)$ is not an inflection point. Choice $\boxed{\text{C}}$ is false.

f does have a vertical tangent at $x = 1$, since f is continuous everywhere but $f'(x) \to \pm\infty$ as $x \to \pm\infty$. Choice D is true.

CHOICE C

13. All of the extrema are less than 0, so any possible zeros must be to the left of -5 or to the right of 2.

Since $(-5, -3)$ is a local minimum and f is a polynomial, $f(x) \to +\infty$ as $x \to -\infty$, and there is a zero to the left of -5.

Since $(2, -6)$ is a local minimum and f is a polynomial, $f(x) \to +\infty$ as $x \to +\infty$, and there is a zero to the right of 2.

Therefore f has $\boxed{\text{two}}$ zeros.

CHOICE B

15. $x(t) = \dfrac{\ln t}{t}$

$$\begin{aligned} x'(t) &= \frac{t \cdot \frac{1}{t} - (\ln t) \cdot 1}{t^2} \\ &= \frac{1 - \ln t}{t^2} \end{aligned}$$

Set $x'(t) = 0$ and solve for t.

$$\begin{aligned} \frac{1 - \ln t}{t^2} &= 0 \\ 1 - \ln t &= 0, \quad \text{since } t^2 > 0, \ \text{ since } t \neq 0 \\ \ln t &= 1 \\ t &= e \end{aligned}$$

$x(t)$ has a critical number where $x'(t) = 0$, which is at $t = e$. The domain is $t > 0$, so there are no values for which $x'(t)$ does not exist.

Time Interval	Sign of $x'(t)$	Signed Distance	Motion of Object
$(0, e)$	$+$	Increasing	To the Right
(e, ∞)	$-$	Decreasing	To the Left

The object is furthest from the origin at $t = \boxed{e \text{ seconds}}$ (at which time it reverses direction and heads back left).

CHOICE D

4.5 Indeterminate Forms and L'Hôpital's Rule

Concepts and Vocabulary

1. $\boxed{\text{False}}$. $\dfrac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\dfrac{0}{0}$ if $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$.

3. $\boxed{\text{False}}$. $\dfrac{1}{x}$ is not an indeterminate form at 0 because, in the numerator, $\lim\limits_{x\to 0} 1 = 1 \neq 0$.

5. Answers will vary, but here is one possible response. Suppose

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = \infty.$$

The following argument would also hold if both limits were $-\infty$. Consider the expression $f(x)-g(x)$ as x approaches c. If the value of f becomes unbounded faster than the value of g, the value of $f(x)-g(x)$ should approach ∞, whereas if the value of g becomes unbounded faster than the value of f, the value of $f(x)-g(x)$ should approach $-\infty$. Moreover, if f and g become unbounded at similar rates, there could be cancellation leading the value of $f(x)-g(x)$ to approach any real number. In other words, in an expression of the form $\infty-\infty$, there is competition between the two terms in the expression, and an analysis of these competing forces is needed to determine the value of the limit. This is why $\infty-\infty$ is an indeterminate form.

Next, consider the expression $f(x)+g(x)$. In this expression, there is no competition between the two terms. The values of f and g are working together to make the value of $f(x)+g(x)$ approach ∞. This is why $\infty+\infty$ is not an indeterminate form.

Skill Building

7. (a) Because

$$\lim_{x\to 0}(1-e^x) = 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

the expression $\dfrac{1-e^x}{x}$ $\boxed{\text{is an indeterminate form at } 0}$.

(b) Based on the limits in part (a), the expression $\dfrac{1-e^x}{x}$ is an indeterminate form at 0 of the type $\boxed{\dfrac{0}{0}}$.

9. (a) Because

$$\lim_{x\to 0} e^x = 1 \neq 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

the expression $\dfrac{e^x}{x}$ $\boxed{\text{is not an indeterminate form at } 0}$.

(b) The value of a limit of the form $\boxed{\dfrac{1}{0} \text{ approaches } \pm\infty}$, so it is not an indeterminate form at 0.

11. (a) Because

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to\infty} x^2 = \infty,$$

the expression $\dfrac{\ln x}{x^2}$ $\boxed{\text{is an indeterminate form at } \infty}$.

(b) Based on the limits in part (a), the expression $\dfrac{\ln x}{x^2}$ is an indeterminate form at ∞ of the type $\boxed{\dfrac{\infty}{\infty}}$.

13. (a) Because

$$\lim_{x\to 0} \sec x = 1 \neq 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

the expression $\dfrac{\sec x}{x}$ $\boxed{\text{is not an indeterminate form at } 0}$.

(b) The value of a limit of the form $\boxed{\dfrac{1}{0} \text{ approaches } \pm\infty}$, so it is not an indeterminate form at 0.

15. (a) Because

$$\lim_{x\to 0}[\sin x(1-\cos x)] = 0 \quad \text{and} \quad \lim_{x\to 0} x^2 = 0,$$

the expression $\dfrac{\sin x(\cos x - 1)}{x^2}$ $\boxed{\text{is an indeterminate form at } 0}$.

(b) Based on the limits in part (a), the expression $\dfrac{\sin x(\cos x - 1)}{x^2}$ is an indeterminate form at 0 of the type $\boxed{\dfrac{0}{0}}$.

17. (a) Because

$$\lim_{x\to \frac{\pi}{4}} (\tan x - 1) = 0 \quad \text{and} \quad \lim_{x\to \frac{\pi}{4}} \sin(4x-\pi) = 0,$$

the expression $\dfrac{(\tan x - 1)}{\sin(4x-\pi)}$ $\boxed{\text{is an indeterminate form at } \dfrac{\pi}{4}}$.

(b) Based on the limits in part (a), the expression $\dfrac{(\tan x - 1)}{\sin(4x-\pi)}$ is an indeterminate form at $\dfrac{\pi}{4}$ of the type $\boxed{\dfrac{0}{0}}$.

19. (a) Because

$$\lim_{x\to \infty} x^2 = \infty \quad \text{and} \quad \lim_{x\to \infty} e^{-x} = 0,$$

the expression x^2e^{-x} $\boxed{\text{is an indeterminate form at } \infty}$.

(b) Based on the limits in part (a), the expression x^2e^{-x} is an indeterminate form at ∞ of the
the
type $\boxed{0\cdot\infty}$.

21. (a) Because

$$\lim_{x\to 0^-} \csc\frac{x}{2} = -\infty \quad \text{and} \quad \lim_{x\to 0^-} \cot\frac{x}{2} = -\infty,$$

the expression $\csc\dfrac{x}{2} - \cot\dfrac{x}{2}$ is an indeterminate form at 0^-. Additionally,

$$\lim_{x\to 0^+} \csc\frac{x}{2} = \infty \quad \text{and} \quad \lim_{x\to 0^+} \cot\frac{x}{2} = \infty,$$

so the expression $\csc\dfrac{x}{2} - \cot\dfrac{x}{2}$ is also an indeterminate form at 0^+. Therefore, the expression $\csc\dfrac{x}{2} - \cot\dfrac{x}{2}$ $\boxed{\text{is an indeterminate form at } 0}$.

(b) Based on the limits in part (a), the expression $\csc\dfrac{x}{2} - \cot\dfrac{x}{2}$ is an indeterminate form at 0 of the type $\boxed{\infty - \infty}$.

23. (a) Because

$$\lim_{x\to 0}\frac{1}{x^2}=\infty \quad \text{and} \quad \lim_{x\to 0}\sin x=0,$$

the expression $\left(\frac{1}{x^2}\right)^{\sin x}$ $\boxed{\text{is an indeterminate form at } 0}$.

(b) Based on the limits in part (a), the expression $\left(\frac{1}{x^2}\right)^{\sin x}$ is an indeterminate form at 0 of the type $\boxed{\infty^0}$.

25. (a) Because

$$\lim_{x\to 0}(x^2-1)=-1 \quad \text{and} \quad \lim_{x\to 0}x=0,$$

the expression $(x^2-1)^x$ $\boxed{\text{is not an indeterminate form at } 0}$.

(b) The value of a limit of the form $\boxed{(-1)^0 \text{ approaches } 1}$, so it is not an indeterminate form at 0.

27. Because

$$\lim_{x\to 2}(x^2+x-6)=0 \quad \text{and} \quad \lim_{x\to 2}(x^2-3x+2)=0,$$

the expression $\frac{x^2+x-6}{x^2-3x+2}$ is an indeterminate form at 2 of the type $\boxed{\frac{0}{0}}$. Using L'Hôpital's Rule,

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2}=\lim_{x\to 2}\frac{\frac{d}{dx}(x^2+x-6)}{\frac{d}{dx}(x^2-3x+2)}=\lim_{x\to 2}\frac{2x+1}{2x-3}=\frac{5}{1}=\boxed{5}.$$

29. Because

$$\lim_{x\to 1}\ln x=0 \quad \text{and} \quad \lim_{x\to 1}(x^2-1)=0,$$

the expression $\frac{\ln x}{x^2-1}$ is an indeterminate form at 1 of the type $\boxed{\frac{0}{0}}$. Using L'Hôpital's Rule,

$$\lim_{x\to 1}\frac{\ln x}{x^2-1}=\lim_{x\to 1}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(x^2-1)}=\lim_{x\to 1}\frac{\frac{1}{x}}{2x}=\boxed{\frac{1}{2}}.$$

31. Because

$$\lim_{x\to 0}(e^x-e^{-x})=0 \quad \text{and} \quad \lim_{x\to 0}\sin x=0,$$

the expression $\frac{e^x-e^{-x}}{\sin x}$ is an indeterminate form at 0 of the type $\boxed{\frac{0}{0}}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{\frac{d}{dx}(e^x-e^{-x})}{\frac{d}{dx}\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{2}{1}=\boxed{2}.$$

33. Because

$$\lim_{x\to 1}\sin(\pi x)=0 \quad \text{and} \quad \lim_{x\to 1}(x-1)=0,$$

the expression $\frac{\sin(\pi x)}{x-1}$ is an indeterminate form at 1 of the type $\boxed{\frac{0}{0}}$. Using L'Hôpital's Rule,

$$\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}=\lim_{x\to 1}\frac{\frac{d}{dx}\sin(\pi x)}{\frac{d}{dx}(x-1)}=\lim_{x\to 1}\frac{\pi\cos(\pi x)}{1}=\boxed{-\pi}.$$

35. Because

$$\lim_{x\to\infty} x^2 = \infty \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty,$$

the expression $\dfrac{x^2}{e^x}$ is an indeterminate form at ∞ of the type $\boxed{\dfrac{\infty}{\infty}}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Because

$$\lim_{x\to\infty} (2x) = \infty \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty,$$

the expression $\dfrac{2x}{e^x}$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule again,

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}.$$

37. Because

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty,$$

the expression $\dfrac{\ln x}{e^x}$ is an indeterminate form at ∞ of the type $\boxed{\dfrac{\infty}{\infty}}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \boxed{0}.$$

39. Because

$$\lim_{x\to 0} (e^x - 1 - \sin x) = 0 \quad \text{and} \quad \lim_{x\to 0} (1 - \cos x) = 0,$$

the expression $\dfrac{e^x - 1 - \sin x}{1 - \cos x}$ is an indeterminate form at 0 of the type $\boxed{\dfrac{0}{0}}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{e^x - 1 - \sin x}{1 - \cos x} = \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1 - \sin x)}{\frac{d}{dx}(1 - \cos x)} = \lim_{x\to 0} \frac{e^x - \cos x}{\sin x}.$$

Because

$$\lim_{x\to 0} (e^x - \cos x) = 0 \quad \text{and} \quad \lim_{x\to 0} \sin x = 0,$$

the expression $\dfrac{e^x - \cos x}{\sin x}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x\to 0} \frac{e^x - 1 - \sin x}{1 - \cos x} = \lim_{x\to 0} \frac{e^x - \cos x}{\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - \cos x)}{\frac{d}{dx}\sin x} = \lim_{x\to 0} \frac{e^x + \sin x}{\cos x} = \frac{1}{1} = \boxed{1}.$$

41. Because

$$\lim_{x\to 0} (\sin x - x) = 0 \quad \text{and} \quad \lim_{x\to 0} x^3 = 0,$$

the expression $\dfrac{\sin x - x}{x^3}$ is an indeterminate form at 0 of the type $\boxed{\dfrac{0}{0}}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x - x)}{\frac{d}{dx}x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2}.$$

Because

$$\lim_{x\to 0}(\cos x - 1) = 0 \quad \text{and} \quad \lim_{x\to 0}(3x^2) = 0,$$

the expression $\dfrac{\cos x - 1}{3x^2}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(3x^2)} = \lim_{x\to 0}\frac{-\sin x}{6x}.$$

Now,

$$\lim_{x\to 0}(-\sin x) = 0 \quad \text{and} \quad \lim_{x\to 0}(6x) = 0,$$

so the expression $\dfrac{-\sin x}{6x}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's rule for a third time,

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{-\sin x}{6x} = \lim_{x\to 0}\frac{\frac{d}{dx}(-\sin x)}{\frac{d}{dx}(6x)} = \lim_{x\to 0}\frac{-\cos x}{6} = \boxed{-\frac{1}{6}}.$$

43. Because

$$\lim_{x\to 0^+} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty,$$

the expression $x^2 \ln x$ is an indeterminate form at 0^+ of the type $\boxed{0\cdot\infty}$. Rewrite

$$x^2 \ln x \quad \text{as} \quad \frac{\ln x}{\frac{1}{x^2}} = \frac{\ln x}{x^{-2}},$$

which is an indeterminate form at 0^+ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0^+}(x^2\ln x) = \lim_{x\to 0^+}\frac{\ln x}{x^{-2}} = \lim_{x\to 0^+}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^{-2}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x\to 0^+}\frac{x^2}{-2} = \boxed{0}.$$

45. Because

$$\lim_{x\to\infty} x = \infty \quad \text{and} \quad \lim_{x\to\infty}(e^{1/x} - 1) = 0,$$

the expression $x(e^{1/x} - 1)$ is an indeterminate form at ∞ of the type $\boxed{0\cdot\infty}$. Rewrite

$$x(e^{1/x} - 1) \quad \text{as} \quad \frac{e^{1/x} - 1}{1/x},$$

which is an indeterminate form at ∞ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty}[x(e^{1/x}-1)] = \lim_{x\to\infty}\frac{e^{1/x} - 1}{1/x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(e^{1/x} - 1)}{\frac{d}{dx}(1/x)} = \lim_{x\to\infty}\frac{-x^{-2}e^{1/x}}{-x^{-2}} = \lim_{x\to\infty} e^{1/x} = e^0 = \boxed{1}.$$

47. Because

$$\lim_{x\to\pi/2^-} \sec x = \infty \quad \text{and} \quad \lim_{x\to\pi/2^-} \tan x = \infty,$$

while

$$\lim_{x\to\pi/2^+} \sec x = -\infty \quad \text{and} \quad \lim_{x\to\pi/2^+} \tan x = -\infty,$$

the expression $\sec x - \tan x$ is an indeterminate form at $\pi/2$ of the type $\boxed{\infty - \infty}$. Rewrite

$$\sec x - \tan x \quad \text{as} \quad \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x},$$

which is an indeterminate form at $\pi/2$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to\pi/2}(\sec x - \tan x) = \lim_{x\to\pi/2}\frac{1-\sin x}{\cos x} = \lim_{x\to\pi/2}\frac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx}\cos x} = \lim_{x\to\pi/2}\frac{-\cos x}{-\sin x} = \frac{0}{-1} = \boxed{0}.$$

49. Because

$$\lim_{x\to1^-}\frac{1}{\ln x} = -\infty \quad \text{and} \quad \lim_{x\to1^-}\frac{x}{\ln x} = -\infty,$$

while

$$\lim_{x\to1^+}\frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x\to1^+}\frac{x}{\ln x} = \infty,$$

the expression $\dfrac{1}{\ln x} - \dfrac{x}{\ln x}$ is an indeterminate form at 1 of the type $\boxed{\infty - \infty}$. Rewrite

$$\frac{1}{\ln x} - \frac{x}{\ln x} \quad \text{as} \quad \frac{1-x}{\ln x},$$

which is an indeterminate form at 1 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to1}\left(\frac{1}{\ln x} - \frac{x}{\ln x}\right) = \lim_{x\to1}\frac{1-x}{\ln x} = \lim_{x\to1}\frac{\frac{d}{dx}(1-x)}{\frac{d}{dx}\ln x} = \lim_{x\to1}\frac{-1}{\frac{1}{x}} = \frac{-1}{1} = \boxed{-1}.$$

51. Because

$$\lim_{x\to0^+}(2x) = 0 \quad \text{and} \quad \lim_{x\to0^+}(3x) = 0,$$

the expression $(2x)^{3x}$ is an indeterminate form at 0^+ of the type $\boxed{0^0}$. Let $y = (2x)^{3x}$. Then

$$\ln y = \ln(2x)^{3x} = (3x)\ln(2x),$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$(3x)\ln(2x) \quad \text{as} \quad \frac{\ln(2x)}{\frac{1}{3x}},$$

which is now an indeterminate form at 0^+ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to0^+}\ln y &= \lim_{x\to0^+}[(3x)\ln(2x)] = \lim_{x\to0^+}\frac{\ln(2x)}{\frac{1}{3x}} = \lim_{x\to0^+}\frac{\frac{d}{dx}\ln(2x)}{\frac{d}{dx}\frac{1}{3x}} \\ &= \lim_{x\to0^+}\frac{\frac{1}{2x}\cdot 2}{-\frac{1}{(3x)^2}\cdot 3} = \lim_{x\to0^+}(-3x) = 0.\end{aligned}$$

Finally, because $\lim_{x\to0^+}\ln y = 0$, it follows that

$$\lim_{x\to0^+} y = \lim_{x\to0^+}(2x)^{3x} = e^0 = \boxed{1}.$$

53. Because
$$\lim_{x\to\infty}(x+1)=\infty \quad \text{and} \quad \lim_{x\to\infty} e^{-x}=0,$$
the expression $(x+1)^{e^{-x}}$ is an indeterminate form at ∞ of the type $\boxed{\infty^0}$. Let $y=(x+1)^{e^{-x}}$. Then
$$\ln y=\ln(x+1)^{e^{-x}}=e^{-x}\ln(x+1),$$
which is an indeterminate form at ∞ of the type $0\cdot\infty$. Rewrite
$$e^{-x}\ln(x+1) \quad \text{as} \quad \frac{\ln(1+x)}{e^x},$$
which is now an indeterminate form ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,
$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}[e^{-x}\ln(1+x)]=\lim_{x\to\infty}\frac{\ln(1+x)}{e^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}e^x}=\lim_{x\to\infty}\frac{\frac{1}{1+x}}{e^x}=0.$$
Finally, because $\lim\limits_{x\to\infty}\ln y=0$, it follows that
$$\lim_{x\to\infty} y=\lim_{x\to\infty}(x+1)^{e^{-x}}=e^0=\boxed{1}.$$

55. Because
$$\lim_{x\to 0^+}\csc x=\infty \quad \text{and} \quad \lim_{x\to 0^+}\sin x=0,$$
the expression $(\csc x)^{\sin x}$ is an indeterminate form at 0^+ of the type $\boxed{\infty^0}$. Let $y=(\csc x)^{\sin x}$. Then
$$\ln y=\ln(\csc x)^{\sin x}=\sin x\ln(\csc x),$$
which is an indeterminate form at 0^+ of the type $0\cdot\infty$. Rewrite
$$\sin x\ln(\csc x) \quad \text{as} \quad \frac{\ln(\csc x)}{\frac{1}{\sin x}}=\frac{\ln(\csc x)}{\csc x},$$
which is now an indeterminate form at 0^+ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,
$$\begin{aligned}\lim_{x\to 0^+}\ln y &= \lim_{x\to 0^+}[\sin x\ln(\csc x)]=\lim_{x\to 0^+}\frac{\ln(\csc x)}{\csc x}=\lim_{x\to 0^+}\frac{\frac{d}{dx}\ln(\csc x)}{\frac{d}{dx}\csc x}\\ &= \lim_{x\to 0^+}\frac{\sin x\cdot -\csc x\cot x}{-\csc x\cot x}=\lim_{x\to 0+}\sin x=0.\end{aligned}$$
Finally, because $\lim\limits_{x\to 0^+}\ln y=0$, it follows that
$$\lim_{x\to 0^+} y=\lim_{x\to 0^+}(\csc x)^{\sin x}=e^0=\boxed{1}.$$

57. Because
$$\lim_{x\to\pi/2^-}\sin x=1 \quad \text{and} \quad \lim_{x\to\pi/2^-}\tan x=\infty,$$
the expression $(\sin x)^{\tan x}$ is an indeterminate form at $\pi/2^-$ of the type $\boxed{1^\infty}$. Let $y=(\sin x)^{\tan x}$. Then
$$\ln y=\ln(\sin x)^{\tan x}=\tan x\ln(\sin x),$$

which is an indeterminate form at $\pi/2^-$ of the type $0\cdot\infty$. Rewrite

$$\tan x\ln(\sin x) \quad \text{as} \quad \frac{\ln(\sin x)}{\cot x},$$

which is now an indeterminate form at $\pi/2^-$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\pi/2^-}\ln y &= \lim_{x\to\pi/2^-}[\tan x\ln(\sin x)] = \lim_{x\to\pi/2^-}\frac{\ln(\sin x)}{\cot x}\\ &= \lim_{x\to\pi/2^-}\frac{\frac{d}{dx}\ln(\sin x)}{\frac{d}{dx}\cot x} = \lim_{x\to\pi/2^-}\frac{\frac{1}{\sin x}\cdot\cos x}{-\csc^2 x} = \lim_{x\to\pi/2^-}(-\sin x\cos x) = 0.\end{aligned}$$

Finally, because $\lim_{x\to\pi/2^-}\ln y = 0$, it follows that

$$\lim_{x\to\pi/2^-} y = \lim_{x\to\pi/2^-}(\sin x)^{\tan x} = e^0 = \boxed{1}.$$

59. Note that

$$\frac{\cot x}{\cot(2x)} = \frac{\tan(2x)}{\tan x};$$

because

$$\lim_{x\to0^+}\tan(2x) = 0 \quad \text{and} \quad \lim_{x\to0^+}\tan x = 0,$$

the expression $\frac{\tan(2x)}{\tan x}$ is an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to0^+}\frac{\cot x}{\cot(2x)} = \lim_{x\to0^+}\frac{\tan(2x)}{\tan x} = \lim_{x\to0^+}\frac{\frac{d}{dx}\tan(2x)}{\frac{d}{dx}\tan x} = \lim_{x\to0^+}\frac{2\sec^2(2x)}{\sec^2 x} = \frac{2}{1} = \boxed{2}.$$

61. Because

$$\lim_{x\to1/2^-}\ln(1-2x) = -\infty \quad \text{and} \quad \lim_{x\to1/2^-}\tan(\pi x) = \infty,$$

the expression $\frac{\ln(1-2x)}{\tan(\pi x)}$ is an indeterminate form at $1/2^-$ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x\to1/2^-}\frac{\ln(1-2x)}{\tan(\pi x)} = \lim_{x\to1/2^-}\frac{\frac{d}{dx}\ln(1-2x)}{\frac{d}{dx}\tan(\pi x)} = \lim_{x\to1/2^-}\frac{\frac{1}{1-2x}\cdot -2}{\pi\sec^2(\pi x)} = \lim_{x\to1/2^-}\frac{-2\cos^2(\pi x)}{\pi(1-2x)}.$$

Now,

$$\lim_{x\to1/2^-}[-2\cos^2(\pi x)] = 0 \quad \text{and} \quad \lim_{x\to1/2^-}[\pi(1-2x)] = 0,$$

so the expression $\frac{-2\cos^2(\pi x)}{\pi(1-2x)}$ is an indeterminate form at $1/2^-$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned}\lim_{x\to1/2^-}\frac{\ln(1-2x)}{\tan(\pi x)} &= \lim_{x\to1/2^-}\frac{-2\cos^2(\pi x)}{\pi(1-2x)} = \lim_{x\to1/2^-}\frac{\frac{d}{dx}[-2\cos^2(\pi x)]}{\frac{d}{dx}[\pi(1-2x)]}\\ &= \lim_{x\to1/2^-}\frac{-4\pi\cos(\pi x)\cdot -\sin(\pi x)}{-2\pi} = \lim_{x\to1/2^-}[-2\sin(\pi x)\cos(\pi x)] = \boxed{0}.\end{aligned}$$

63. Because
$$\lim_{x\to\infty}(x^4+x^3)=\infty \quad \text{and} \quad \lim_{x\to\infty}(e^x+1)=\infty,$$
the expression $\dfrac{x^4+x^3}{e^x+1}$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,
$$\lim_{x\to\infty}\frac{x^4+x^3}{e^x+1}=\lim_{x\to\infty}\frac{\frac{d}{dx}(x^4+x^3)}{\frac{d}{dx}(e^x+1)}=\lim_{x\to\infty}\frac{4x^3+3x^2}{e^x}.$$
Because
$$\lim_{x\to\infty}(4x^3+3x^2)=\infty \quad \text{and} \quad \lim_{x\to\infty}e^x=\infty,$$
the expression $\dfrac{4x^3+3x^2}{e^x}$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule again,
$$\lim_{x\to\infty}\frac{x^4+x^3}{e^x+1}=\lim_{x\to\infty}\frac{4x^3+3x^2}{e^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(4x^3+3x^2)}{\frac{d}{dx}e^x}=\lim_{x\to\infty}\frac{12x^2+6x}{e^x}.$$
Now,
$$\lim_{x\to\infty}(12x^2+6x)=\infty \quad \text{and} \quad \lim_{x\to\infty}e^x=\infty,$$
so the expression $\dfrac{12x^2+6x}{e^x}$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule a third time,
$$\lim_{x\to\infty}\frac{x^4+x^3}{e^x+1}=\lim_{x\to\infty}\frac{12x^2+6x}{e^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(12x^2+6x)}{\frac{d}{dx}e^x}=\lim_{x\to\infty}\frac{24x+6}{e^x}.$$
Finally, because
$$\lim_{x\to\infty}(24x+6)=\infty \quad \text{and} \quad \lim_{x\to\infty}e^x=\infty,$$
the expression $\dfrac{24x+6}{e^x}$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule a fourth time,
$$\lim_{x\to\infty}\frac{x^4+x^3}{e^x+1}=\lim_{x\to\infty}\frac{24x+6}{e^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(24x+6)}{\frac{d}{dx}e^x}=\lim_{x\to\infty}\frac{24}{e^x}=\boxed{0}.$$

65. Because
$$\lim_{x\to 0}(xe^{4x}-x)=0 \quad \text{and} \quad \lim_{x\to 0}[1-\cos(2x)]=0,$$
the expression $\dfrac{xe^{4x}-x}{1-\cos(2x)}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,
$$\lim_{x\to 0}\frac{xe^{4x}-x}{1-\cos(2x)}=\lim_{x\to 0}\frac{\frac{d}{dx}(xe^{4x}-x)}{\frac{d}{dx}[1-\cos(2x)]}=\lim_{x\to 0}\frac{4xe^{4x}+e^{4x}-1}{2\sin(2x)}.$$
Now,
$$\lim_{x\to 0}(4xe^{4x}+e^{4x}-1)=0 \quad \text{and} \quad \lim_{x\to 0}[2\sin(2x)]=0,$$
so the expression $\dfrac{4xe^{4x}+e^{4x}-1}{2\sin(2x)}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,
$$\begin{aligned}\lim_{x\to 0}\frac{xe^{4x}-x}{1-\cos(2x)} &= \lim_{x\to 0}\frac{4xe^{4x}+e^{4x}-1}{2\sin(2x)}=\lim_{x\to 0}\frac{\frac{d}{dx}(4xe^{4x}+e^{4x}-1)}{\frac{d}{dx}[2\sin(2x)]}\\ &= \lim_{x\to 0}\frac{16xe^{4x}+4e^{4x}+4e^{4x}}{4\cos(2x)}=\frac{8}{4}=\boxed{2}.\end{aligned}$$

67. Because

$$\lim_{x\to 0} \tan^{-1} x = 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

the expression $\dfrac{\tan^{-1} x}{x}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\tan^{-1} x}{\frac{d}{dx} x} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{1} = \boxed{1}.$$

69. Because

$$\lim_{x\to 0} (\cos x - 1) = 0 \quad \text{and} \quad \lim_{x\to 0} [\cos(2x) - 1] = 0,$$

the expression $\dfrac{\cos x - 1}{\cos(2x) - 1}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\cos x - 1}{\cos(2x) - 1} = \lim_{x\to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}[\cos(2x) - 1]} = \lim_{x\to 0} \frac{-\sin x}{-2\sin(2x)} = \lim_{x\to 0} \frac{\sin x}{2\sin(2x)}.$$

Now,

$$\lim_{x\to 0} \sin x = 0 \quad \text{and} \quad \lim_{x\to 0} [2\sin(2x)] = 0,$$

so the expression $\dfrac{\sin x}{2\sin(2x)}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x\to 0} \frac{\cos x - 1}{\cos(2x) - 1} = \lim_{x\to 0} \frac{\sin x}{2\sin(2x)} = \lim_{x\to 0} \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}[2\sin(2x)]} = \lim_{x\to 0} \frac{\cos x}{4\cos(2x)} = \boxed{\frac{1}{4}}.$$

71. Because

$$\lim_{x\to 0^+} x^{1/2} = 0 \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty,$$

the expression $x^{1/2} \ln x$ is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$x^{1/2} \ln x \quad \text{as} \quad \frac{\ln x}{\frac{1}{x^{1/2}}} = \frac{\ln x}{x^{-1/2}},$$

which is an indeterminate form at 0^+ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0^+} (x^{1/2} \ln x) = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^{-1/2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} (-2x^{1/2}) = \boxed{0}.$$

73. Because

$$\lim_{x\to \pi/2^-} \tan x = \infty \quad \text{and} \quad \lim_{x\to \pi/2^-} \ln(\sin x) = 0,$$

while

$$\lim_{x\to \pi/2^+} \tan x = -\infty \quad \text{and} \quad \lim_{x\to \pi/2^+} \ln(\sin x) = 0,$$

the expression $\tan x \ln(\sin x)$ is an indeterminate form at $\pi/2$ of the type $0 \cdot \infty$. Rewrite

$$\tan x \ln(\sin x) \quad \text{as} \quad \frac{\ln(\sin x)}{\frac{1}{\tan x}} = \frac{\ln(\sin x)}{\cot x},$$

which is an indeterminate form at $\pi/2$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\pi/2}[\tan x\ln(\sin x)] &= \lim_{x\to\pi/2}\frac{\ln(\sin x)}{\cot x} = \lim_{x\to\pi/2}\frac{\frac{d}{dx}\ln(\sin x)}{\frac{d}{dx}\cot x}\\ &= \lim_{x\to\pi/2}\frac{\frac{1}{\sin x}\cdot\cos x}{-\csc^2 x} = \lim_{x\to\pi/2}(-\sin x\cos x) = \boxed{0}.\end{aligned}$$

75. Because

$$\lim_{x\to 0^-}\csc x = -\infty \quad\text{and}\quad \lim_{x\to 0^-}\ln(x+1) = 0,$$

while

$$\lim_{x\to 0^+}\csc x = \infty \quad\text{and}\quad \lim_{x\to 0^+}\ln(x+1) = 0,$$

the expression $\csc x\ln(x+1)$ is an indeterminate form at 0 of the type $0\cdot\infty$. Rewrite

$$\csc x\ln(x+1) \quad\text{as}\quad \frac{\ln(x+1)}{\sin x},$$

which is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0}[\csc x\ln(x+1)] = \lim_{x\to 0}\frac{\ln(x+1)}{\sin x} = \lim_{x\to 0}\frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}\sin x} = \lim_{x\to 0}\frac{\frac{1}{x+1}}{\cos x} = \frac{1}{1} = \boxed{1}.$$

77. Because

$$\lim_{x\to a^-}(a^2-x^2) = 0 \quad\text{and}\quad \lim_{x\to a^-}\tan\left(\frac{\pi x}{2a}\right) = \infty,$$

while

$$\lim_{x\to a^+}(a^2-x^2) = 0 \quad\text{and}\quad \lim_{x\to a^+}\tan\left(\frac{\pi x}{2a}\right) = -\infty,$$

the expression $(a^2-x^2)\tan\left(\dfrac{\pi x}{2a}\right)$ is an indeterminate form at a of the type $0\cdot\infty$. Rewrite

$$(a^2-x^2)\tan\left(\frac{\pi x}{2a}\right) \quad\text{as}\quad \frac{a^2-x^2}{\cot\left(\frac{\pi x}{2a}\right)},$$

which is an indeterminate form at a of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to a}\left[(a^2-x^2)\tan\left(\frac{\pi x}{2a}\right)\right] = \lim_{x\to a}\frac{a^2-x^2}{\cot\left(\frac{\pi x}{2a}\right)} = \lim_{x\to a}\frac{\frac{d}{dx}(a^2-x^2)}{\frac{d}{dx}\cot\left(\frac{\pi x}{2a}\right)} = \lim_{x\to a}\frac{-2x}{-\frac{\pi}{2a}\csc^2\left(\frac{\pi x}{2a}\right)} = \frac{-2a}{-\frac{\pi}{2a}} = \boxed{\frac{4a^2}{\pi}}.$$

79. Because

$$\lim_{x\to 1^-}\frac{1}{\ln x} = -\infty \quad\text{and}\quad \lim_{x\to 1^-}\frac{1}{x-1} = -\infty,$$

while

$$\lim_{x\to 1^+}\frac{1}{\ln x} = \infty \quad\text{and}\quad \lim_{x\to 1^+}\frac{1}{x-1} = \infty,$$

the expression $\dfrac{1}{\ln x} - \dfrac{1}{x-1}$ is an indeterminate form at 1 of the type $\infty-\infty$. Rewrite

$$\frac{1}{\ln x} - \frac{1}{x-1} \quad\text{as}\quad \frac{x-1-\ln x}{(x-1)\ln x},$$

which is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to1}\frac{x-1-\ln x}{(x-1)\ln x}\\ &= \lim_{x\to1}\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}[(x-1)\ln x]}=\lim_{x\to1}\frac{1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x}+\ln x}=\lim_{x\to1}\frac{x-1}{x-1+x\ln x}.\end{aligned}$$

Now,

$$\lim_{x\to1}(x-1)=0 \quad\text{and}\quad \lim_{x\to1}(x-1+x\ln x)=0,$$

so the expression $\frac{x-1}{x-1+x\ln x}$ is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x\to1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to1}\frac{x-1}{x-1+x\ln x}=\lim_{x\to1}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x-1+x\ln x)}=\lim_{x\to1}\frac{1}{1+x\cdot\frac{1}{x}+\ln x}=\boxed{\frac{1}{2}}.$$

81. Because

$$\lim_{x\to\pi/2^-}(x\tan x)=\infty \quad\text{and}\quad \lim_{x\to\pi/2^-}\frac{\pi}{2}\sec x=\infty,$$

while

$$\lim_{x\to\pi/2^+}(x\tan x)=-\infty \quad\text{and}\quad \lim_{x\to\pi/2^+}\frac{\pi}{2}\sec x=-\infty,$$

the expression $x\tan x-\frac{\pi}{2}\sec x$ is an indeterminate form at $\pi/2$ of the type $\infty-\infty$. Rewrite

$$x\tan x-\frac{\pi}{2}\sec x \quad\text{as}\quad x\frac{\sin x}{\cos x}-\frac{\pi}{2}\frac{1}{\cos x}=\frac{x\sin x-\frac{\pi}{2}}{\cos x},$$

which is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\pi/2}\left(x\tan x-\frac{\pi}{2}\sec x\right) &= \lim_{x\to\pi/2}\frac{x\sin x-\frac{\pi}{2}}{\cos x}=\lim_{x\to\pi/2}\frac{\frac{d}{dx}\left(x\sin x-\frac{\pi}{2}\right)}{\frac{d}{dx}\cos x}\\ &= \lim_{x\to\pi/2}\frac{x\cos x+\sin x}{-\sin x}=\frac{1}{-1}=\boxed{-1}.\end{aligned}$$

83. Because

$$\lim_{x\to1^-}(1-x)=0 \quad\text{and}\quad \lim_{x\to1^-}\tan(\pi x)=0,$$

the expression $(1-x)^{\tan(\pi x)}$ is an indeterminate form at 1^- of the type 0^0. Let $y=(1-x)^{\tan(\pi x)}$. Then

$$\ln y=\ln(1-x)^{\tan(\pi x)}=\tan(\pi x)\ln(1-x),$$

which is an indeterminate form at 1^- of the type $0\cdot\infty$. Rewrite

$$\tan(\pi x)\ln(1-x) \quad\text{as}\quad \frac{\ln(1-x)}{\cot(\pi x)},$$

which is now an indeterminate form at 1^- of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to1^-}\ln y &= \lim_{x\to1^-}[\tan(\pi x)\ln(1-x)]=\lim_{x\to1^-}\frac{\ln(1-x)}{\cot(\pi x)}\\ &= \lim_{x\to1^-}\frac{\frac{d}{dx}\ln(1-x)}{\frac{d}{dx}\cot(\pi x)}=\lim_{x\to1^-}\frac{\frac{-1}{1-x}}{-\pi\csc^2(\pi x)}=\lim_{x\to1^-}\frac{\sin^2(\pi x)}{\pi(1-x)}.\end{aligned}$$

Now,

$$\lim_{x\to 1^-} \sin^2(\pi x) = 0 \quad \text{and} \quad \lim_{x\to 1^-} [\pi(1-x)] = 0,$$

so the expression $\dfrac{\sin^2(\pi x)}{\pi(1-x)}$ is an indeterminate form at 1^- of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x\to 1^-} \ln y = \lim_{x\to 1^-} \frac{\sin^2(\pi x)}{\pi(1-x)} = \lim_{x\to 1^-} \frac{\frac{d}{dx}\sin^2(\pi x)}{\frac{d}{dx}[\pi(1-x)]} = \lim_{x\to 1^-} \frac{2\pi \sin(\pi x)\cos(\pi x)}{-\pi} = 0.$$

Finally, because $\lim_{x\to 1^-} \ln y = 0$, it follows that

$$\lim_{x\to 1^-} y = \lim_{x\to 1^-} (1-x)^{\tan(\pi x)} = e^0 = \boxed{1}.$$

85. Because

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty,$$

while

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = \infty,$$

the expression $\left(\dfrac{\sin x}{x}\right)^{1/x}$ is an indeterminate form at 0 of the type 1^∞. Let $y = \left(\dfrac{\sin x}{x}\right)^{1/x}$. Then

$$\ln y = \ln\left(\frac{\sin x}{x}\right)^{1/x} = \frac{1}{x}\ln\left(\frac{\sin x}{x}\right) = \frac{\ln\left(\frac{\sin x}{x}\right)}{x},$$

which is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln\left(\frac{\sin x}{x}\right)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\ln\left(\frac{\sin x}{x}\right)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{1} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x}.$$

Now,

$$\lim_{x\to 0}(x\cos x - \sin x) = 0 \quad \text{and} \quad \lim_{x\to 0}(x\sin x) = 0,$$

so the expression $\dfrac{x\cos x - \sin x}{x\sin x}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{x\cos x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}(x\cos x - \sin x)}{\frac{d}{dx}(x\sin x)} \\
&= \lim_{x\to 0} \frac{-x\sin x + \cos x - \cos x}{x\cos x + \sin x} = \lim_{x\to 0} \frac{-x\sin x}{x\cos x + \sin x}.
\end{aligned}$$

Continuing,

$$\lim_{x\to 0}(-x\sin x) = 0 \quad \text{and} \quad \lim_{x\to 0}(x\cos x + \sin x) = 0,$$

so the expression $\dfrac{-x\sin x}{x\cos x + \sin x}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule a third time,

$$\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{-x\sin x}{x\cos x + \sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}(-x\sin x)}{\frac{d}{dx}(x\cos x + \sin x)} \\
&= \lim_{x\to 0} \frac{-x\cos x - \sin x}{-x\sin x + \cos x + \cos x} = 0.
\end{aligned}$$

Finally, because $\lim\limits_{x\to 0} \ln y = 0$, it follows that

$$\lim_{x\to 0} y = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x} = e^0 = \boxed{1}.$$

87. Because

$$\lim_{x\to \pi/2^-} \tan x = \infty \quad \text{and} \quad \lim_{x\to \pi/2^-} \cos x = 0,$$

the expression $(\tan x)^{\cos x}$ is an indeterminate form at $\pi/2^-$ of the type ∞^0. Let $y = (\tan x)^{\cos x}$. Then

$$\ln y = \ln(\tan x)^{\cos x} = \cos x \ln(\tan x),$$

which is an indeterminate form at $\pi/2^-$ of the type $0 \cdot \infty$. Rewrite

$$\cos x \ln(\tan x) \quad \text{as} \quad \frac{\ln(\tan x)}{\sec x},$$

which is now an indeterminate form at $\pi/2^-$ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to \pi/2^-} \ln y &= \lim_{x\to \pi/2^-} [\cos x \ln(\tan x)] = \lim_{x\to \pi/2^-} \frac{\ln(\tan x)}{\sec x} \\
&= \lim_{x\to \pi/2^-} \frac{\frac{d}{dx}\ln(\tan x)}{\frac{d}{dx}\sec x} = \lim_{x\to \pi/2^-} \frac{\cot x \cdot \sec^2 x}{\sec x \tan x} = \lim_{x\to \pi/2^-} \frac{\cos x}{\sin^2 x} = 0.
\end{aligned}$$

Finally, because $\lim\limits_{x\to \pi/2^-} \ln y = 0$, it follows that

$$\lim_{x\to \pi/2^-} y = \lim_{x\to \pi/2^-} (\tan x)^{\cos x} = e^0 = \boxed{1}.$$

89. Because

$$\lim_{x\to 0} \cosh x = 1 \quad \text{and} \quad \lim_{x\to 0} e^x = 1,$$

the expression $(\cosh x)^{e^x}$ is of the form 1^1, which is not an indeterminate form; rather the value of this expression tends toward 1. Therefore,

$$\lim_{x\to 0} (\cosh x)^{e^x} = \boxed{1}.$$

Applications and Extensions

91. Let

$$w(t) = \frac{Ke^{rt}}{\frac{K}{40} + e^{rt} - 1},$$

where $K = 366$, $r = 0.283$, and $t = 0$ represents the year 2000.

(a) Because $r > 0$,

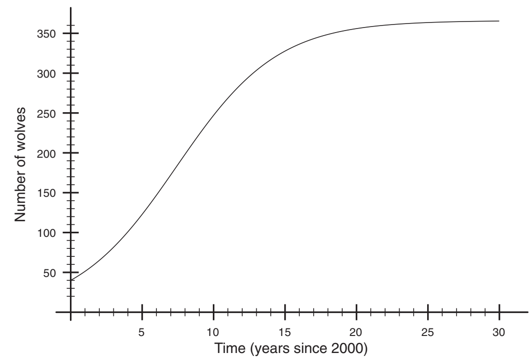
$$\lim_{t\to\infty} Ke^{rt} = \infty, \quad \text{and} \quad \lim_{t\to\infty} \left(\frac{K}{40} + e^{rt} - 1\right) = \infty,$$

$w(t)$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \frac{Ke^{rt}}{\frac{K}{40} + e^{rt} - 1} = \lim_{t\to\infty} \frac{\frac{d}{dt}Ke^{rt}}{\frac{d}{dt}\left(\frac{K}{40} + e^{rt} - 1\right)} = \lim_{t\to\infty} \frac{Kre^{rt}}{re^{rt}} = \lim_{t\to\infty} K = \boxed{K = 366}.$$

(b) If the population of wolves in Wyoming outside of Yellowstone National Park continues to follow the given logistic curve, $\boxed{\text{in the long run, the population will reach } K = 366 \text{ wolves}}$.

(c) The figure below displays the graph of the wolf population.



93. (a) Let

$$I = \frac{E}{R}(1 - e^{-Rt/L}).$$

First,

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{E}{R}(1 - e^{-Rt/L}) = \frac{E}{R} \cdot 1 = \boxed{\frac{E}{R}}.$$

Next, for the limit as $R \to 0^+$, note that

$$\lim_{R\to 0^+} E(1 - e^{-Rt/L}) = 0 \quad \text{and} \quad \lim_{R\to 0^+} R = 0,$$

so $I(t)$ is an indeterminate form at $R = 0^+$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{R\to 0^+} I(t) = \lim_{R\to 0^+} \frac{E(1 - e^{-Rt/L})}{R} = \lim_{R\to 0^+} \frac{\frac{d}{dR}[E(1 - e^{-Rt/L})]}{\frac{d}{dR}R} = \lim_{R\to 0^+} \frac{E\frac{t}{L}e^{-Rt/L}}{1} = \boxed{\frac{Et}{L}}.$$

(b) After the circuit has been active for a long time, the current will level out at the value E/R. On the other hand, in the limit as the resistance goes to zero, the current becomes a linear function of time, $I(t) = Et/L$.

95. Let $n \geq 1$ be an integer. Because

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to\infty} x^n = \infty,$$

the expression $\frac{\ln x}{x^n}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{x^n} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^n} = \lim_{x\to\infty} \frac{\frac{1}{x}}{nx^{n-1}} = \lim_{x\to\infty} \frac{1}{nx^n} = 0,$$

because $n \geq 1$.

97. Because

$$\lim_{x\to 0^+} (\cos x + 2\sin x) = 1 \quad \text{and} \quad \lim_{x\to 0^+} \cot x = \infty,$$

the expression $(\cos x + 2\sin x)^{\cot x}$ is an indeterminate form at 0^+ of the type 1^∞. Let $y = (\cos x + 2\sin x)^{\cot x}$. Then

$$\ln y = \ln(\cos x + 2\sin x)^{\cot x} = \cot x \ln(\cos x + 2\sin x),$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$\cot x \ln(\cos x + 2\sin x) \quad \text{as} \quad \frac{\ln(\cos x + 2\sin x)}{\tan x},$$

which is now an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} [\cot x \ln(\cos x + 2\sin x)] = \lim_{x\to 0^+} \frac{\ln(\cos x + 2\sin x)}{\tan x} \\ &= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(\cos x + 2\sin x)}{\frac{d}{dx}\tan x} = \lim_{x\to 0^+} \frac{\frac{1}{\cos x + 2\sin x}\cdot(-\sin x + 2\cos x)}{\sec^2 x} = \frac{1(2)}{1} = 2.\end{aligned}$$

Finally, because $\lim_{x\to 0^+} \ln y = 2$, it follows that

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} (\cos x + 2\sin x)^{\cot x} = e^2.$$

99. Using the properties of the natural logarithm function,

$$\ln(x+1) - \ln(x-1) = \ln\frac{x+1}{x-1},$$

so

$$\lim_{x\to\infty} [\ln(x+1) - \ln(x-1)] = \lim_{x\to\infty} \ln\frac{x+1}{x-1} = \ln\left(\lim_{x\to\infty} \frac{x+1}{x-1}\right).$$

Now,

$$\lim_{x\to\infty}(x+1) = \infty \quad \text{and} \quad \lim_{x\to\infty}(x-1) = \infty,$$

so the expression $\frac{x+1}{x-1}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x+1}{x-1} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(x-1)} = \lim_{x\to\infty} \frac{1}{1} = 1.$$

Therefore,

$$\lim_{x\to\infty} [\ln(x+1) - \ln(x-1)] = \ln\left(\lim_{x\to\infty} \frac{x+1}{x-1}\right) = \ln 1 = \boxed{0}.$$

101. Let n be an integer. If $n < 0$, then $-n > 0$, $\lim_{x\to 0^+} x^{-n} = 0$, and

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{x\to 0^+} (x^{-n} e^{-1/x^2}) = 0.$$

If $n = 0$, then

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{x\to 0^+} e^{-1/x^2} = 0.$$

Finally, suppose $n > 0$. Further suppose that k is the smallest positive integer such that $-n + 2k \geq 0$. Because

$$\lim_{x\to 0^+} e^{1/x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^+} x^{-c} = \infty$$

for $c > 0$, the expression $\dfrac{x^{-c}}{e^{1/x^2}} = \dfrac{e^{-1/x^2}}{x^c}$ is an indeterminate form at 0^+ of the type $\dfrac{\infty}{\infty}$ for all $c > 0$. Using L'Hôpital's Rule k times yields

$$\begin{aligned}
\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} &= \lim_{x\to 0^+} \frac{x^{-n}}{e^{1/x^2}} = \lim_{x\to 0^+} \frac{-nx^{-n-1}}{e^{1/x^2}\cdot -\frac{2}{x^3}} = \frac{n}{2}\lim_{x\to 0^+} \frac{x^{-n+2}}{e^{1/x^2}} \\
&= \frac{n(n-2)}{2^2}\lim_{x\to 0^+} \frac{x^{-n+4}}{e^{1/x^2}} \\
&\vdots \\
&= \frac{n(n-2)\cdots(n-2k+2)}{2^k}\lim_{x\to 0^+} \frac{x^{-n+2k}}{e^{1/x^2}}.
\end{aligned}$$

If $-n + 2k = 0$, then

$$\lim_{x\to 0^+} \frac{x^{-n+2k}}{e^{1/x^2}} = \lim_{x\to 0^+} \frac{1}{e^{1/x^2}} = 0;$$

otherwise,

$$\lim_{x\to 0^+} \frac{x^{-n+2k}}{e^{1/x^2}} = \lim_{x\to 0^+} \frac{1}{x^{n-2k}e^{1/x^2}} = 0.$$

Therefore,

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0.$$

103. Because

$$\lim_{x\to\infty}\left(1 + \frac{a}{x}\right) = 1 \quad \text{and} \quad \lim_{x\to\infty} x = \infty,$$

the expression $\left(1 + \dfrac{a}{x}\right)^x$ is an indeterminate form at ∞ of the type 1^∞. Let $y = \left(1 + \dfrac{a}{x}\right)^x$. Then

$$\ln y = \ln\left(1 + \frac{a}{x}\right)^x = x\ln\left(1 + \frac{a}{x}\right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$x\ln\left(1 + \frac{a}{x}\right) \quad \text{as} \quad \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}},$$

which is now an indeterminate form at ∞ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty}\left[x\ln\left(1 + \frac{a}{x}\right)\right] = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln\left(1 + \frac{a}{x}\right)}{\frac{d}{dx}\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}\cdot -\frac{a}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{a}{1 + \frac{a}{x}} = \frac{a}{1} = a.
\end{aligned}$$

Finally, because $\lim_{x\to\infty} \ln y = a$, it follows that

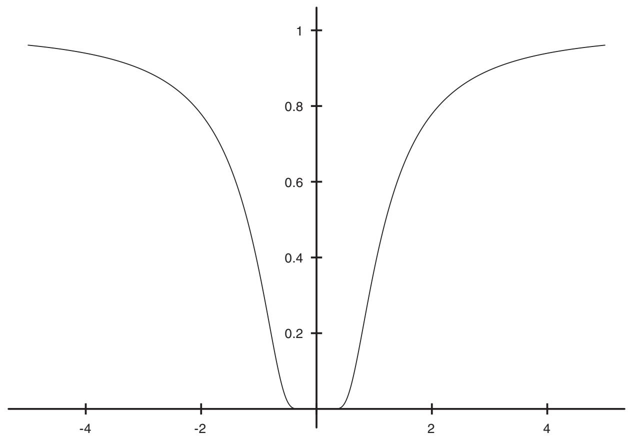
$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1 + \frac{a}{x}\right)^x = e^a.$$

105. (a) Note that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2} - 0}{x} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = 0$$

by Problem 100. Because this limit exists, it follows that f is differentiable at 0; moreover, $\boxed{f'(0) = 0}$.

(b) The figure below displays the graph of f.



107. Suppose the expression $\dfrac{f(x)}{g(x)}$ is an indeterminate form at $-\infty$ of the type $\dfrac{0}{0}$. Then

$$\lim_{x\to-\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} g(x) = 0.$$

Define

$$F(t) = f\left(\frac{1}{t}\right) \quad \text{and} \quad G(t) = g\left(\frac{1}{t}\right).$$

Now, consider the limit

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)}.$$

Make the substitution $t = 1/x$. Then, as $x \to -\infty$, $t \to 0^-$, and

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{t\to0^-} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to0^-} \frac{F(t)}{G(t)}.$$

Because

$$\lim_{t\to0^-} F(t) = \lim_{t\to0^-} f\left(\frac{1}{t}\right) = \lim_{x\to-\infty} f(x) = 0$$

and

$$\lim_{t\to0^-} G(t) = \lim_{t\to0^-} g\left(\frac{1}{t}\right) = \lim_{x\to-\infty} g(x) = 0,$$

the expression $\dfrac{F(t)}{G(t)}$ is an indeterminate form at 0^- of the type $\dfrac{0}{0}$. By the partial proof of L'Hôpital's Rule provided in the text, it follows that

$$\lim_{t\to0^-} \frac{F(t)}{G(t)} = \lim_{t\to0^-} \frac{F'(t)}{G'(t)}.$$

By the definition of F and G

$$F'(t) = -\frac{1}{t^2} f'\left(\frac{1}{t}\right) \quad \text{and} \quad G'(t) = -\frac{1}{t^2} g'\left(\frac{1}{t}\right),$$

so

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{t\to0^-} \frac{F(t)}{G(t)} = \lim_{t\to0^-} \frac{F'(t)}{G'(t)} = \lim_{t\to0^-} \frac{-\frac{1}{t^2} f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2} g'\left(\frac{1}{t}\right)} = \lim_{t\to0^-} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)},$$

as desired.

Challenge Problems

109. (a) Because

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1 \quad \text{and} \quad \lim_{x\to\infty}(-x^2)=-\infty,$$

the expression $\left(1+\frac{1}{x}\right)^{-x^2}$ is an indeterminate form at ∞ of the type 1^∞. Let $y=\left(1+\frac{1}{x}\right)^{-x^2}$. Then

$$\ln y=\ln\left(1+\frac{1}{x}\right)^{-x^2}=-x^2\ln\left(1+\frac{1}{x}\right),$$

which is an indeterminate form at ∞ of the type $0\cdot\infty$. Rewrite

$$-x^2\ln\left(1+\frac{1}{x}\right) \quad \text{as} \quad \frac{\ln\left(1+\frac{1}{x}\right)}{-\frac{1}{x^2}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\left[-x^2\ln\left(1+\frac{1}{x}\right)\right]=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\left(-\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot-\frac{1}{x^2}}{\frac{2}{x^3}}=\lim_{x\to\infty}\frac{-x}{2\left(1+\frac{1}{x}\right)}=-\infty.\end{aligned}$$

Finally, because $\lim_{x\to\infty}\ln y=-\infty$, it follows that

$$\lim_{x\to\infty}y=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-x^2}=\lim_{x\to\infty}e^{\ln y}=\boxed{0}.$$

(b) Because

$$\lim_{x\to\infty}\left(1+\frac{\ln a}{x}\right)=1 \quad \text{and} \quad \lim_{x\to\infty}x=\infty,$$

the expression $\left(1+\frac{\ln a}{x}\right)^{x}$ is an indeterminate form at ∞ of the type 1^∞. Let $y=\left(1+\frac{\ln a}{x}\right)^{x}$. Then

$$\ln y=\ln\left(1+\frac{\ln a}{x}\right)^{x}=x\ln\left(1+\frac{\ln a}{x}\right),$$

which is an indeterminate form at ∞ of the type $0\cdot\infty$. Rewrite

$$x\ln\left(1+\frac{\ln a}{x}\right) \quad \text{as} \quad \frac{\ln\left(1+\frac{\ln a}{x}\right)}{\frac{1}{x}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{\ln a}{x}\right)\right]=\lim_{x\to\infty}\frac{\ln\left(1+\frac{\ln a}{x}\right)}{\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1+\frac{\ln a}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{\ln a}{x}}\cdot-\frac{\ln a}{x^2}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\ln a}{1+\frac{\ln a}{x}}=\ln a.\end{aligned}$$

Finally, because $\lim_{x\to\infty} \ln y = \ln a$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{\ln a}{x}\right)^x = e^{\ln a} = \boxed{a}.$$

(c) Because

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1 \quad \text{and} \quad \lim_{x\to\infty} x^2 = \infty,$$

the expression $\left(1+\frac{1}{x}\right)^{x^2}$ is an indeterminate form at ∞ of the type 1^∞. Let $y = \left(1+\frac{1}{x}\right)^{x^2}$. Then

$$\ln y = \ln\left(1+\frac{1}{x}\right)^{x^2} = x^2\ln\left(1+\frac{1}{x}\right),$$

which is an indeterminate form at ∞ of the type $0\cdot\infty$. Rewrite

$$x^2\ln\left(1+\frac{1}{x}\right) \quad \text{as} \quad \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\left[x^2\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \\ &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}}{-\frac{2}{x^3}} = \lim_{x\to\infty}\frac{x}{2\left(1+\frac{1}{x}\right)} = \infty.\end{aligned}$$

Finally, because $\lim_{x\to\infty}\ln y = \infty$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2} = \lim_{x\to\infty} e^{\ln y} = \boxed{\infty}.$$

(d) This limit does not exist. To show this, first let $x = n\pi$, where n is a natural number. As $n\to\infty$, $x\to\infty$, and

$$\left(1+\frac{\sin x}{x}\right)^x = \left(1+\frac{\sin(n\pi)}{n\pi}\right)^{n\pi} = 1^{n\pi} = 1.$$

Next, let $x = \frac{\pi}{2} + 2n\pi$, where n is a natural number. Again, as $n\to\infty$, $x\to\infty$, but

$$\left(1+\frac{\sin x}{x}\right)^x = \left(1+\frac{\sin(\pi/2+2n\pi)}{\pi/2+2n\pi}\right)^{\pi/2+2n\pi} = \left(1+\frac{1}{\pi/2+2n\pi}\right)^{\pi/2+2n\pi} \to e.$$

Because the value of

$$\left(1+\frac{\sin x}{x}\right)^x$$

does not approach a single number as $x\to\infty$, the indicated limit $\boxed{\text{does not exist}}$.

(e) Because

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to\infty} -\frac{1}{\ln x} = 0,$$

the expression $(e^x)^{-1/\ln x}$ is an indeterminate form at ∞ of the type ∞^0. Let $y = (e^x)^{-1/\ln x}$. Then

$$\ln y = \ln(e^x)^{-1/\ln x} = -\frac{1}{\ln x}\ln e^x = -\frac{x}{\ln x},$$

which is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} -\frac{x}{\ln x} = \lim_{x\to\infty} -\frac{\frac{d}{dx}x}{\frac{d}{dx}\ln x} = \lim_{x\to\infty} -\frac{1}{\frac{1}{x}} = \lim_{x\to\infty} -x = -\infty.$$

Finally, because $\lim\limits_{x\to\infty} \ln y = -\infty$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty} (e^x)^{1-/\ln x} = \lim_{x\to\infty} e^{\ln y} = \boxed{0}.$$

(f) Because

$$\left[\left(\frac{1}{a}\right)^x\right]^{-1/x} = \left(\frac{1}{a}\right)^{-1} = a$$

for all x,

$$\lim_{x\to\infty} \left[\left(\frac{1}{a}\right)^x\right]^{-1/x} = \boxed{a}.$$

(g) Because

$$\lim_{x\to\infty} x = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} = 0,$$

the expression $x^{1/x}$ is an indeterminate form at ∞ of the type ∞^0. Let $y = x^{1/x}$. Then

$$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x = \frac{\ln x}{x},$$

which is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Finally, because $\lim\limits_{x\to\infty} \ln y = 0$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}.$$

(h) Because

$$(a^x)^{1/x} = a$$

for all x,

$$\lim_{x\to\infty} (a^x)^{1/x} = \boxed{a}.$$

(i) Because

$$[(2+\sin x)^x]^{1/x} = (2+\sin x)$$

for all x,

$$\lim_{x\to\infty} [(2+\sin x)^x]^{1/x} = \lim_{x\to\infty} (2+\sin x)$$

which $\boxed{\text{does not exist}}$ because $\sin x$ never approaches a single number as $x \to \infty$.

(j) For $x > 0$,

$$x^{-1/\ln x} = \left(e^{\ln x}\right)^{-1/\ln x} = e^{-1},$$

so

$$\lim_{x\to 0^+} x^{-1/\ln x} = \lim_{x\to 0^+} e^{-1} = \boxed{e^{-1}}.$$

111. Let f be a function whose derivatives of all orders exist.

(a) Because

$$\lim_{h\to 0}[f(x+2h) - 2f(x+h) + f(x)] = 0 \quad \text{and} \quad \lim_{h\to 0} h^2 = 0,$$

the expression $\dfrac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$ is an indeterminate form at $h = 0$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} &= \lim_{h\to 0} \frac{\frac{d}{dh}[f(x+2h) - 2f(x+h) + f(x)]}{\frac{d}{dh}h^2} \\ &= \lim_{h\to 0} \frac{2f'(x+2h) - 2f'(x+h)}{2h} \\ &= \lim_{h\to 0} \frac{f'(x+2h) - f'(x+h)}{h}.\end{aligned}$$

Now,

$$\lim_{h\to 0}[f'(x+2h) - f'(x+h)] = 0 \quad \text{and} \quad \lim_{h\to 0} h = 0,$$

so the expression $\dfrac{f'(x+2h) - f'(x+h)}{h}$ is an indeterminate form at $h = 0$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} &= \lim_{h\to 0} \frac{f'(x+2h) - f'(x+h)}{h} \\ &= \lim_{h\to 0} \frac{\frac{d}{dh}[f'(x+2h) - f'(x+h)]}{\frac{d}{dh}h} \\ &= \lim_{h\to 0} \frac{2f''(x+2h) - f''(x+h)}{1} = \boxed{f''(x)}.\end{aligned}$$

(b) Because

$$\lim_{h\to 0}[f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)] = 0 \quad \text{and} \quad \lim_{h\to 0} h^3 = 0,$$

the expression $\dfrac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}$ is an indeterminate form at $h = 0$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}&\lim_{h\to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} \\ &\quad = \lim_{h\to 0} \frac{\frac{d}{dh}[f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)]}{\frac{d}{dh}h^3} \\ &\quad = \lim_{h\to 0} \frac{3f'(x+3h) - 6f'(x+2h) + 3f'(x+h)}{3h^2} \\ &\quad = \lim_{h\to 0} \frac{f'(x+3h) - 2f'(x+2h) + f'(x+h)}{h^2}.\end{aligned}$$

Now,

$$\lim_{h\to 0}[f'(x+3h)-2f'(x+2h)+f'(x+h)]=0 \quad \text{and} \quad \lim_{h\to 0} h^2=0,$$

so the expression $\dfrac{f'(x+3h)-2f'(x+2h)+f'(x+h)}{h^2}$ is an indeterminate form at $h=0$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{h^3}\\
&\qquad= \lim_{h\to 0}\frac{f'(x+3h)-2f'(x+2h)+f'(x+h)}{h^2}\\
&\qquad= \lim_{h\to 0}\frac{\frac{d}{dh}[f'(x+3h)-2f'(x+2h)+f'(x+h)]}{\frac{d}{dh}h^2}\\
&\qquad= \lim_{h\to 0}\frac{3f''(x+3h)-4f''(x+2h)+f''(x+h)}{2h}.
\end{aligned}$$

Finally,

$$\lim_{h\to 0}[3f''(x+3h)-4f''(x+2h)+f''(x+h)]=0 \quad \text{and} \quad \lim_{h\to 0}(2h)=0,$$

so the expression $\dfrac{3f''(x+3h)-4f''(x+2h)+f''(x+h)}{2h}$ is an indeterminate form at $h=0$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule a third time

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{h^3}\\
&\qquad= \lim_{h\to 0}\frac{3f''(x+3h)-4f''(x+2h)+f''(x+h)}{2h}\\
&\qquad= \lim_{h\to 0}\frac{\frac{d}{dh}[3f''(x+3h)-4f''(x+2h)+f''(x+h)]}{\frac{d}{dh}(2h)}\\
&\qquad= \lim_{h\to 0}\frac{9f'''(x+3h)-8f'''(x+2h)+f'''(x+h)}{2}\\
&\qquad= \frac{2f'''(x)}{2} = \boxed{f'''(x)}.
\end{aligned}$$

(c) Recognize that in the numerator of each limit, the coefficients on the values of f (1, -2, 1 in part (a) and 1, -3, 3, -1 in part (b)) are the coefficients in $(x-1)^2$ and $(x-1)^3$, respectively; that is, they are binomial coefficients. Therefore, for a positive integer n,

$$\boxed{\lim_{h\to 0}\frac{\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}f(x+(n-k)h)}{h^n}=f^{(n)}(x)}.$$

113. Let $f(t,x)=\dfrac{x^{t+1}-1}{t+1}$, where $x>0$ and $t\neq -1$.

(a) Let $x=x_0$. Because

$$\lim_{t\to -1}(x_0^{t+1}-1)=0 \quad \text{and} \quad \lim_{t\to -1}(t+1)=0,$$

the expression $f(t,x_0) = \dfrac{x_0^{t+1}-1}{t+1}$ is an indeterminate form at $t=-1$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{t\to -1} f(t,x_0) &= \lim_{t\to -1}\frac{x_0^{t+1}-1}{t+1} = \lim_{t\to -1}\frac{\frac{d}{dt}(x_0^{t+1}-1)}{\frac{d}{dt}(t+1)}\\ &= \lim_{t\to -1}\frac{x_0^{t+1}\ln x_0}{1} = \ln x_0.\end{aligned}$$

(b) The function $f(t,x)$ is continuous for $t\neq -1$. Based on the result from part (a), define

$$\boxed{F(t,x) = \begin{cases} f(t,x), & t\neq -1\\ \ln x, & t=-1.\end{cases}}$$

The function $F(t,x)$ is then continuous for all t.

(c) Based on part (b), for t fixed but not equal to -1,

$$\frac{d}{dx}F(t,x) = \frac{d}{dx}f(t,x) = \frac{d}{dx}\left(\frac{x^{t+1}-1}{t+1}\right) = \frac{1}{t+1}\cdot(t+1)x^t = x^t.$$

For $t=-1$,

$$\frac{d}{dx}F(t,x) = \frac{d}{dx}\ln x = \frac{1}{x} = x^{-1} = x^t.$$

AP® Practice Problems

1. $\displaystyle\lim_{x\to 0}\frac{e^{4x}-1}{\sin(2x)}$

Since $\lim_{x\to 0}\left(e^{4x}-1\right)=0$ and $\lim_{x\to 0}[\sin(2x)]=0$, $\lim_{x\to 0}\frac{e^{4x}-1}{\sin(2x)}$ is an indeterminate form at 0 of the type $\frac{0}{0}$.

Using L'Hôpital's Rule, $\lim_{x\to 0}\frac{e^{4x}-1}{\sin(2x)} = \lim_{x\to 0}\frac{4e^{4x}}{2\cos(2x)} = \dfrac{4}{2} = \boxed{2}$.

CHOICE B

3. $\displaystyle\lim_{x\to 0} x\csc x = \lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{1}{\left(\frac{x}{\sin x}\right)} = \frac{\lim(1)}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = \boxed{1}$.

Alternatively, for $\lim_{x\to 0}\frac{x}{\sin x}$, $\lim_{x\to 0}(x)=0$ and $\lim_{x\to 0}(\sin x)=0$, so $\lim_{x\to 0}\frac{x}{\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$ and L'Hôpital's Rule is applicable.

Therefore, using L'Hôpital's Rule, $\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{1}{\cos x} = \dfrac{1}{1} = \boxed{1}$.

CHOICE B

5. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^k}$

Since $\lim_{x\to\infty}(\ln x)=\infty$ and $\lim_{x\to\infty}\left(x^k\right)=\infty$, $\lim_{x\to\infty}\frac{\ln x}{x^k}$ is an indeterminate form of the type $\frac{\infty}{\infty}$ and L'Hôpital's Rule is applicable.

Therefore, $\lim\limits_{x\to\infty} \frac{\ln x}{x^k} = \lim\limits_{x\to\infty} \frac{1/x}{kx^{k-1}} = \lim\limits_{x\to\infty} \frac{1}{kx^k} = \boxed{0}$.

CHOICE A

7. $\lim\limits_{x\to 0^+} (1-4x)^{\cot x}$

$$\begin{aligned}
&\text{Let } y = (1-4x)^{\cot x}. \text{ Then}\\
\ln y &= \ln (1-4x)^{\cot x}\\
&= \cot x \ln (1-4x)\\
&= \frac{\ln (1-4x)}{\tan x}\\
\lim_{x\to 0^+} (\ln y) &= \lim_{x\to 0^+} \frac{\ln (1-4x)}{\tan x}.
\end{aligned}$$

Since $\lim\limits_{x\to 0^+} [\ln (1-4x)] = \ln [1-4(0)] = 0$ and $\lim\limits_{x\to 0^+} (\tan x) = 0$, $\lim\limits_{x\to 0^+} \left(\frac{\ln (1-4x)}{\tan x}\right)$ is an indeterminate form at 0^+ of the type $\frac{0}{0}$ and L'Hôpital's Rule is applicable.

Therefore, $\lim\limits_{x\to 0^+} (\ln y) = \lim\limits_{x\to 0^+} \frac{\ln (1-4x)}{\tan x} = \lim\limits_{x\to 0^+} \frac{\frac{-4}{1-4x}}{\sec^2 x} = \lim\limits_{x\to 0^+} \frac{4\cos^2 x}{4x-1} = -4$.

Since $\lim\limits_{x\to 0^+} (\ln y) = -4$, we can conclude that $\lim\limits_{x\to 0^+} y = e^{-4}$, or $\lim\limits_{x\to 0^+} (1-4x)^{\cot x} = e^{-4} = \boxed{\frac{1}{e^4}}$.

CHOICE D

4.6 Using Calculus to Graph Functions

Skill Building

1. Let $f(x) = x^4 - 6x^2 + 10$.

Step 1 The polynomial function f has a domain of $\boxed{\text{all real numbers}}$. $f(0) = 10$, so the $\boxed{y\text{-intercept is } 10}$. To find the x-intercepts, solve the equation $f(x) = 0$. Because

$$x^4 - 6x^2 + 10 = (x^2-3)^2 + 1,$$

it follows that there are no real solutions to the equation $f(x) = 0$, so the graph of f has $\boxed{\text{no } x\text{-intercepts}}$.

Step 2 The $\boxed{\text{graphs of polynomial functions do not have asymptotes}}$, but the end behavior of the graph of f will resemble the power function $y = x^4$.

Step 3 Now

$$\begin{aligned}
f'(x) &= 4x^3 - 12x = 4x(x^2-3); \text{ and}\\
f''(x) &= 12x^2 - 12 = 12(x-1)(x+1).
\end{aligned}$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so $\boxed{0 \text{ and } \pm\sqrt{3}}$ are the critical numbers. At the points $(-\sqrt{3}, 1)$, $(0, 10)$, and $(\sqrt{3}, 1)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers 0 and $\pm\sqrt{3}$ to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\sqrt{3})$	$-$	f is decreasing on $(-\infty, -\sqrt{3})$
$(-\sqrt{3}, 0)$	$+$	f is increasing on $(-\sqrt{3}, 0)$
$(0, \sqrt{3})$	$-$	f is decreasing on $(0, \sqrt{3})$
$(\sqrt{3}, \infty)$	$+$	f is increasing on $(\sqrt{3}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\sqrt{3}$, a local maximum value at 0, and a local minimum value at $\sqrt{3}$. The

$$\boxed{\text{local minimum values are } f(-\sqrt{3}) = 1 \text{ and } f(\sqrt{3}) = 1},$$

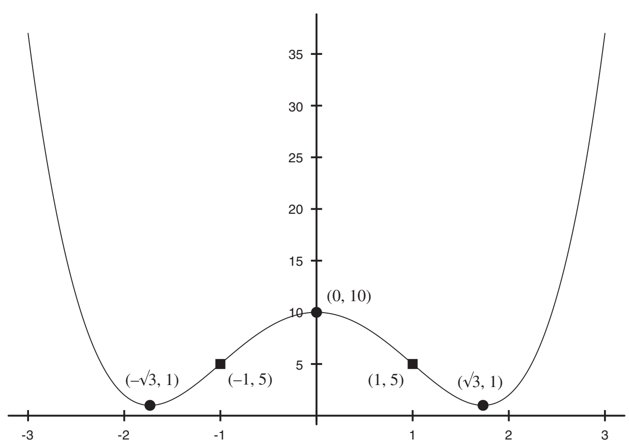
while the $\boxed{\text{local maximum value is } f(0) = 10}$.

Step 6 The second derivative is equal to zero at $x = \pm 1$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	$+$	f is concave up on $(-\infty, -1)$
$(-1, 1)$	$-$	f is concave down on $(-1, 1)$
$(1, \infty)$	$+$	f is concave up on $(1, \infty)$

The concavity of f changes at ± 1, so the points $\boxed{(-1, 5) \text{ and } (1, 5)}$ are points of inflection of f.

Step 7 The figure below displays the graph of f. Local extreme values are highlighted by closed circles, and points of inflection are highlighted by closed squares.



3. Let $f(x) = x^5 - 10x^2$

Step 1 The polynomial function f has a domain of $\boxed{\text{all real numbers}}$. $f(0) = 0$, so the $\boxed{y\text{-intercept is } 0}$. To find the x-intercepts, solve the equation $f(x) = 0$. Because

$$x^5 - 10x^2 = x^2\left(x^3 - 10\right)$$

it follows that the real solutions to the equation $f(x) = 0$ are $\boxed{x = 0 \text{ and } x = \sqrt[3]{10}}$, so those are the x-intercepts.

Step 2 The graphs of polynomial functions do not have asymptotes, but the end behavior of the graph of f will resemble the power function $y = x^5$.

Step 3 Now

$$f'(x) = 5x^4 - 20x = 5x\left(x^3 - 4\right)$$
$$f''(x) = 20x^3 - 20$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so 0 and $\sqrt[3]{4}$ are the critical numbers. At the points $(0, 0)$ and $\left(\sqrt[3]{4}, f\left(\sqrt[3]{4}\right)\right) = \left(\sqrt[3]{4}, -12\sqrt[3]{2}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers 0 and $\sqrt[3]{4}$ to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	$+$	f is increasing on $(-\infty, 0]$
$\left(0, \sqrt[3]{4}\right)$	$-$	f is decreasing on $\left[0, \sqrt[3]{4}\right]$
$\left(\sqrt[3]{4}, \infty\right)$	$+$	f is increasing on $\left[\sqrt[3]{4}, \infty\right)$

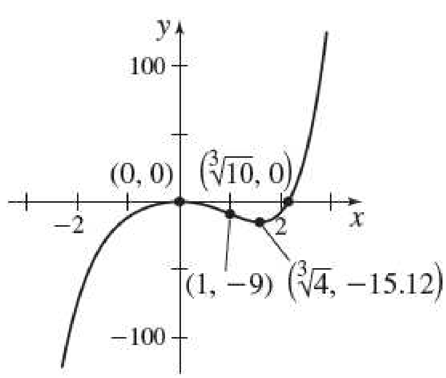
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0 and a local minimum value at $\sqrt[3]{4}$. The local maximum value is $(0, 0)$ and the local minimum value is $\left(\sqrt[3]{4}, -12\sqrt[3]{2}\right)$.

Step 6 The second derivative is equal to zero at $x = 1$. Use this number to divide the number line into two intervals, and determine the sign of $f''(x)$ on each interval.

Interval	Sign of f'	Conclusion
$(-\infty, 1)$	$-$	f is concave down on $(-\infty, 1)$
$(1, \infty)$	$+$	f is concave up on $(1, \infty)$

The concavity of f changes at 1, so the point $(1, -9)$ is a point of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme values are highlighted by closed circles and the point of inflection is highlighted by a closed square.



5. Let $f(x) = 3x^5 + 5x^4$

Step 1 The polynomial function f has a domain of $\boxed{\text{all real numbers}}$. $f(0) = 0$, so the $\boxed{y\text{-intercept is } 0}$. To find the x-intercepts, solve the equation $f(x) = 0$. Because

$$3x^5 + 5x^4 = x^4(3x + 5)$$

it follows that the real solutions to the equation $f(x) = 0$ are $\boxed{x = 0 \text{ and } x = -\frac{5}{3}}$, so those are the x-intercepts.

Step 2 The $\boxed{\text{graphs of polynomial functions do not have asymptotes}}$, but the end behavior of the graph of f will resemble the power function $y = x^5$.

Step 3 Now

$$f'(x) = 15x^4 + 20x^3 = 5x^3(3x + 4)$$
$$f''(x) = 60x^3 + 60x^2 = 60x^2(x + 1)$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so $\boxed{-\frac{4}{3} \text{ and } 0}$ are the critical numbers. At the points $\left(-\frac{4}{3}, \frac{256}{81}\right)$ and $(0, 0)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers 0 and $-\frac{4}{3}$ to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$\left(-\infty, -\frac{4}{3}\right)$	$+$	f is increasing on $\left(-\infty, -\frac{4}{3}\right]$
$\left(0, -\frac{4}{3}\right)$	$-$	f is decreasing on $\left[-\frac{4}{3}, 0\right]$
$(0, \infty)$	$+$	f is increasing on $[0, \infty)$

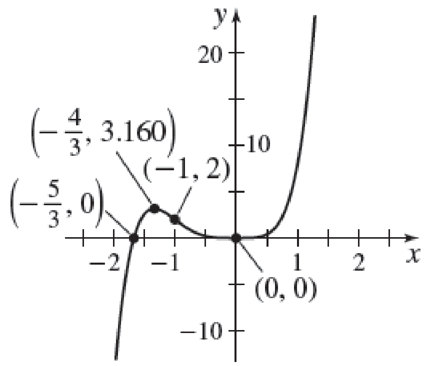
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\boxed{-\frac{4}{3}}$ and a local minimum at $\boxed{0}$. The $\boxed{\text{local maximum value is } \left(-\frac{4}{3}, \frac{256}{81}\right)}$ and the $\boxed{\text{local minimum value is } (0, 0)}$.

Step 6 The second derivative is equal to zero at $x = -1$ and $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of $f''(x)$ on each interval.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is concave down on $(-\infty, -1)$
$(-1, 0)$	$+$	f is concave up on $(-1, 0)$
$(0, \infty)$	$+$	f is concave up on $(0, \infty)$

The concavity of f changes at -1, so the point $(-1, 2)$ is a point of inflection of f. The concavity does not change at 0, so the point $(0, 0)$ is not a point of inflection.

Step 7 The figure below displays the graph of f. The local extreme values are highlighted by closed circles and the point of inflection is highlighted by a closed square.



7. Let $f(x) = \dfrac{2}{x^2 - 4}$.

Step 1 The domain of the rational function f is the set $\boxed{\{x \mid x \neq \pm 2\}}$. There are $\boxed{\text{no } x\text{-intercepts}}$, and the $\boxed{y\text{-intercept is } f(0) = -\frac{1}{2}}$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \to \pm\infty} \frac{2}{x^2 - 4} = \lim_{x \to \pm\infty} \frac{2}{x^2 - 4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \pm\infty} \frac{\frac{2}{x^2}}{1 - \frac{4}{x^2}} = 0,$$

the line $\boxed{y = 0 \text{ is a horizontal asymptote}}$. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f. As

$$\lim_{x \to -2^-} \frac{2}{x^2 - 4} = \infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{2}{x^2 - 4} = -\infty,$$

and

$$\lim_{x \to 2^-} \frac{2}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{2}{x^2 - 4} = \infty,$$

the lines $\boxed{x = \pm 2 \text{ are vertical asymptotes}}$.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} 2(x^2 - 4)^{-1} = -2(x^2 - 4)^{-2} \cdot 2x = -\frac{4x}{(x^2 - 4)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx}\left[-\frac{4x}{(x^2 - 4)^2}\right] = -\frac{(x^2 - 4)^2 \cdot 4 - 4x \cdot 2(x^2 - 4)(2x)}{(x^2 - 4)^4} \\ &= -\frac{4(x^2 - 4) - 16x^2}{(x^2 - 4)^3} = \frac{12x^2 + 16}{(x^2 - 4)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 0$ and does not exist when $x = \pm 2$. However, ± 2 are not in the domain of f, so ± 2 are not critical numbers. Therefore, $\boxed{0}$ is the only critical number of f. The tangent line at the point $\left(0, -\frac{1}{2}\right)$ is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and ± 2 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	$+$	f is increasing on $(-\infty, -2)$
$(-2, 0)$	$+$	f is increasing on $(-2, 0)$
$(0, 2)$	$-$	f is decreasing on $(0, 2)$
$(2, \infty)$	$-$	f is decreasing on $(2, \infty)$

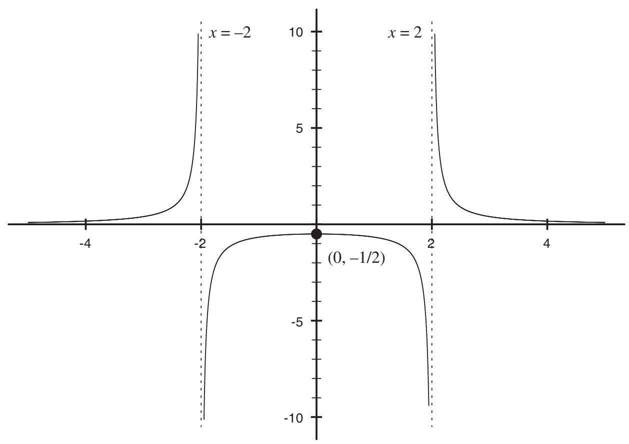
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The $\boxed{\text{local maximum value is } f(0) = -\frac{1}{2}}$.

Step 6 The second derivative is never equal to zero and does not exist when $x = \pm 2$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	$+$	f is concave up on $(-\infty, -2)$
$(-2, 2)$	$-$	f is concave down on $(-2, 2)$
$(2, \infty)$	$+$	f is concave up on $(2, \infty)$

Although the concavity of f changes at ± 2, there is $\boxed{\text{no point of inflection}}$ at ± 2 because ± 2 are not in the domain of f.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle.



9. Let $f(x) = \dfrac{2x-1}{x+1}$.

Step 1 The domain of the rational function f is the set $\boxed{\{x | x \neq -1\}}$. The $\boxed{x\text{-intercept is } \frac{1}{2}}$, and the $\boxed{y\text{-intercept is } f(0) = -1}$.

Step 2 The degree of the numerator is equal to the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \to \pm\infty} \frac{2x-1}{x+1} = \lim_{x \to \pm\infty} \frac{2x-1}{x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \pm\infty} \frac{2 - \frac{1}{x}}{1 + \frac{1}{x}} = 2,$$

the line $\boxed{y = 2 \text{ is a horizontal asymptote}}$. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f. As

$$\lim_{x \to -1^-} \frac{2x-1}{x+1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{2x-1}{x+1} = -\infty,$$

the line $\boxed{x = -1 \text{ is a vertical asymptote}}$.

Step 3 Now

$$f'(x) = \frac{d}{dx}\left(\frac{2x-1}{x+1}\right) = \frac{(x+1)\cdot 2-(2x-1)\cdot 1}{(x+1)^2} = \frac{3}{(x+1)^2}; \text{ and}$$
$$f''(x) = \frac{d}{dx}[3(x+1)^{-2}] = -6(x+1)^{-3} = -\frac{6}{(x+1)^3}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = -1$. However, -1 is not in the domain of f, so -1 is not a critical number. Therefore, f has **no critical numbers**.

Step 4 To apply the Increasing/Decreasing Function Test, use the number -1 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$+$	f is increasing on $(-\infty, -1)$
$(-1, \infty)$	$+$	f is increasing on $(-1, \infty)$

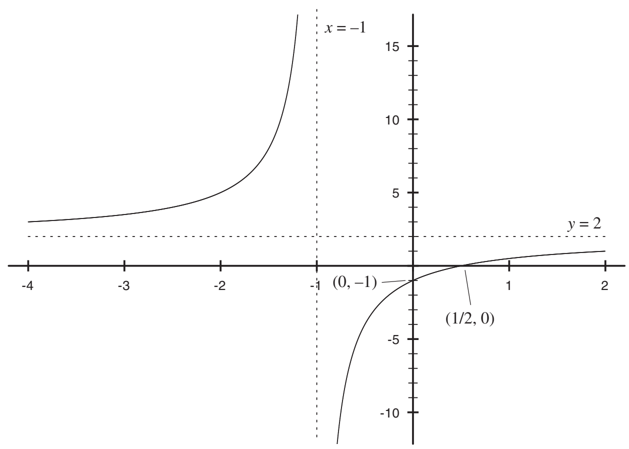
Step 5 Because there are no critical numbers, f has **no local extreme values**.

Step 6 The second derivative is never equal to zero and does not exist when $x = -1$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	$+$	f is concave up on $(-\infty, -1)$
$(-1, \infty)$	$-$	f is concave down on $(-1, \infty)$

Although the concavity of f changes at -1, there is **no point of inflection** at -1 because -1 is not in the domain of f.

Step 7 The figure below displays the graph of f.



11. Let $f(x) = \dfrac{x}{x^2+1}$.

Step 1 The domain of the rational function f is the set of **all real numbers** (because $1+x^2$ is never equal to zero for any real x). The **x-intercept is 0**, and the **y-intercept is $f(0) = 0$**.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x\to\pm\infty} \frac{x}{x^2+1} = \lim_{x\to\pm\infty} \frac{x}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\pm\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = 0,$$

the line $\boxed{y = 0 \text{ is a horizontal asymptote}}$. Because f is defined for all real numbers, there are $\boxed{\text{no vertical asymptotes}}$.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx}\left(\frac{1-x^2}{(x^2+1)^2}\right) = \frac{(x^2+1)^2\cdot(-2x) - (1-x^2)\cdot 2(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3} = \frac{2x^3-6x}{(x^2+1)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ exists everywhere and is equal to 0 when $x = \pm 1$. Therefore, $\boxed{\pm 1}$ are the critical numbers of f. At the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers ± 1 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is decreasing on $(-\infty, -1)$
$(-1, 1)$	$+$	f is increasing on $(-1, 1)$
$(1, \infty)$	$-$	f is decreasing on $(0, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -1 and a local maximum value at 1. The $\boxed{\text{local minimum value is } f(-1) = -\frac{1}{2}}$, and the $\boxed{\text{local maximum value is } f(1) = \frac{1}{2}}$.

Step 6 The second derivative exists everywhere and is equal to 0 when $x = 0$ and when $x = \pm\sqrt{3}$. Use these numbers to divide the number line into four intervals, and determine the sign of f'' on each interval.

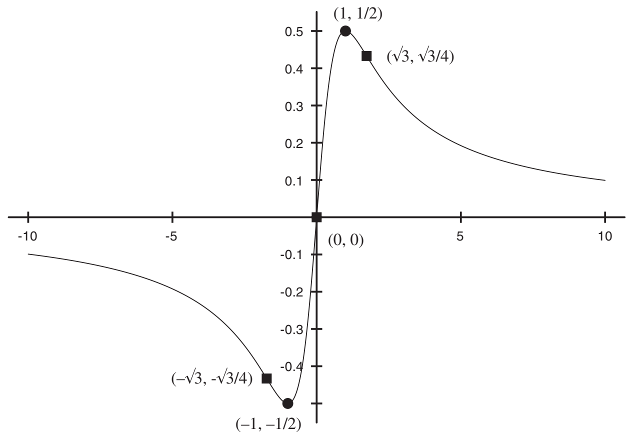
Interval	Sign of f''	Conclusion
$(-\infty, -\sqrt{3})$	$-$	f is concave down on $(-\infty, -\sqrt{3})$
$(-\sqrt{3}, 0)$	$+$	f is concave up on $(-\sqrt{3}, 0)$
$(0, \sqrt{3})$	$-$	f is concave down on $(0, \sqrt{3})$
$(\sqrt{3}, \infty)$	$+$	f is concave up on $(\sqrt{3}, \infty)$

The concavity of f changes at $-\sqrt{3}$, 0, and $\sqrt{3}$, so the points

$$\boxed{\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0,0), \quad \text{and} \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)}$$

are points of inflection of f.

Step 7 The figure below displays the graph of f. Local extreme values are highlighted by closed circles, and points of inflection are highlighted by closed squares.



13. Let $f(x) = \dfrac{x^2+1}{2x}$.

Step 1 The domain of the rational function f is the set $\boxed{\{x|x \neq 0\}}$. There are $\boxed{\text{no } x\text{-intercepts}}$ because x^2+1 is never equal to zero for any real x, and there is $\boxed{\text{no } y\text{-intercept}}$ because f is not defined at 0.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{x^2+1}{2x} = \frac{1}{2}x + \frac{1}{2x},$$

it follows that

$$\lim_{x\to\infty}\left[f(x) - \frac{1}{2}x\right] = \lim_{x\to\infty}\frac{1}{2x} = 0$$

so the line $\boxed{y = \frac{1}{2}x \text{ is an oblique asymptote}}$. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f. As

$$\lim_{x\to 0^-}\frac{x^2+1}{2x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+}\frac{x^2+1}{2x} = \infty,$$

the line $\boxed{x = 0 \text{ is a vertical asymptote}}$.

Step 3 Now

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{2x}\right) = \frac{2x\cdot 2x - (x^2+1)\cdot 2}{4x^2} = \frac{2x^2-2}{4x^2} = \frac{(x-1)(x+1)}{2x^2}; \text{ and}$$

$$f''(x) = \frac{d}{dx}\left[\frac{x^2-1}{2x^2}\right] = \frac{2x^2\cdot 2x - (x^2-1)\cdot 4x}{4x^4} = \frac{4x}{4x^4} = \frac{1}{x^3}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \pm 1$ and does not exist when $x = 0$. However, 0 is not in the domain of f, so 0 is not a critical number. Therefore, $\boxed{\pm 1}$ are the only critical numbers of f. At the points $(-1,-1)$ and $(1,1)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and ± 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty,-1)$	$+$	f is increasing on $(-\infty,-1)$
$(-1,0)$	$-$	f is decreasing on $(-1,0)$
$(0,1)$	$-$	f is decreasing on $(0,1)$
$(1,\infty)$	$+$	f is increasing on $(1,\infty)$

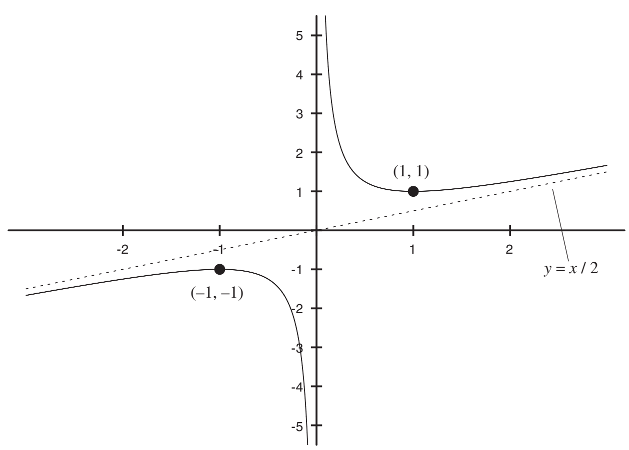
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at -1 and a local minimum value at 1. The $\boxed{\text{local maximum value is } f(-1)=-1}$, and the $\boxed{\text{local minimum value is } f(1)=1}$.

Step 6 The second derivative is never equal to zero and does not exist when $x=0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty,0)$	$-$	f is concave down on $(-\infty,0)$
$(0,\infty)$	$+$	f is concave up on $(0,\infty)$

Although the concavity of f changes at 0, there is $\boxed{\text{no point of inflection}}$ at 0 because 0 is not in the domain of f.

Step 7 The figure below displays the graph of f. The local extreme values are highlighted by a closed circles.



15. Let $f(x)=\frac{x^2-1}{x^2+2x+3}$

Step 1 The function f has a domain of $\boxed{\text{all real numbers}}$ (the quadratic formula shows that the denominator has no real zeros). $f(0)=-\frac{1}{3}$, so the $\boxed{y\text{-intercept is } -\frac{1}{3}}$. To find the x-intercepts, solve the equation $f(x)=0$. Because the denominator is always positive, $f(x)=0$ whenever the numerator is 0, that is, at $\boxed{x=\pm 1}$, so those are the x-intercepts.

Step 2 The degree of the numerator is the same as the degree of the denominator, so f has a horizontal asymptote. Because

$$\lim_{x\to\pm\infty}\frac{x^2-1}{x^2+2x+3}=\lim_{x\to\pm\infty}\frac{x^2-1}{x^2+2x+3}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\pm\infty}\frac{1-\frac{1}{x^2}}{1+\frac{2}{x}+\frac{e}{x^2}}=1$$

$\boxed{\text{the horizontal asymptote is } y=1}$. Because the domain is all real numbers, there is $\boxed{\text{no vertical asymptote}}$.

Step 3 Now

$$f'(x) = \frac{2\left(x^2+4x+1\right)}{\left(x^2+2x+3\right)^2}$$
$$f''(x) = -\frac{4\left(x^3+6x^2+3x-4\right)}{\left(x^2+2x+3\right)^3}$$

The critical numbers of the polynomial function f occur where $f'(x)=0$. $\frac{2\left(x^2+4x+1\right)}{\left(x^2+2x+3\right)^2}=0$ when $\left(x^2+4x+1\right)=0$, since the denominator is always positive. Using the quadratic formula, the critical points are $\boxed{-2\pm\sqrt{3}}$. At the points $\left(-2-\sqrt{3}, f\left(-2-\sqrt{3}\right)\right) = \left(-2-\sqrt{3}, \frac{1+\sqrt{3}}{2}\right)$ and $\left(-2+\sqrt{3}, f\left(-2+\sqrt{3}\right)\right) = \left(-2-\sqrt{3}, \frac{1-\sqrt{3}}{2}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers $-2\pm\sqrt{3}$ to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$\left(-\infty, -2-\sqrt{3}\right)$	$+$	f is increasing on $\left(-\infty, -2-\sqrt{3}\right]$
$\left(-2-\sqrt{3}, -2+\sqrt{3}\right)$	$-$	f is decreasing on $\left[-2-\sqrt{3}, -2+\sqrt{3}\right]$
$\left(-2+\sqrt{3}, \infty\right)$	$+$	f is increasing on $\left[-2+\sqrt{3}, \infty\right)$

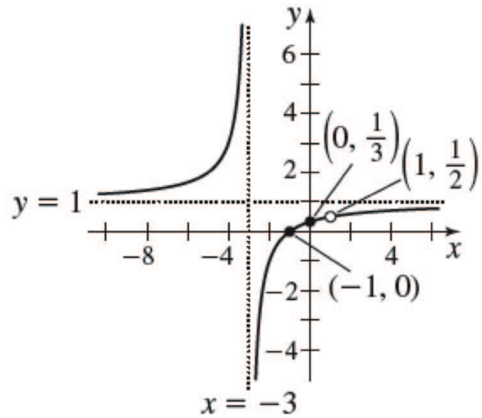
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\boxed{-2-\sqrt{3}}$ and a local minimum at $\boxed{-2+\sqrt{3}}$. The $\boxed{\text{local maximum value is } \left(-2-\sqrt{3}, \frac{1+\sqrt{3}}{2}\right)}$ and the $\boxed{\text{local minimum value is } \left(-2+\sqrt{3}, \frac{1-\sqrt{3}}{2}\right)}$.

Step 6 The second derivative is equal to zero when $x^3+6x^2+3x-4=0$, since the denominator is always positive. Using technology, this happens when $x \approx -5.290$, -1.294, or 0.584. Use these numbers to divide the number line into three intervals, and determine the sign of $f''(x)$ on each interval.

Interval	Sign of f'	Conclusion
$(-\infty, -5.290)$	$+$	f is concave up on $(-\infty, -5.290]$
$(-5.290, -1.294)$	$-$	f is concave down on $[-5.290, -1.294]$
$(-1.294, 0.584)$	$+$	f is concave up on $[-1.294, 0.584]$
$(0.584, \infty)$	$-$	f is concave down on $[0.584, \infty)$

The concavity of f changes at each of these points, so each point is a point of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme values are highlighted by closed circles and the point of inflection is highlighted by a closed square.



17. Let $f(x) = \frac{x(x^3+1)}{(x^2-4)(x+1)}$

Step 1 The function f has a domain of $\boxed{\{x\,|x \neq -1, x \neq \pm 2\}}$. Next, $f(0) = 0$, so the $\boxed{y\text{-intercept is } 0}$. To find the $x$intercepts, solve the equation $f(x) = 0$. This happens when the numerator equals 0 and the denominator does not equal 0. The numerator equals 0 only for $x = 0$, and the denominator does not equal 0 there. Therefore, the only x-intercept is $\boxed{x = 0}$.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but it does have an oblique asymptote. If $x \neq 1$, then

$$f(x) = \frac{x(x^3+1)}{(x^2-4)(x+1)} = \frac{x(x+1)(x^2-x+1)}{(x^2-4)(x+1)} = \frac{x(x^2-x+1)}{x^2-4} = \frac{x^3-x^2+x}{x^2-4}$$
$$= x - 1 + \frac{5x-4}{x^2-4}$$

which behaves at the endpoints like $y = x - 1$, so $\boxed{\text{the oblique asymptote is } y = x - 1}$. Also,

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\left(x - 1 + \frac{5x-4}{x^2-4}\right) = x - 1 + \lim_{x\to -2^-}\frac{5x-4}{x^2-4} = -\infty$$
$$\text{and } \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\left(x - 1 + \frac{5x-4}{x^2-4}\right) = x - 1 + \lim_{x\to -2^+}\frac{5x-4}{x^2-4} = +\infty$$

so $\boxed{x = -2 \text{ is a vertical asymptote}}$; and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(x - 1 + \frac{5x-4}{x^2-4}\right) = x - 1 + \lim_{x\to 2^-}\frac{5x-4}{x^2-4} = -\infty$$
$$\text{and } \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(x - 1 + \frac{5x-4}{x^2-4}\right) = x - 1 + \lim_{x\to 2^+}\frac{5x-4}{x^2-4} == \infty$$

so $\boxed{x = 2 \text{ is a vertical asymptote}}$; but

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(x - 1 + \frac{5x-4}{x^2-4}\right) = (-1) - 1 + \frac{5(-1)-4}{(1)^2-4} = -2 + \frac{-9}{-3} = 1$$

so $\left(1, \lim_{x\to -1} f(x)\right) = \boxed{(-1, 1) \text{ is a hole}}$.

Step 3 Now

$$f'(x) = \frac{x^4 - 13x^2 + 8x - 4}{(x^2 - 4)^2}$$
$$f''(x) = -\frac{2\big(5x^3 - 12x^2 + 60x - 16\big)}{(x^2 - 4)^3}$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, that is, when $\frac{x^4-13x^2+8x-4}{(x^2-4)^2} = 0$. This happens when the numerator equals zero and the denominator does not equal 0. Using technology, the numerator equals 0 when $\boxed{x \approx -3.912 \text{ or } 3.309}$ (the denominator does not equal 0 there), so these are the critical points. At the points $\approx (-3.912, -6.700)$ and $\approx (3.309, 4.114)$, the tangent lines are horizontal. The other critical points are where $f'(x)$ is not defined, namely at $\boxed{x = \pm 2 \text{ or } -1}$.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers $-3.912, -2$, -1 , 2, and 3.309 to divide the number line into five intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -3.912)$	$+$	f is increasing on $(-\infty, -3.912]$
$(-3.912, -2)$	$-$	f is decreasing on $[-3.912, -2)$
$(-2, -1)$	$-$	f is decreasing on $(-2, -1)$
$(-1, 2)$	$-$	f is decreasing on $(-1, 2)$
$(2, 3.309)$	$-$	f is decreasing on $(2, 3.309]$
$(3.309, \infty)$	$+$	f is increasing on $[3.309, \infty)$

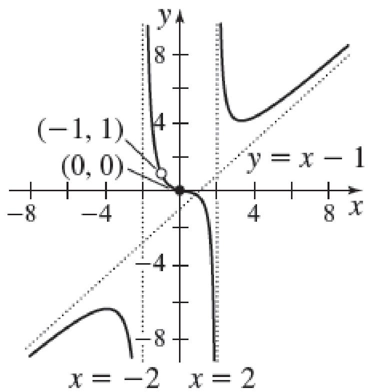
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\boxed{-3.912}$ and a local minimum at $\boxed{3.309}$. The $\boxed{\text{local maximum value is } (-3.912, -6.700)}$ and the $\boxed{\text{local minimum value is } (3.309, 4.114)}$.

Step 6 The second derivative is equal to zero when the numerator of $f''(x)$ equals 0 and the denominator does not equal 0. Using technology, this happens when $x \approx 0.281$. Use this number and the numbers where $f''(x)$ is not defined, that is, $x = \pm 2$ or -1, to divide the number line into five intervals, and determine the sign of $f''(x)$ on each interval.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	$-$	f is concave down on $(-\infty, -2)$
$(-2, 0.281)$	$+$	f is concave up on $(-2, 0.281]$
$(0.281, -1)$	$-$	f is concave down on $= [0.281, -1)$
$(-1, 2)$	$-$	f is concave down on $(-1, 2)$
$(2, \infty)$	$+$	f is concave up on $(2, \infty)$

The only point where f is continuous and the concavity of f changes is at 0.281, so $(0.281, f(0.281)) = \boxed{(0.281, -0.057) \text{ is the inflection point}}$.

Step 7 The figure below displays the graph of f.



19. Let $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2} = \frac{x^2+x+1}{x^2}$.

Step 1 The domain of the rational function f is the set $\boxed{\{x|x \neq 0\}}$. There are $\boxed{\text{no } x\text{-intercepts}}$ because $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ is never equal to zero for any real x, and there is $\boxed{\text{no } y\text{-intercept}}$ because f is not defined at 0.

Step 2 The degree of the numerator is equal to the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x\to\pm\infty}\left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = 1,$$

the line $\boxed{y = 1 \text{ is a horizontal asymptote}}$. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f. As

$$\lim_{x\to 0^-}\frac{x^2+x+1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^+}\frac{x^2+x+1}{x^2} = \infty,$$

the line $\boxed{x = 0 \text{ is a vertical asymptote}}$.

Step 3 Now

$$f'(x) = \frac{d}{dx}\left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = -\frac{1}{x^2} - \frac{2}{x^3} = -\frac{x+2}{x^3}; \text{ and}$$

$$f''(x) = \frac{d}{dx}\left(-\frac{1}{x^2} - \frac{2}{x^3}\right) = \frac{2}{x^3} + \frac{6}{x^4} = \frac{2x+6}{x^4}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -2$ and does not exist when $x = 0$. However, 0 is not in the domain of f, so 0 is not a critical number. Therefore, $\boxed{-2}$ is the only critical number of f. At the point $\left(-2, \frac{3}{4}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -2 and 0 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	$-$	f is decreasing on $(-\infty, -2)$
$(-2, 0)$	$+$	f is increasing on $(-2, 0)$
$(0, \infty)$	$-$	f is decreasing on $(0, \infty)$

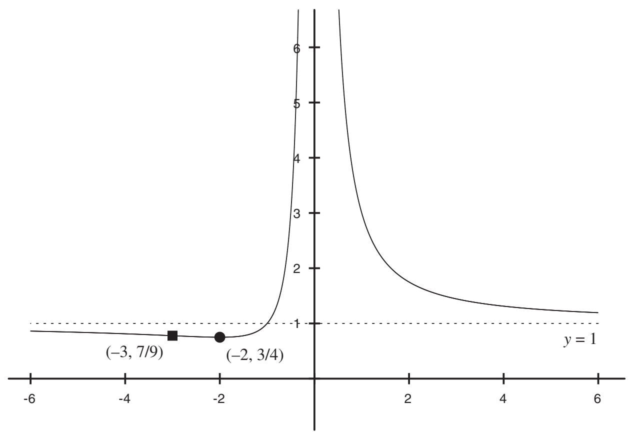
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -2. The $\boxed{\text{local minimum value is } f(-2) = \frac{3}{4}}$.

Step 6 The second derivative is equal to zero when $x = -3$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -3)$	$-$	f is concave down on $(-\infty, -3)$
$(-3, 0)$	$+$	f is concave up on $(-3, 0)$
$(0, \infty)$	$+$	f is concave up on $(0, \infty)$

The concavity of f changes at -3 so the point $\boxed{\left(-3, \frac{7}{9}\right)}$ is a point of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



21. Let $f(x) = \sqrt{3-x}$.

Step 1 The domain of f is given by the solution to the inequality $3 - x \geq 0$; that is, the set $\boxed{\{x|x \leq 3\}}$. The $\boxed{x\text{-intercept is } 3}$, and the $\boxed{y\text{-intercept is } f(0) = \sqrt{3}}$.

Step 2 Because
$$\lim_{x\to-\infty} \sqrt{3-x} = \infty,$$
the graph of f $\boxed{\text{does not have a horizontal asymptote}}$. The graph also has $\boxed{\text{no vertical asymptotes}}$.

Step 3 Now,
$$\begin{aligned} f'(x) &= -\frac{1}{2}(3-x)^{-1/2} = -\frac{1}{2\sqrt{3-x}}; \text{ and} \\ f''(x) &= -\frac{1}{4}(3-x)^{-3/2} = -\frac{1}{4\sqrt{(3-x)^3}}. \end{aligned}$$

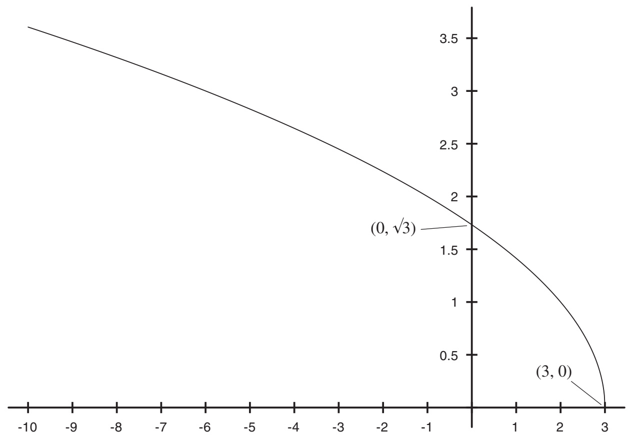
The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 3$. Therefore, $\boxed{3}$ is the only critical number of f. At the point $(3, 0)$, the tangent line is vertical.

Step 4 Because $f'(x) < 0$ for all $x < 3$, f is $\boxed{\text{decreasing on the interval } (-\infty, 3)}$.

Step 5 Because the only critical number of f is an endpoint of the domain of f, f has $\boxed{\text{no local extreme values}}$.

Step 6 Because $f''(x) < 0$ for all $x < 3$, f is $\boxed{\text{concave down on the interval } (-\infty, 3)}$. As the concavity of f never changes, f has $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f.



23. Let $f(x) = x + \sqrt{x}$.

Step 1 The domain of f is the set $\boxed{\{x|x \geq 0\}}$. The $\boxed{x\text{-intercept is } 0}$, and the $\boxed{y\text{-intercept is } f(0) = 0}$.

Step 2 Because

$$\lim_{x\to\infty} (x + \sqrt{x}) = \infty,$$

the graph of f $\boxed{\text{does not have a horizontal asymptote}}$. The graph also has $\boxed{\text{no vertical asymptotes}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= 1 + \frac{1}{2\sqrt{x}}; \text{ and} \\ f''(x) &= -\frac{1}{4x^{3/2}}. \end{aligned}$$

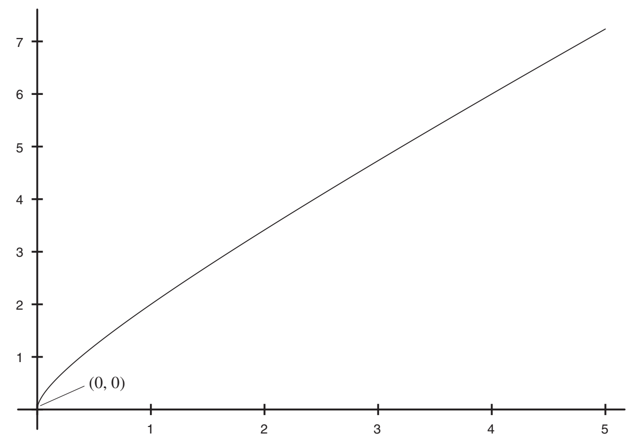
The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 0$. Therefore, $\boxed{0}$ is the only critical number of f. At the point $(0, 0)$, the tangent line is vertical.

Step 4 Because $f'(x) > 1 > 0$ for all $x > 0$, f is $\boxed{\text{increasing on the interval } (0, \infty)}$.

Step 5 Because the only critical number of f is an endpoint of the domain of f, f has $\boxed{\text{no local extreme values}}$.

Step 6 Because $f''(x) < 0$ for all $x > 0$, f is $\boxed{\text{concave down on the interval } (0, \infty)}$. As the concavity of f never changes, f has $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f.



25. Let $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Step 1 The domain of f is given by the solution to the inequality $x + 1 > 0$; that is, the set $\boxed{\{x|x > -1\}}$. The $\boxed{x\text{-intercept is } 0}$, and the $\boxed{y\text{-intercept is } f(0) = 0}$.

Step 2 Because

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \infty,$$

the graph of f $\boxed{\text{does not have a horizontal asymptote}}$. On the other hand,

$$\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty,$$

so the line $\boxed{x = -1 \text{ is a vertical asymptote}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2}{\sqrt{x+1}}\right) = \frac{\sqrt{x+1}\cdot 2x - x^2 \cdot \frac{1}{2}(x+1)^{-1/2}}{x+1} \\ &= \frac{4x(x+1) - x^2}{2(x+1)^{3/2}} = \frac{3x^2+4x}{2(x+1)^{3/2}}; \text{ and} \\ f''(x) &= \frac{d}{dx}\left(\frac{3x^2+4x}{2(x+1)^{3/2}}\right) = \frac{2(x+1)^{3/2}\cdot(6x+4) - (3x^2+4x)\cdot 3(x+1)^{1/2}}{4(x+1)^3} \\ &= \frac{2(x+1)(6x+4) - 3(3x^2+4x)}{4(x+1)^{5/2}} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 0$ and when $x = -\dfrac{4}{3}$ and does not exist when $x = -1$. As $-\dfrac{4}{3}$ and -1 are not in the domain of f, $-\dfrac{4}{3}$ and -1 are not critical numbers of f. Therefore, $\boxed{0}$ is the only critical number of f. At the point $(0, 0)$, the tangent line is horizontal.

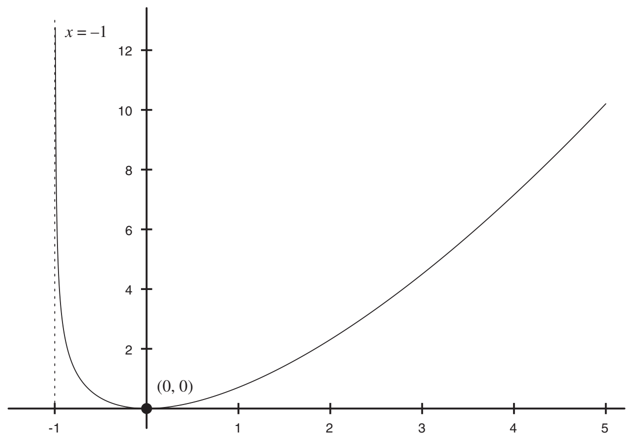
Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide $(-1, \infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(-1, 0)$	$-$	f is decreasing on $(-1, 0)$
$(0, \infty)$	$+$	f is increasing on $(0, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at 0. The $\boxed{\text{local minimum value is } f(0) = 0}$.

Step 6 Because $f''(x) > 0$ for all $x > -1$, f is $\boxed{\text{concave up on the interval } (-1, \infty)}$. As the concavity of f never changes, f has $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle.



27. Let $f(x) = \dfrac{1}{(x+1)(x-2)} = \dfrac{1}{x^2 - x - 2}$.

Step 1 The domain of the rational function f is the set $\boxed{\{x | x \neq -1, x \neq 2\}}$. There are $\boxed{\text{no } x\text{-intercepts}}$, and the $\boxed{y\text{-intercept is } f(0) = -\dfrac{1}{2}}$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - x - 2} = \lim_{x \to \pm\infty} \frac{1}{x^2 - x - 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \pm\infty} \frac{\frac{1}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 0,$$

the line $\boxed{y = 0 \text{ is a horizontal asymptote}}$. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f. As

$$\lim_{x \to -1^-} \frac{1}{(x+1)(x-2)} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{(x+1)(x-2)} = -\infty,$$

and

$$\lim_{x \to 2^-} \frac{1}{(x+1)(x-2)} = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{(x+1)(x-2)} = \infty,$$

the lines $\boxed{x = -1 \text{ and } x = 2 \text{ are vertical asymptotes}}$.

Step 3 Now,

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x^2-x-2}\right) = -\frac{2x-1}{(x^2-x-2)^2}; \text{ and}$$

$$f''(x) = -\frac{d}{dx}\left(\frac{2x-1}{(x^2-x-2)^2}\right) = -\frac{(x^2-x-2)^2\cdot 2-(2x-1)\cdot 2(x^2-x-2)(2x-1)}{(x^2-x-2)^4}$$

$$= -\frac{2(x^2-x-2)-2(4x^2-4x+1)}{(x^2-x-2)^3} = \frac{6(x^2-x+1)}{(x^2-x-2)^3}.$$

The critical numbers of f occur where $f'(x)=0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x=\frac{1}{2}$ and does not exist when $x=-1$ and when $x=2$. As -1 and 2 are not in the domain of f, -1 and 2 are not critical numbers. Therefore, $\boxed{\frac{1}{2}}$ is the only critical number of f. At the point $\left(\frac{1}{2},-\frac{4}{9}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -1, $\frac{1}{2}$, and 2 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty,-1)$	$+$	f is increasing on $(-\infty,-1)$
$(-1,\frac{1}{2})$	$+$	f is increasing on $(-1,\frac{1}{2})$
$(\frac{1}{2},2)$	$-$	f is decreasing on $(\frac{1}{2},2)$
$(2,\infty)$	$-$	f is decreasing on $(2,\infty)$

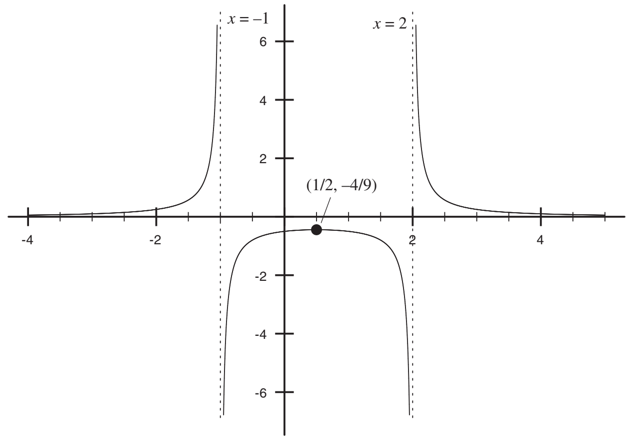
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\frac{1}{2}$. The $\boxed{\text{local maximum value is } f\left(\frac{1}{2}\right)=-\frac{4}{9}}$.

Step 6 The second derivative is never equal to zero and does not exist for $x=-1$ and $x=2$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty,-1)$	$+$	f is concave up on $(-\infty,-1)$
$(-1,2)$	$-$	f is concave down on $(-1,2)$
$(2,\infty)$	$+$	f is concave up on $(2,\infty)$

Although the concavity of f changes at -1 and 2, there is $\boxed{\text{no point of inflection}}$ at either -1 or 2 because -1 and 2 are not in the domain of f.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle.



29. Let $f(x) = x^{2/3} + 3x^{1/3} + 2$.

Step 1 The domain of f is the set of $\boxed{\text{all real numbers}}$. Solving

$$x^{2/3} + 3x^{1/3} + 2 = (x^{1/3} + 2)(x^{1/3} + 1) = 0,$$

yields $\boxed{-8 \text{ and } -1 \text{ as the } x\text{-intercepts}}$; the $\boxed{y\text{-intercept is } f(0) = 2}$.

Step 2 Because

$$\lim_{x \to \pm\infty} (x^{2/3} + 3x^{1/3} + 2) = \lim_{x \to \pm\infty} \left[x^{2/3} \left(1 + \frac{3}{x^{1/3}} + \frac{2}{x^{2/3}} \right) \right] = \infty,$$

the graph of f $\boxed{\text{does not have a horizontal asymptote}}$. The graph also has $\boxed{\text{no vertical asymptotes}}$ because f is defined for all x.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3} + x^{-2/3} = \frac{2x^{1/3} + 3}{3x^{2/3}}; \text{ and} \\ f''(x) &= -\frac{2}{9}x^{-4/3} - \frac{2}{3}x^{-5/3} = -\frac{2x^{1/3} + 6}{9x^{5/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -\frac{27}{8}$ and does not exist when $x = 0$. Therefore, $\boxed{-\frac{27}{8} \text{ and } 0}$ are critical numbers of f. At the point $\left(-\frac{27}{8}, -\frac{1}{4} \right)$, the tangent line is horizontal; at the point $(0, 2)$, the tangent line is vertical.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers $-\frac{27}{8}$ and 0 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\frac{27}{8})$	$-$	f is decreasing on $(-\infty, -\frac{27}{8})$
$(-\frac{27}{8}, 0)$	$+$	f is increasing on $(-\frac{27}{8}, 0)$
$(0, \infty)$	$+$	f is increasing on $(0, \infty)$

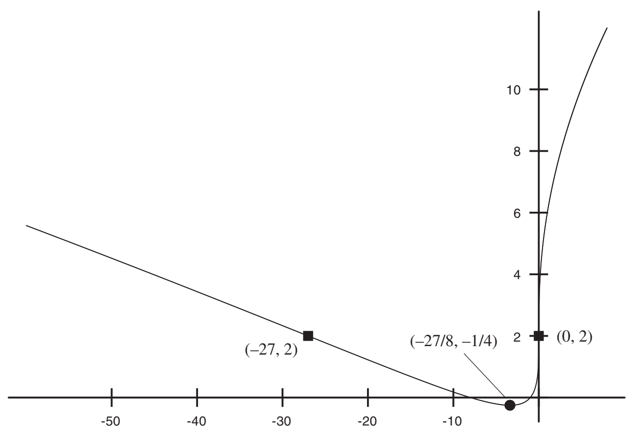
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\dfrac{27}{8}$ and has neither a local maximum value nor a local minimum value at 0. The $\boxed{\text{local minimum value is } f\left(-\dfrac{27}{8}\right) = -\dfrac{1}{4}}$.

Step 6 The second derivative is equal to zero when $x = -27$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -27)$	$-$	f is concave down on $(-\infty, -27)$
$(-27, 0)$	$+$	f is concave up on $(-27, 0)$
$(0, \infty)$	$-$	f is concave down on $(0, \infty)$

The concavity of f changes at -27 and 0, so the points $\boxed{(-27, 2) \text{ and } (0, 2)}$ are points of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



31. Let $f(x) = \sin x - \cos x$. Note the function f is periodic with period 2π.

Step 1 The domain of f is the set of $\boxed{\text{all real numbers}}$. The x-intercepts satisfy the equation

$$\sin x - \cos x = 0 \quad \text{or} \quad \tan x = 1;$$

the solutions to this equation are $\boxed{\dfrac{\pi}{4} + k\pi}$ for any integer k. The $\boxed{y\text{-intercept is } f(0) = -1}$.

Step 2 The function f has $\boxed{\text{no asymptotes}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \cos x + \sin x; \text{ and} \\ f''(x) &= -\sin x + \cos x. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$, which is where $\cos x = -\sin x$ or $\tan x = -1$. Therefore, the critical numbers of f are $\boxed{\dfrac{3\pi}{4} + k\pi}$ for any integer k. At the points $\left(\dfrac{3\pi}{4} + k\pi, (-1)^k\sqrt{2}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, consider the intervals $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. The increasing/decreasing pattern on these two intervals will repeat indefinitely in either direction in increments of the period 2π. The results are shown in the table below.

Interval	Sign of f'	Conclusion
$(-\frac{\pi}{4}, \frac{3\pi}{4})$	$+$	f is increasing on $(-\frac{\pi}{4}, \frac{3\pi}{4})$
$(\frac{3\pi}{4}, \frac{7\pi}{4})$	$-$	f is decreasing on $(\frac{3\pi}{4}, \frac{7\pi}{4})$

Therefore, f is

$$\boxed{\text{increasing on intervals of the form } \left(-\frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi\right)}$$

and is

$$\boxed{\text{decreasing on intervals of the form } \left(\frac{3\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi\right)}$$

for any integer k.

Step 5 By the First Derivative Test and the information above, f has a local maximum value at $\frac{3\pi}{4} + 2k\pi$ and a local minimum value at $\frac{7\pi}{4} + 2k\pi$ for any integer k. The

$$\boxed{\text{local maximum value is } f\left(\frac{3\pi}{4} + 2k\pi\right) = \sqrt{2}},$$

and the

$$\boxed{\text{local minimum value is } f\left(\frac{7\pi}{4} + 2k\pi\right) = -\sqrt{2}}.$$

Step 6 The second derivative exists everywhere and is equal to zero when $\tan x = 1$, which is when $x = \frac{\pi}{4} + k\pi$ for any integer k. Consider the intervals $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\frac{3\pi}{4}, \frac{\pi}{4})$	$+$	f is concave up on $(-\frac{3\pi}{4}, \frac{\pi}{4})$
$(\frac{\pi}{4}, \frac{5\pi}{4})$	$-$	f is concave down on $(\frac{\pi}{4}, \frac{5\pi}{4})$

Taking into account the period of the function f, it follows that f is

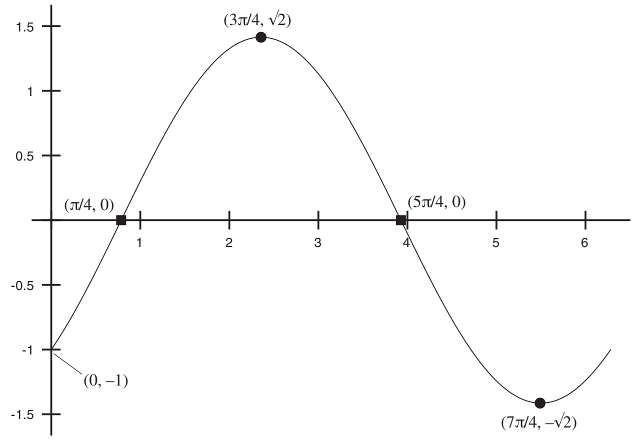
$$\boxed{\text{concave up on intervals of the form } \left(-\frac{3\pi}{4} + 2k\pi, \frac{\pi}{4} + 2k\pi\right)}$$

and is

$$\boxed{\text{concave down on intervals of the form } \left(\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi\right)}$$

for any integer k. The concavity of f changes at $\frac{\pi}{4} + k\pi$ for each integer k, so $\boxed{\left(\frac{\pi}{4} + k\pi, 0\right)}$ is a point of inflection for each integer k.

Step 7 The figure below displays the graph of f over the interval $[0, 2\pi]$. The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares. The graph repeats indefinitely in either direction with period 2π.



33. Let $f(x) = \sin^2 x - \cos x$. Note the function f is periodic with period 2π.

Step 1 The domain of f is the set of $\boxed{\text{all real numbers}}$. The x-intercepts satisfy the equation

$$\sin^2 x - \cos x = 1 - \cos^2 x - \cos x = 0.$$

This equation is quadratic in form, so by the quadratic formula,

$$\cos x = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)(1)}}{-2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Now, $\dfrac{-1-\sqrt{5}}{2} < -1$ and $-1 \le \cos x \le 1$, so the x-intercepts are

$$\boxed{\cos^{-1}\left(\frac{-1+\sqrt{5}}{2}\right) + 2k\pi \approx 0.905 + 2k\pi \quad \text{and} \quad 2\pi - \cos^{-1}\left(\frac{-1+\sqrt{5}}{2}\right) + 2k\pi \approx 5.379 + 2k\pi}$$

for any integer k. The $\boxed{y\text{-intercept is } f(0) = -1}$.

Step 2 The graph of the function f has $\boxed{\text{no asymptotes}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= 2\sin x\cos x + \sin x = \sin x(2\cos x + 1); \text{ and} \\ f''(x) &= -2\sin^2 x + \cos x(2\cos x + 1) = 2\cos^2 x - 2\sin^2 x + \cos x \\ &= 4\cos^2 x + \cos x - 2. \end{aligned}$$

As f is differentiable everywhere, the critical numbers of f occur where $f'(x) = 0$; that is, where $\sin x = 0$ and $2\cos x + 1 = 0$. Therefore, $\boxed{k\pi,\ \frac{2\pi}{3} + 2k\pi, \text{ and } \frac{4\pi}{3} + 2k\pi}$ are the critical numbers of f for any integer k. At the points $(k\pi, (-1)^{k+1})$, $\left(\frac{2\pi}{3} + 2k\pi, \frac{5}{4}\right)$, and $\left(\frac{4\pi}{3} + 2k\pi, \frac{5}{4}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, consider the intervals $\left(0, \frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3}, \pi\right)$, $\left(\pi, \frac{4\pi}{3}\right)$, and $\left(\frac{4\pi}{3}, 2\pi\right)$. The increasing/decreasing pattern on these two intervals will repeat indefinitely in either direction in increments of the period 2π. The results are shown in the table below.

Interval	Sign of f'	Conclusion
$(0, \frac{2\pi}{3})$	$+$	f is increasing on $(0, \frac{2\pi}{3})$
$(\frac{2\pi}{3}, \pi)$	$-$	f is decreasing on $(\frac{2\pi}{3}, \pi)$
$(\pi, \frac{4\pi}{3})$	$+$	f is increasing on $(\pi, \frac{4\pi}{3})$
$(\frac{4\pi}{3}, 2\pi)$	$-$	f is decreasing on $(\frac{4\pi}{3}, 2\pi)$

Therefore, f is

$$\boxed{\text{increasing on intervals of the form } \left(2k\pi, \frac{2\pi}{3} + 2k\pi\right) \text{ and } \left((2k+1)\pi, \frac{4\pi}{3} + 2k\pi\right)}$$

and is

$$\boxed{\text{decreasing on intervals of the form } \left(\frac{2\pi}{3} + 2k\pi, (2k+1)\pi\right) \text{ and } \left(\frac{4\pi}{3} + 2k\pi, (2k+2)\pi\right)}$$

for any integer k.

Step 5 By the First Derivative Test and the information above, f has a local maximum value at $\frac{2\pi}{3} + 2k\pi$ and at $\frac{4\pi}{3} + 2k\pi$ and a local minimum value at $k\pi$ for each integer k. The

$$\boxed{\text{local maximum values are } f\left(\frac{2\pi}{3} + 2k\pi\right) = \frac{5}{4} \text{ and } f\left(\frac{4\pi}{3} + 2k\pi\right) = \frac{5}{4}},$$

and the $\boxed{\text{local minimum values are } f(k\pi) = (-1)^k}$.

Step 6 The second derivative exists everywhere and is equal to zero when $4\cos^2 x + \cos x - 2 = 0$. This equation is quadratic in form, so by the quadratic formula,

$$\cos x = \frac{-1 \pm \sqrt{1^2 - 4(4)(-2)}}{8} = \frac{-1 \pm \sqrt{33}}{8}.$$

Therefore, $f''(x) = 0$ when

$$x = \cos^{-1}\left(\frac{-1-\sqrt{33}}{8}\right) + 2k\pi \approx 2.574 + 2k\pi, \quad x = 2\pi - \cos^{-1}\left(\frac{-1-\sqrt{33}}{8}\right) + 2k\pi \approx 3.709 + 2k\pi,$$

$$x = \cos^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) + 2k\pi \approx 0.936 + 2k\pi, \quad x = 2\pi - \cos^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) + 2k\pi \approx 5.347 + 2k\pi.$$

Consider the approximate intervals $(0.936, 2.574)$, $(2.574, 3.709)$, $(3.709, 5.347)$, and $(5, 347, 7.219)$, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(0.936, 2.574)$	$-$	f is concave down on $(0.936, 2.574)$
$(2.574, 3.709)$	$+$	f is concave up on $(2.574, 3.709)$
$(3.709, 5.347)$	$-$	f is concave down on $(3.709, 5.347)$
$(5.347, 7.219)$	$+$	f is concave up on $(5.347, 7.219)$

Taking into account the period of the function f, it follows that f is

$\boxed{\text{concave down on intervals of the form } (0.936+2k\pi, 2.574+2k\pi) \text{ and}}$

$\boxed{(3.709+2k\pi, 5.347+2k\pi)}$

and is

$\boxed{\text{concave up on intervals of the form } (2.574+2k\pi, 3.709+2k\pi) \text{ and}}$

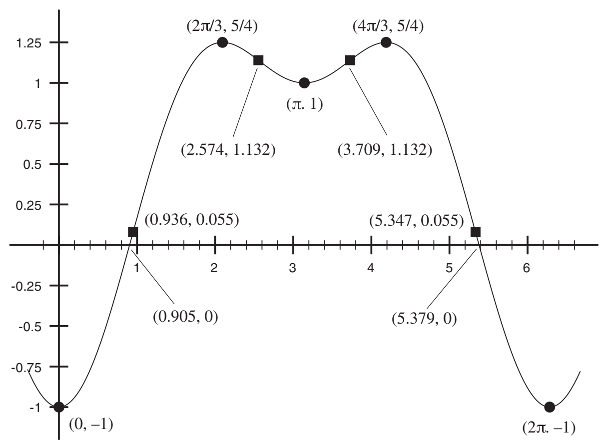
$\boxed{(5.347+2k\pi, 7.219+2k\pi)}$

for any integer k. The concavity of f changes at $0.936+2k\pi$, $2.574+2k\pi$, $3.709+2k\pi$, and $5.347+2k\pi$ for each integer k, so the points

$$\boxed{(0.936+2k\pi, 0.055),\ (2.574+2k\pi, 1.132),\ (3.709+2k\pi, 1.132),\ \text{and}\ (5.347+2k\pi, 0.055)}$$

are points of inflection of f for each integer k.

Step 7 The figure below displays the graph of f over the interval $[0, 2\pi]$. The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares. The graph repeats indefinitely in either direction with period 2π.



35. Let $y = f(x) = \ln x - x$.

Step 1 The domain of f is the set $\boxed{\{x|x>0\}}$. There are $\boxed{\text{no } x\text{-intercepts}}$ as the graphs of $y=\ln x$ and $y=x$ never intersect, and there is also $\boxed{\text{no } y\text{-intercept}}$ because the function is not defined at 0.

Step 2 Because

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to\infty} x = \infty,$$

the expression $\ln x - x$ is an indeterminate form at ∞ of the type $\infty - \infty$. Rewrite

$$\ln x - x \quad \text{as} \quad x\left(\frac{\ln x}{x} - 1\right).$$

Now, the expression $\dfrac{\ln x}{x}$ is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$; by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

Therefore,

$$\lim_{x\to\infty}\left(\frac{\ln x}{x}-1\right)=-1 \quad\text{and}\quad \lim_{x\to\infty}(\ln x - x)=\lim_{x\to\infty}\left[x\left(\frac{\ln x}{x}-1\right)\right]=-\infty,$$

so the graph of f has $\boxed{\text{no horizontal asymptote}}$. On the other hand,

$$\lim_{x\to 0^+}(\ln x - x)=-\infty,$$

so the line $\boxed{x=0 \text{ is a vertical asymptote}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{1}{x}-1=\frac{1-x}{x};\ \text{and} \\ f''(x) &= -\frac{1}{x^2}. \end{aligned}$$

The critical numbers of f occur where $f'(x)=0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x=1$ and does not exist when $x=0$; however, 0 is not a critical number because 0 is not in the domain of f. Therefore, $\boxed{1}$ is the only critical number. At the point $(1,-1)$, the tangent line is horizontal.

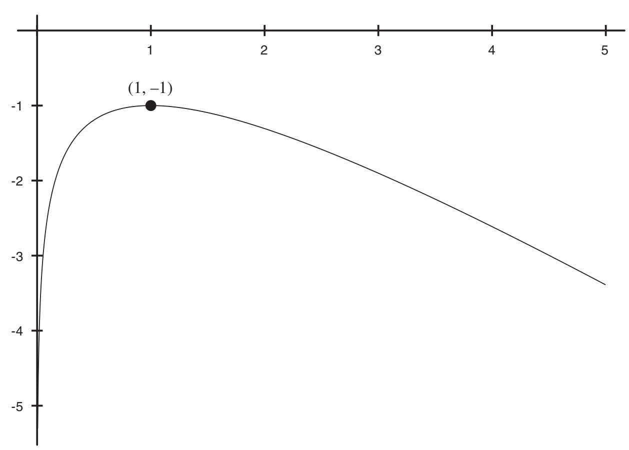
Step 4 To apply the Increasing/Decreasing Function Test, use the number 1 to divide $(0,\infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(0,1)$	$+$	f is increasing on $(0,1)$
$(1,\infty)$	$-$	f is decreasing on $(1,\infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 1. The $\boxed{\text{local maximum value is } f(1)=-1}$.

Step 6 Because $f''(x)<0$ for all $x>0$, f is $\boxed{\text{concave down on the interval } (0,\infty)}$, so f has $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle.



37. Let $f(x) = \ln(4 - x^2)$.

Step 1 The domain of f is given by the solution to the inequality $4 - x^2 > 0$; that is, the set $\boxed{\{x| -2 < x < 2\}}$. The $\boxed{x\text{-intercepts are } \pm\sqrt{3}}$, and the $\boxed{y\text{-intercept is } f(0) = \ln 4}$.

Step 2 Because f is not defined as x becomes unbounded in either direction, the graph of f has $\boxed{\text{no horizontal asymptote}}$. As x approaches the endpoints of the domain

$$\lim_{x \to -2^+} \ln(4 - x^2) = -\infty \quad \text{and} \quad \lim_{x \to 2^-} \ln(4 - x^2) = -\infty,$$

so the lines $\boxed{x = \pm 2 \text{ are vertical asymptotes}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{1}{4 - x^2} \cdot (-2x) = -\frac{2x}{4 - x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx}\left(-\frac{2x}{4 - x^2}\right) = -\frac{(4 - x^2) \cdot 2 - 2x \cdot (-2x)}{(4 - x^2)^2} = -\frac{8 + 2x^2}{(4 - x^2)^2}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = 0$ and exists everywhere on the domain of f. Therefore, $\boxed{0}$ is the only critical number. At the point $(0, \ln 4)$, the tangent line is horizontal.

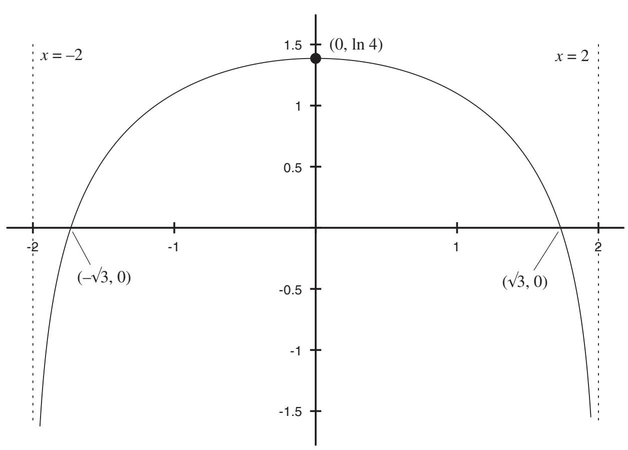
Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide $(-2, 2)$ into two intervals.

Interval	Sign of f'	Conclusion
$(-2, 0)$	$+$	f is increasing on $(-2, 0)$
$(0, 2)$	$-$	f is decreasing on $(0, 2)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The $\boxed{\text{local maximum value is } f(0) = \ln 4}$.

Step 6 Because $f''(x) < 0$ for all $-2 < x < 2$, f is $\boxed{\text{concave down on the interval } (-2, 2)}$, so f has $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle.



39. Let $f(x) = 3e^{3x}(5 - x)$.

Step 1 The domain of f is the set of $\boxed{\text{all real numbers}}$. The $\boxed{x\text{-intercept is } 5}$, and the $\boxed{y\text{-intercept is } f(0) = 15}$.

Step 2 Because the domain of f is the set of all real numbers, the graph of f has $\boxed{\text{no vertical asymptotes}}$. To determine if there is a horizontal asymptote, consider the limits at infinity:

$$\lim_{x\to-\infty}[3e^{3x}(5-x)] = \lim_{x\to-\infty}\frac{3(5-x)}{e^{-3x}} = \lim_{x\to-\infty}\frac{-3}{-3e^{-3x}} = 0$$

and

$$\lim_{x\to\infty}[3e^{3x}(5-x)] = -\infty,$$

where L'Hôpital's Rule was used in the first limit. Therefore, the graph of f has the line $\boxed{y = 0 \text{ as a horizontal asymptote}}$ as $x \to -\infty$ and no horizontal asymptote as $x \to \infty$.

Step 3 Now,

$$\begin{aligned} f'(x) &= -3e^{3x} + 9e^{3x}(5-x) = 3e^{3x}(14-3x); \text{ and} \\ f''(x) &= -9e^{3x} + 9e^{3x}(14-3x) = 9e^{3x}(13-3x). \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $\boxed{x = \frac{14}{3}}$. At the point $\left(\frac{14}{3}, e^{14}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number $\frac{14}{3}$ to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, \frac{14}{3})$	$+$	f is increasing on $(-\infty, \frac{14}{3})$
$(\frac{14}{3}, \infty)$	$-$	f is decreasing on $(\frac{14}{3}, \infty)$

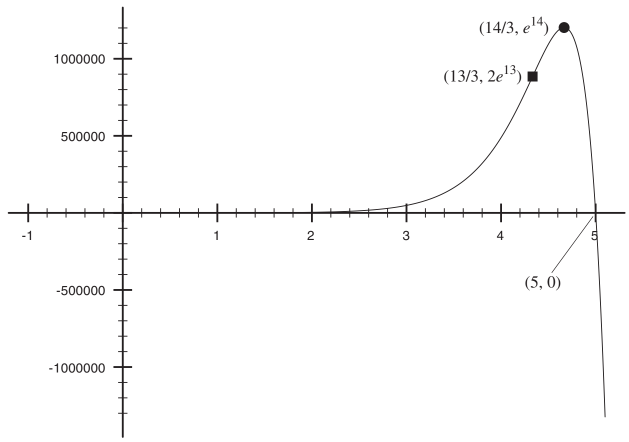
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\frac{14}{3}$. The $\boxed{\text{local maximum value is } f\left(\frac{14}{3}\right) = e^{14}}$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = \frac{13}{3}$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, \frac{13}{3})$	$+$	f is concave up on $(-\infty, \frac{13}{3})$
$(\frac{13}{3}, \infty)$	$-$	f is concave down on $(\frac{13}{3}, \infty)$

The concavity of f changes at $\frac{13}{3}$, so the point $\boxed{\left(\frac{13}{3}, 2e^{13}\right)}$ is a point of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



41. Let $f(x) = e^{-x^2}$.

Step 1 The domain of f is the set of $\boxed{\text{all real numbers}}$. There is $\boxed{\text{no } x\text{-intercept}}$, and the $\boxed{y\text{-intercept is } f(0) = 1}$.

Step 2 Because

$$\lim_{x \to \pm\infty} e^{-x^2} = 0,$$

the line $\boxed{y = 0 \text{ is a horizontal asymptote}}$. The graph of f has $\boxed{\text{no vertical asymptotes}}$ because f is defined for all real x.

Step 3 Now,

$$\begin{aligned} f'(x) &= -2xe^{-x^2}; \text{ and} \\ f''(x) &= 4x^2e^{-x^2} - 2e^{-x^2} = 2(2x^2 - 1)e^{-x^2}. \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $\boxed{x = 0}$. At the point $(0, 1)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	$+$	f is increasing on $(-\infty, 0)$
$(0, \infty)$	$-$	f is decreasing on $(0, \infty)$

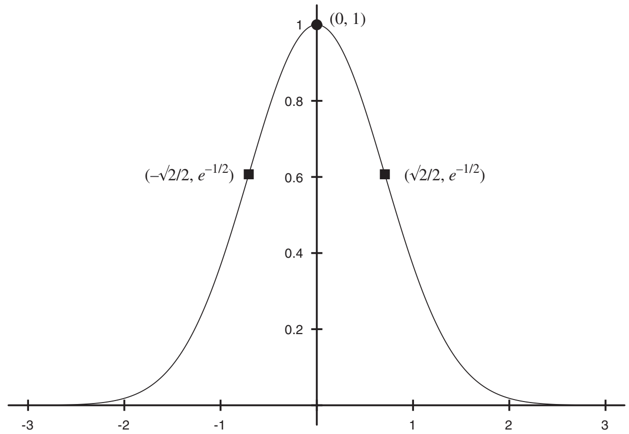
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The $\boxed{\text{local maximum value is } f(0) = 1}$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = \pm\dfrac{\sqrt{2}}{2}$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{\sqrt{2}}{2})$	$+$	f is concave up on $(-\infty, -\frac{\sqrt{2}}{2})$
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$-$	f is concave down on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$(\frac{\sqrt{2}}{2}, \infty)$	$+$	f is concave up on $(\frac{\sqrt{2}}{2}, \infty)$

The concavity of f changes at $\pm\frac{\sqrt{2}}{2}$, so the points $\boxed{\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right) \text{ and } \left(\frac{\sqrt{2}}{2}, e^{-1/2}\right)}$ are points of inflection of f.

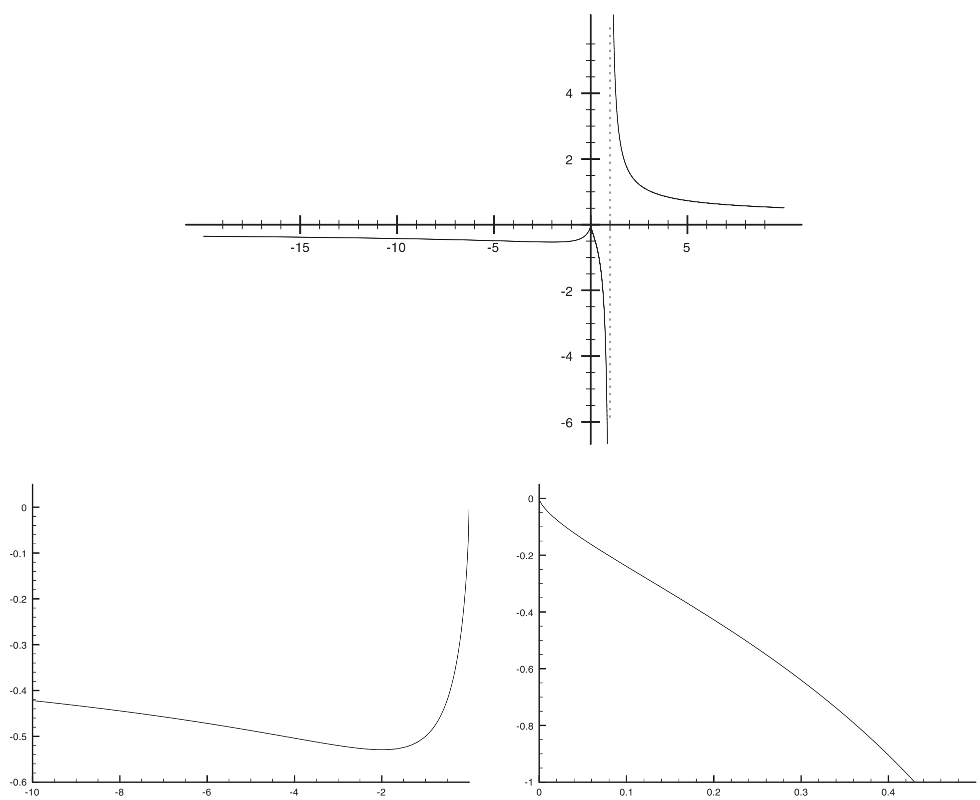
Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



Applications and Extensions

43. Let $f(x) = \dfrac{x^{2/3}}{x-1}$.

(a) The figures below display the graph of f. In the top figure, the graph is shown for $-20 \le x \le 10$. In the figure at the bottom left, attention is focused on $-10 \le x \le 0$. Attention is focused on $0 \le x \le 0.5$ in the figure at the bottom right.



(b) The graph of f appears to have a $\boxed{\text{vertical asymptote at } x = 1}$ and a $\boxed{\text{horizontal asymptote at } y = 0}$.

(c) The function appears to be

$$\boxed{\text{decreasing on the intervals } (-\infty, -2),\ (0, 1), \text{ and } (1, \infty)}$$

and $\boxed{\text{increasing on the interval } (-2, 0)}$. Additionally, the function appears to be

$$\boxed{\text{concave down, approximately, on the intervals } (-\infty, -4) \text{ and } (0.1, 1)}$$

and

$$\boxed{\text{concave up, approximately, on the intervals } (-4, 0),\ (0, 0.1), \text{ and } (1, \infty)}.$$

(d) The function appears to have a $\boxed{\text{local minimum value at } -2}$ and a $\boxed{\text{local maximum value at } 0}$.

(e) Note that

$$f'(x) = \frac{(x-1)\cdot \frac{2}{3}x^{-1/3} - x^{2/3}}{(x-1)^2} = \frac{2x - 2 - 3x}{3x^{1/3}(x-1)^2} = -\frac{x+2}{3x^{1/3}(x-1)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x) = 0$ when $x = -2$ and does not exist when $x = 0$ and when $x = 1$; however, 1 is not in the domain of f, so 1 is not a critical number. To apply the Increasing/Decreasing Function Test, use the numbers -2, 0, and 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	$-$	f is decreasing on $(-\infty, -2)$
$(-2, 0)$	$+$	f is increasing on $(-2, 0)$
$(0, 1)$	$-$	f is decreasing on $(0, 1)$
$(1, \infty)$	$-$	f is decreasing on $(1, \infty)$

By the First Derivative Test and the information in the table above, f has a $\boxed{\text{local minimum value at } -2}$ and a $\boxed{\text{local maximum value at } 0}$, in agreement with the approximations given in part (d).

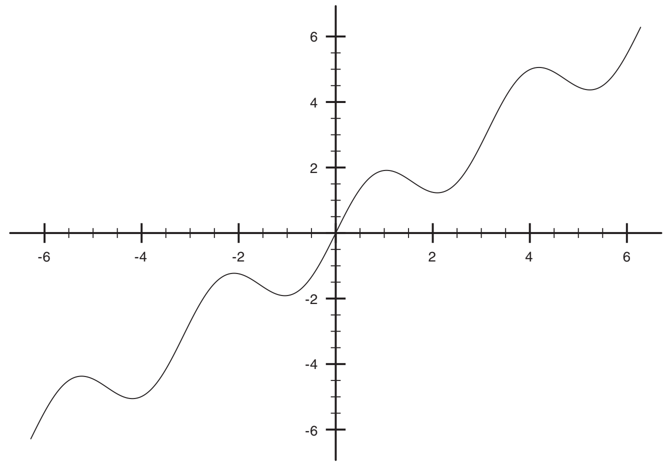
(f) Concavity appears to change at approximately $x = -4$ and $x = 0.1$; therefore,

$$\boxed{(-4, f(-4)) = \left(-4, -\frac{2\sqrt[3]{2}}{5}\right) \quad \text{and} \quad (0.1, f(0.1)) \approx (0.1, -0.239)}$$

are approximate points of inflection of f.

45. Let $f(x) = x + \sin(2x)$.

(a) The figure below displays the graph of f. Though the function f is not periodic, observe that the graph consists of one basic shape that is repeated every π units along the x-axis.



(b) The graph of f does not appear to have any asymptotes.

(c) The function appears to be increasing on the interval $(-1,1) \approx \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on the interval $(1,2) \approx \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$. Given the repetitive structure of the graph, it follows that f is

$$\boxed{\text{increasing on intervals of the form } \left(-\frac{\pi}{3}+k\pi, \frac{\pi}{3}+k\pi\right)}$$

and

$$\boxed{\text{decreasing on intervals of the form } \left(\frac{\pi}{3}+k\pi, \frac{2\pi}{3}+k\pi\right)}$$

for each integer k. Additionally, f appears to be concave up on the interval $(-1.5,0) \approx \left(-\frac{\pi}{2},0\right)$ and concave down on the interval $(0,1.5) \approx \left(0,\frac{\pi}{2}\right)$. Again taking into account the repetitive structure of the graph, it follows that f is $\boxed{\text{concave up on intervals of the form } \left(-\frac{\pi}{2}+k\pi, k\pi\right)}$ and $\boxed{\text{concave down on intervals of the form } \left(k\pi, \frac{\pi}{2}+k\pi\right)}$ for each integer k.

(d) The graph appears to have a $\boxed{\text{local maximum value at approximately } \frac{\pi}{3}+k\pi}$ and a $\boxed{\text{local minimum value at approxiamtely } \frac{2\pi}{3}+k\pi}$ for each integer k.

(e) Note

$$f'(x) = 1 + 2\cos(2x).$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = \frac{\pi}{3} + k\pi$ and when $x = \frac{2\pi}{3} + k\pi$ for each integer k. To apply the Increasing/Decreasing Function Test, use the numbers $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ to divide $(0,\pi)$ into three intervals.

Interval	Sign of f'	Conclusion
$(0, \frac{\pi}{3})$	$+$	f is increasing on $(0, \frac{\pi}{3})$
$(\frac{\pi}{3}, \frac{2\pi}{3})$	$-$	f is decreasing on $(\frac{\pi}{3}, \frac{2\pi}{3})$
$(\frac{2\pi}{3}, \pi)$	$+$	f is increasing on $(\frac{2\pi}{3}, \pi)$

By the First Derivative Test, the information in the table above and the repetitive structure of the graph, f has a $\boxed{\text{local maximum value at } \dfrac{\pi}{3} + k\pi}$ and a $\boxed{\text{local minimum value at } \dfrac{2\pi}{3} + k\pi}$ for each integer k, in agreement with the approximations given in part (d).

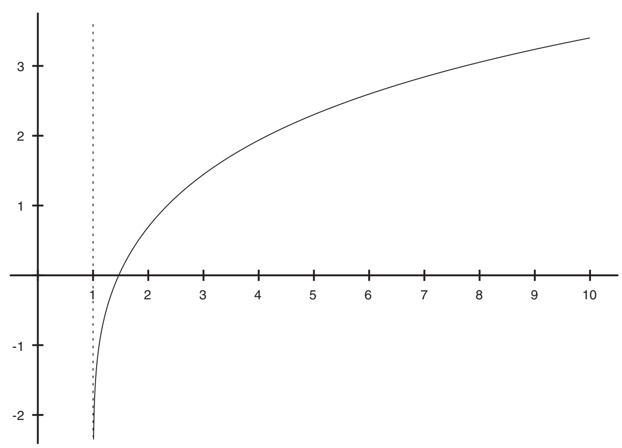
(f) The concavity of f appears to change at multiples of $\dfrac{\pi}{2}$; that is, there appear to be points of inflection at

$$\boxed{\left(\frac{k\pi}{2}, f\left(\frac{k\pi}{2}\right)\right) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)}$$

for each integer k.

47. Let $f(x) = \ln(x\sqrt{x-1})$.

(a) The figure below displays the graph of f.



(b) The graph of f has $\boxed{\text{no horizontal asymptote}}$, but has a $\boxed{\text{vertical asymptote at } x = 1}$.

(c) The function appears to be $\boxed{\text{increasing and concave down on the interval } (1, \infty)}$.

(d) The function f $\boxed{\text{does not appear to have any local extreme values}}$.

(e) Note that

$$f(x) = \ln(x\sqrt{x-1}) = \ln x + \ln\sqrt{x-1} = \ln x + \frac{1}{2}\ln(x-1),$$

so

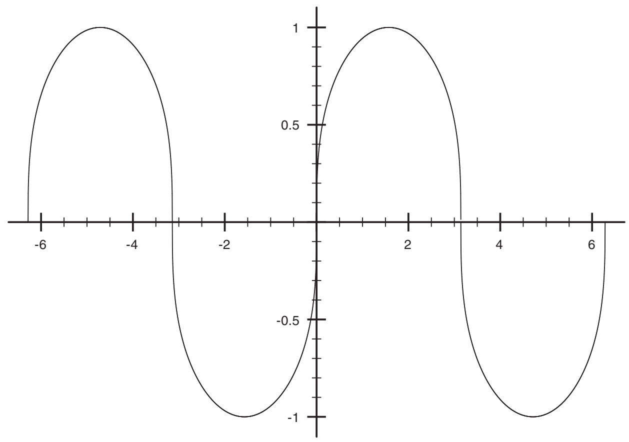
$$f'(x) = \frac{1}{x} + \frac{1}{2(x-1)} = \frac{3x-2}{2x(x-1)}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x) = 0$ when $x = \dfrac{2}{3}$ and does not exist when $x = 0$ and when $x = 1$; however, none of these numbers is in the domain of f, so none are critical numbers. Because the function has no critical numbers, there are $\boxed{\text{no local extreme values}}$, in agreement with the result of part (d).

(f) Because the concavity of f never changes, there are $\boxed{\text{no points of inflection}}$.

49. Let $f(x) = \sqrt[3]{\sin x}$.

(a) The figure below displays the graph of f. Note that the graph is periodic with period 2π.



(b) The graph of f $\boxed{\text{does not appear to have any asymptotes}}$.

(c) Taking into account the period of the function, f appears to be

$$\boxed{\text{increasing on intervals of the form } \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)}$$

and

$$\boxed{\text{decreasing on intervals of the form } \left(\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right)}$$

for each integer k. Additionally, f appears to be

$$\boxed{\text{concave up on intervals of the form } (-\pi + 2k\pi, 2k\pi)}$$

and

$$\boxed{\text{concave down on intervals of the form } (2k\pi, \pi + 2k\pi)},$$

again for each integer k.

(d) The graph appears to have a $\boxed{\text{local maximum value at } x = \frac{\pi}{2} + 2k\pi}$ and a $\boxed{\text{local minimum value at } x = \frac{3\pi}{2} + 2k\pi}$ for each integer k.

(e) Note

$$f'(x) = \frac{1}{3}(\sin x)^{-2/3} \cos x.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = \frac{\pi}{2} + k\pi$ for each integer k and does not exist when $x = k\pi$ for each integer k. To apply the Increasing/Decreasing Function Test, use the numbers $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ to divide $(0, 2\pi)$ into four intervals.

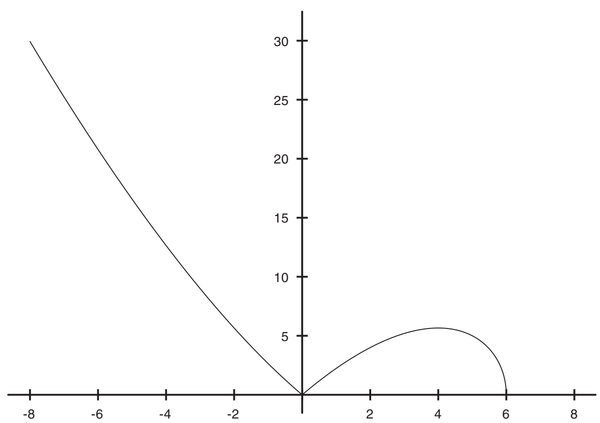
Interval	Sign of f'	Conclusion
$(0, \frac{\pi}{2})$	$+$	f is increasing on $(0, \frac{\pi}{2})$
$(\frac{\pi}{2}, \pi)$	$-$	f is decreasing on $(\frac{\pi}{2}, \pi)$
$(\pi, \frac{3\pi}{2})$	$-$	f is decreasing on $(\pi, \frac{3\pi}{2})$
$(\frac{3\pi}{2}, 2\pi)$	$+$	f is increasing on $(\frac{3\pi}{2}, 2\pi)$

By the First Derivative Test, the information in the table above and the periodic nature of the function, f has a $\boxed{\text{local maximum value at } \frac{\pi}{2} + 2k\pi}$ and a $\boxed{\text{local minimum value at } \frac{3\pi}{2} + 2k\pi}$ for each integer k, in agreement with the approximations given in part (d).

(f) The concavity of f appears to change at integer multiples of π; that is, there appear to be points of inflection at $\boxed{(k\pi, f(k\pi)) = (k\pi, 0)}$ for each integer k.

51. Let $y^2 = x^2(6-x)$, and consider $y \geq 0$. Therefore, $y = |x|\sqrt{6-x}$.

(a) The figure below displays the graph of $y^2 = x^2(6-x)$ for $y \geq 0$.



(b) The graph $\boxed{\text{does not appear to have any asymptotes}}$.

(c) The function appears to be

$$\boxed{\text{decreasing on the intervals } (-\infty, 0) \text{ and } (4, 6)}$$

and $\boxed{\text{increasing on the interval } (0, 4)}$. Additionally, the function appears to be $\boxed{\text{concave down on the interval } (0, 6)}$ and $\boxed{\text{concave up on the interval } (-\infty, 0)}$.

(d) The function appears to have a $\boxed{\text{local minimum value at } 0}$ and a $\boxed{\text{local maximum value at } 4}$.

(e) Differentiating $y^2 = x^2(6-x)$ implicitly with respect to x yields

$$2y\frac{dy}{dx} = x^2(-1) + (6-x)(2x) = -x^2 + 12x - 2x^2 = -3x(x-4),$$

so that

$$\frac{dy}{dx} = -\frac{3x(x-4)}{y} = -\frac{3x(x-4)}{|x|\sqrt{6-x}}.$$

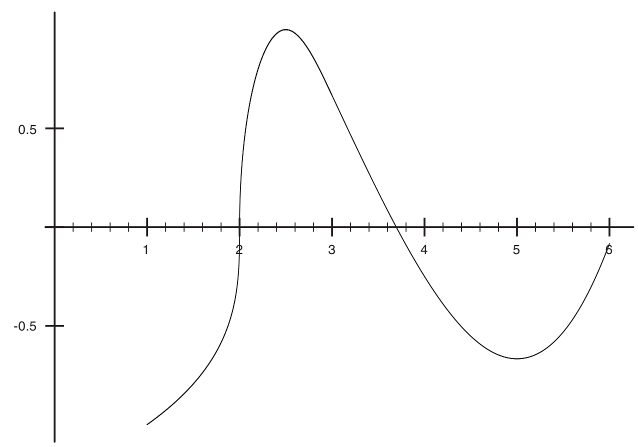
The critical numbers of y occur where $y'(x) = 0$ and where $y'(x)$ does not exist. $y'(x) = 0$ when $x = 4$ and does not exist when $x = 0$ and when $x = 6$. To apply the Increasing/Decreasing Function Test, use the numbers 0 and 4 to divide the domain of y, the set $\{x | x \leq 6\}$, into three intervals.

Interval	Sign of y'	Conclusion
$(-\infty, 0)$	$-$	y is decreasing on $(-\infty, 0)$
$(0, 4)$	$+$	y is increasing on $(0, 4)$
$(4, 6)$	$-$	y is decreasing on $(4, 6)$

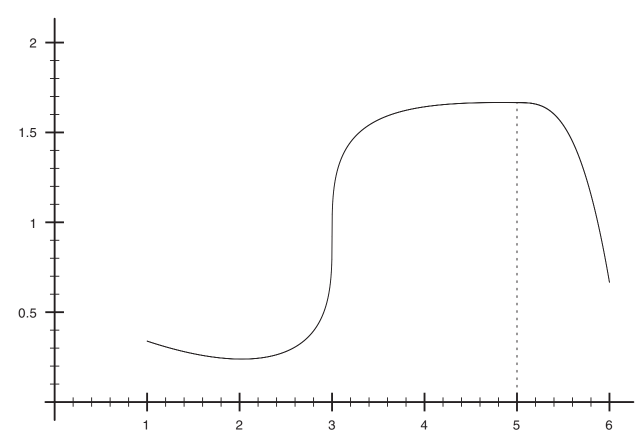
By the First Derivative Test and the information in the table above, y has a local minimum value at 0 and a local maximum value at 4, in agreement with the approximations given in part (d).

(f) The concavity of the graph appears to change at 0; that is, there appears to be a point of inflection at $(0, 0)$.

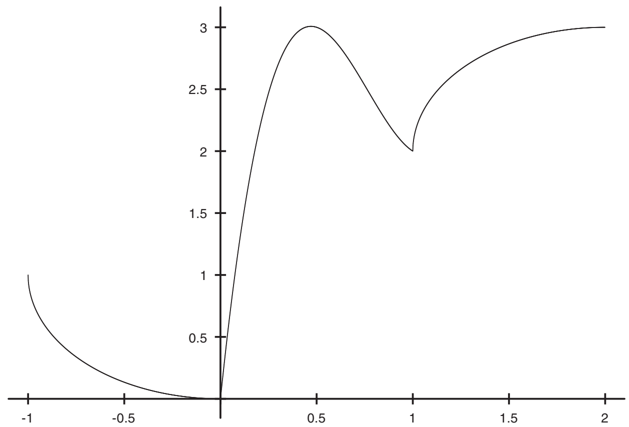
53. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[2, 5]$ and satisfies the following conditions: $f'(2)$ does not exist, $f'(3) = -1$, $f''(3) = 0$, $f'(5) = 0$, $f''(x) < 0$ for $2 < x < 3$, and $f''(x) > 0$ for $x > 3$.



55. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[2, 5]$ and satisfies the following conditions: $f'(2) = 0$, $\lim_{x \to 3^-} f'(x) = \infty$, $\lim_{x \to 3^+} f'(x) = \infty$, $f'(5) = 0$, $f''(x) > 0$ for $x < 3$, and $f''(x) < 0$ for $x > 3$.



57. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[-1, 2]$ and satisfies the following conditions: $f(-1) = 1$, $f(1) = 2$, $f(2) = 3$, $f(0) = 0$, $f\left(\frac{1}{2}\right) = 3$, $\lim_{x\to -1^+} f'(x) = -\infty$, $\lim_{x\to 1^-} f'(x) = -1$, $\lim_{x\to 1^+} f'(x) = \infty$, f has a local minimum at 0, and f has a local maximum at $\frac{1}{2}$.



59. Let $f(x) = \frac{1}{x} + \ln x$, and suppose the function is defined only on the closed interval $\left[\frac{1}{e}, e\right]$.

(a) Absolute extreme values can only occur at endpoints and critical numbers. The function f is differentiable on the domain $\left[\frac{1}{e}, e\right]$, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2},$$

so $f'(x) = 0$ when $x = 1$. Evaluating f at the endpoints of the interval $\left[\frac{1}{e}, e\right]$ and the critical number 1 yields

$$\begin{aligned} f\left(\frac{1}{e}\right) &= e + \ln\left(\frac{1}{e}\right) = e - 1 \approx 1.718 \\ f(1) &= 1 + \ln 1 = 1 \\ f(e) &= \frac{1}{e} + \ln e = \frac{1}{e} + 1 \approx 1.368 \end{aligned}$$

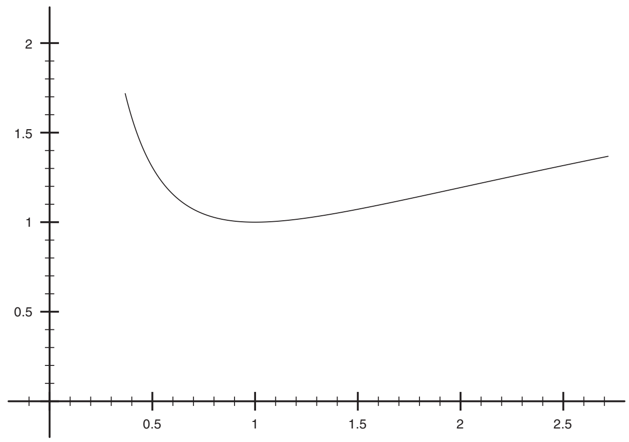
The absolute maximum value of f is $e - 1$, which occurs at $\boxed{x = \frac{1}{e}}$, and the absolute minimum value of f is 1, which occurs at $\boxed{x = 1}$.

(b) With

$$f''(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2-x}{x^3},$$

it follows that $f''(x) > 0$ for $\frac{1}{e} < x < 2$ and $f''(x) < 0$ for $2 < x < e$. Therefore, f is $\boxed{\text{concave up on the interval } \left(\frac{1}{e}, 2\right)}$ and $\boxed{\text{concave down on the interval } (2, e)}$.

(c) The figure below displays the graph of f.



61. Let $f(x) = \dfrac{\sin(3x)}{x\sqrt{4-x^2}}$.

Step 1 The domain of f is the set $\boxed{\{x| -2 < x < 0\} \cup \{x|0 < x < 2\}}$. The $\boxed{x\text{-intercepts are } \pm\frac{\pi}{3}}$, and there is $\boxed{\text{no } y\text{-intercept}}$ because $x = 0$ is not in the domain of f.

Step 2 Because

$$\lim_{x\to -2^+} \frac{\sin(3x)}{x\sqrt{4-x^2}} = -\infty \quad \text{and} \quad \lim_{x\to 2^-} \frac{\sin(3x)}{x\sqrt{4-x^2}} = -\infty,$$

the lines $\boxed{x = \pm 2 \text{ are vertical asymptotes}}$. On the other hand, by L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\sin(3x)}{x\sqrt{4-x^2}} = \lim_{x\to 0} \frac{3\cos(3x)}{x \cdot \left(-\frac{x}{\sqrt{4-x^2}}\right) + \sqrt{4-x^2}} = \lim_{x\to 0} \frac{3\sqrt{4-x^2}\cos(3x)}{4-2x^2} = \frac{3}{2},$$

so $x = 0$ is not a vertical asymptote; rather, the graph of f has a $\boxed{\text{missing point at } \left(0, \frac{3}{2}\right)}$.

The graph has $\boxed{\text{no horizontal asymptotes}}$ because f is not defined for $|x| \geq 2$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{x\sqrt{4-x^2} \cdot 3\cos(3x) - \sin(3x)\left(x \cdot \frac{1}{2}(x^2-4)^{-1/2}(-2x) + \sqrt{4-x^2}\right)}{x^2(4-x^2)} \\ &= \frac{3x(4-x^2)\cos(3x) - (4-2x^2)\sin(3x)}{x^2(4-x^2)^{3/2}}; \text{ and} \\ f''(x) &= \frac{(-9x^6 + 78x^4 - 164x^2 + 32)\sin(3x) - 12x(x^4 - 6x^2 + 8)\cos(3x)}{x^3(4-x^2)^{5/2}}, \end{aligned}$$

where the second derivative was obtained using *Mathematica*. The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Using the command

```
Solve [ D [ Sin[3x]/(x Sqrt[4-x^2]), x ] == 0, x ]
```

in *Wolfram Alpha*, we find that $f'(x) = 0$ when $x \approx \pm 1.642$ and $x \approx \pm 1.908$. Now, $f'(x)$ does not exist when $x = 0$ and when $x = \pm 2$; however, because none of these values are in the domain of f, none are critical numbers. Therefore, $\boxed{\pm 1.642 \text{ and } \pm 1.908}$ are the critical numbers of f. At the points $(\pm 1.642, -0.521)$ and $(\pm 1.908, -0.464)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0, ± 1.642 and ± 1.908 to divide the interval $(-2, 2)$ into six subintervals.

Interval	Sign of f'	Conclusion
$(-2, -1.908)$	$+$	f is increasing on $(-2, -1.908)$
$(-1.908, -1.642)$	$-$	f is decreasing on $(-1.908, -1.642)$
$(-1.642, 0)$	$+$	f is increasing on $(-1.642, 0)$
$(0, 1.642)$	$-$	f is decreasing on $(0, 1.642)$
$(1.642, 1.908)$	$+$	f is increasing on $(1.642, 1.908)$
$(1.908, 2)$	$-$	f is decreasing on $(1.908, 2)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at ± 1.908 and a local minimum value at ± 1.642. The

$$\boxed{\text{local maximum values are } f(\pm 1.908) \approx -0.461},$$

and the

$$\boxed{\text{local minimum values are } f(\pm 1.642) \approx -0.521}.$$

Step 6 The second derivative does not exist when $x = 0$ and when $x = \pm 2$ and is equal to zero when $x \approx \pm 0.763$ and when $x \approx \pm 1.826$. These latter values were obtained using the command

```
Solve [ D [ Sin[3x]/(x Sqrt[4-x^2]), {x, 2} ] == 0, x ]
```

in *Wolfram Alpha*. Use the numbers 0, ± 0.763 and ± 1.826 to divide the interval $(-2, 2)$ into six subintervals, and determine the sign of f'' on each subinterval.

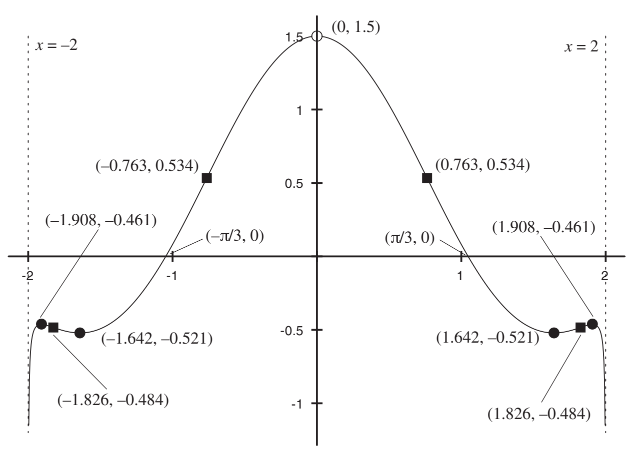
Interval	Sign of f''	Conclusion
$(-2, -1.826)$	$-$	f is concave down on $(-2, -1.826)$
$(-1.826, -0.763)$	$+$	f is concave up on $(-1.826, -0.763)$
$(-0.763, 0)$	$-$	f is concave down on $(-0.763, 0)$
$(0, 0.763)$	$-$	f is concave down on $(0, 0.763)$
$(0.763, 1.826)$	$+$	f is concave up on $(0.763, 1.826)$
$(1.826, 2)$	$-$	f is concave down on $(1.826, 2)$

The concavity of f changes at approximately ± 0.763 and ± 1.826, so the points

$$\boxed{(\pm 0.763, 0.534) \quad \text{and} \quad (\pm 1.826, -0.484)}$$

are approximate points of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares.



63. Let $f(x) = x^{1/x}$.

Step 1 The domain of f is the set $\boxed{\{x|x > 0\}}$. There are $\boxed{\text{no } x\text{-intercepts}}$, and there is $\boxed{\text{no } y\text{-intercept}}$ because $x = 0$ is not in the domain of f.

Step 2 At ∞, the expression $x^{1/x}$ is an indeterminate form of the type ∞^0. Let $y = x^{1/x}$. Then

$$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x = \frac{\ln x}{x},$$

which is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

Because $\lim\limits_{x\to\infty} \ln y = 0$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty} x^{1/x} = e^0 = 1,$$

so $\boxed{y = 1 \text{ is a horizontal asymptote}}$. At 0^+, the expression $x^{1/x}$ is of the form 0^∞, which is not an indeterminate form; rather,

$$\lim_{x\to 0^+} x^{1/x} = 0.$$

The graph of f has a $\boxed{\text{missing point at } (0,0)}$ and not a vertical asymptote at $x = 0$.

Step 3 To determine $f'(x)$, first take the natural logarithm of both sides of $f(x) = x^{1/x}$ to obtain $\ln f(x) = \ln x^{1/x} = \dfrac{1}{x}\ln x$. Now, by implicit differentiation,

$$\frac{1}{f(x)} f'(x) = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\ln x = \frac{1 - \ln x}{x^2},$$

so that

$$f'(x) = x^{1/x}\frac{1 - \ln x}{x^2}.$$

Next,

$$\begin{aligned} f''(x) &= x^{1/x}\frac{d}{dx}\left(\frac{1-\ln x}{x^2}\right) + \frac{1-\ln x}{x^2}\frac{d}{dx}x^{1/x} \\ &= x^{1/x}\frac{x^2\left(-\frac{1}{x}\right) - (1-\ln x)(2x)}{x^4} + x^{1/x}\left(\frac{1-\ln x}{x^2}\right)^2 \\ &= x^{1/x}\left[\frac{2\ln x - 3}{x^3} + \left(\frac{1-\ln x}{x^2}\right)^2\right]. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x = e$ and does not exist when $x = 0$. However, $x = 0$ is not in the domain of f, so $x = 0$ is not a critical number of f. Therefore, $\boxed{e}$ is the only critical number of f. At the point $(e, e^{1/e}) \approx (2.718, 1.445)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number e to divide the interval $(0, \infty)$ into two subintervals.

Interval	Sign of f'	Conclusion
$(0, e)$	$+$	f is increasing on $(0, e)$
(e, ∞)	$-$	f is decreasing on (e, ∞)

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at e. The $\boxed{\text{local maximum value is } f(e) = e^{1/e} \approx 1.445}$.

Step 6 The second derivative exists for all $x > 0$ and is equal to zero when $x \approx 0.582$ and when $x \approx 4.368$. These latter values were obtained using the command

```
Solve [ D [ x^(1/x), {x, 2} ] == 0, x ]
```

in *Wolfram Alpha*. Use the numbers 0.582 and 4.368 to divide the interval $(0, \infty)$ into three subintervals, and determine the sign of f'' on each subinterval.

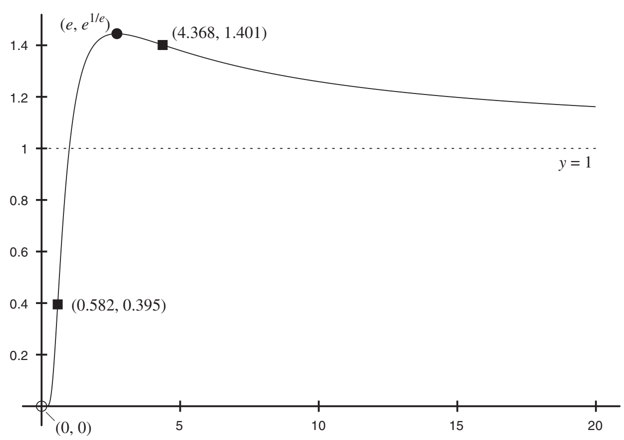
Interval	Sign of f''	Conclusion
$(0, 0.582)$	$+$	f is concave up on $(0, 0.582)$
$(0.582, 4.368)$	$-$	f is concave down on $(0.582, 4.368)$
$(4.368, \infty)$	$+$	f is concave up on $(4.368, \infty)$

The concavity of f changes at approximately 0.582 and 4.368, so the points

$$\boxed{(0.582, 0.395) \quad \text{and} \quad (4.368, 1.401)}$$

are approximate points of inflection of f.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



4.7 Optimization

Applications and Extensions

1. Let x denote the length of fencing used to enclose the plot on each side perpendicular to the highway and y the length of fencing used parallel to the highway (see the diagram below). The area A enclosed by the fencing is then $A = xy$. With 3000 m of fencing available, x and y are related by the equation

$$2x + y = 3000 \quad \text{or} \quad y = 3000 - 2x.$$

Substituting for y in the area formula yields

$$A = x(3000 - 2x) = 3000x - 2x^2.$$

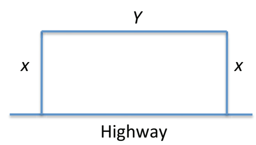
The domain of this function is the closed interval $[0, 1500]$. The function A is differentiable on the open interval $(0, 1500)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 3000 - 4x,$$

so $A'(x) = 0$ when $x = 750$. Evaluating A at the endpoints of the interval $[0, 1500]$ and at the critical number 750 yields

$$A(0) = 0, \quad A(750) = 1,125,000, \quad \text{and} \quad A(1500) = 0.$$

The largest area that can be enclosed is therefore $\boxed{1{,}125{,}000 \text{ m}^2}$, achieved by using 1500 m of fencing parallel to the highway and 750 m of fencing on each side perpendicular to the highway.



3. Let x and y denote the width and length of the rectangular plot enclosed on all sides by fencing (see the diagram below). The area A enclosed by the fencing is then $A = xy$. With L m of fencing available, x and y are related by the equation

$$2x + 2y = L \quad \text{or} \quad y = \frac{L}{2} - x.$$

Substituting for y in the area formula yields

$$A = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2.$$

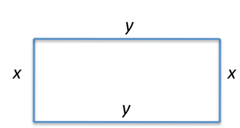
The domain of this function is the closed interval $\left[0, \frac{L}{2}\right]$. The function A is differentiable on the open interval $\left(0, \frac{L}{2}\right)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = \frac{L}{2} - 2x,$$

so $A'(x) = 0$ when $x = \frac{L}{4}$. Evaluating A at the endpoints of the interval $\left[0, \frac{L}{2}\right]$ and at the critical number $\frac{L}{4}$ yields

$$A(0) = 0, \quad A\left(\frac{L}{4}\right) = \frac{L^2}{16}, \quad \text{and} \quad A\left(\frac{L}{2}\right) = 0.$$

The largest area that can be enclosed is therefore $\frac{L^2}{16}$ m^2, achieved by using $L/4$ m of fencing on each side of the rectangle; that is, by making a square plot with dimensions $\boxed{\frac{L}{4} \text{ m} \times \frac{L}{4} \text{ m}}$.



5. Using the diagram in the text, let x denote the length of each of the three vertical sides and y denote the length of each of the two horizontal sides. The area A enclosed by the fencing is then $A = xy$. With 200 m of fencing available, x and y are related by the equation

$$2y + 3x = 200 \quad \text{or} \quad y = 100 - \frac{3}{2}x.$$

Substituting for y in the area formula yields

$$A = x\left(100 - \frac{3}{2}x\right) = 100x - \frac{3}{2}x^2.$$

The domain of this function is the closed interval $\left[0, \dfrac{200}{3}\right]$. The function A is differentiable on the open interval $\left(0, \frac{200}{3}\right)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 100 - 3x,$$

so $A'(x) = 0$ when $x = \dfrac{100}{3}$. Evaluating A at the endpoints of the interval $\left[0, \dfrac{200}{3}\right]$ and at the critical number $\dfrac{100}{3}$ yields

$$A(0) = 0, \quad A\left(\frac{100}{3}\right) = \frac{5000}{3}, \quad \text{and} \quad A\left(\frac{200}{3}\right) = 0.$$

The largest area that can be enclosed is therefore $\boxed{\dfrac{5000}{3}\text{ m}^2}$.

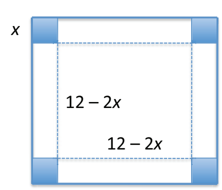
7. Let x denote the side length of the square cut from each corner of the base (see the diagram below). When the sides are turned up, the resulting box will have a square base $12 - 2x$ cm on a side and a height of x cm; the volume will therefore be $V = x(12 - 2x)^2 = 4x(6 - x)^2$. Because both $x \geq 0$ and $12 - 2x \geq 0$, it follows that $x \leq 6$, so the domain of V is the closed interval $[0, 6]$. The function V is differentiable on the open interval $(0, 6)$, so the critical numbers occur where $V'(x) = 0$. Now,

$$V'(x) = 4x \cdot 2(6 - x)(-1) + 4(6 - x)^2 = (6 - x)(-8x + 24 - 4x) = (6 - x)(24 - 12x),$$

so the only critical number inside the open interval $(0, 6)$ is $x = 2$, where $V'(2) = 0$. Note that $x = 6$ is not in the open interval $(0, 6)$. Evaluating V at the endpoints of the interval $[0, 6]$ and at the critical number 2 yields

$$V(0) = 0, \quad V(2) = 128, \quad \text{and} \quad V(6) = 0.$$

The largest volume is therefore achieved when $x = 2$, and the box with the largest volume has a square base $12 - 2(2) = 8$ cm on a side and a height of 2 cm; that is, the box with the largest possible volume has dimensions $\boxed{8 \text{ cm} \times 8 \text{ cm} \times 2 \text{ cm}}$.



9. Let s denote the side length of the square base and h denote the height of the open top box. The surface area A of the box is then

$$A = 4sh + s^2,$$

where the first term accounts for the area of the four vertical rectangular sides of the box and the second term accounts for the area of the base. Given that the box is to have a volume of 2000 cm^3, s and h are related by the equation

$$s^2h = 2000 \quad \text{or} \quad h = \frac{2000}{s^2}.$$

Substituting for h in the area formula yields

$$A = \frac{8000}{s} + s^2.$$

The domain of A is the interval $(0, \infty)$. Now,

$$A'(s) = -\frac{8000}{s^2} + 2s = \frac{2(s^3 - 4000)}{s^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $s = 10\sqrt[3]{4}$, where $A'(s) = 0$. Note that $s = 0$ is not in the domain of A. Using the Second Derivative Test,

$$A''(s) = \frac{16000}{s^3} + 2 \quad \text{so} \quad A''(10\sqrt[3]{4}) = 6 > 0$$

and A has a local minimum value at $10\sqrt[3]{4}$. Because $A''(s) > 0$ for all $s > 0$, the local minimum is also an absolute minimum. Therefore, the minimum amount of material is used to make the box when $s = 10\sqrt[3]{4}$ and

$$h = \frac{2000}{100\sqrt[3]{16}} = \frac{20\sqrt[3]{4}}{4} = 5\sqrt[3]{4}.$$

The box should have a square base measuring $10\sqrt[3]{4} \approx 15.874$ cm on a side and a height of $5\sqrt[3]{4} \approx 7.937$ cm; that is, the dimensions of the box using the minimum amount of material are

$$\boxed{10\sqrt[3]{4} \times 10\sqrt[3]{4} \times 5\sqrt[3]{4} \approx 15.874 \text{ cm } \times 15.874 \text{ cm } \times 7.937 \text{ cm}}.$$

11. Let r denote the radius and h the height of the cylindrical can. The cost C of the material needed to manufacture the can is

$$C = 2\pi r^2(20) + 2\pi rh(15) = 40\pi r^2 + 30\pi rh,$$

where the first term accounts for the cost of the top and bottom of the can and the second term accounts for the lateral surface area of the can. Given that the can is to have a capacity of the 10 m^3, r and h are related by the equation

$$\pi r^2 h = 10 \quad \text{or} \quad h = \frac{10}{\pi r^2}.$$

Substituting for h in the cost formula yields

$$C = 40\pi r^2 + \frac{300}{r}.$$

The domain of C is the interval $(0, \infty)$. Now,

$$C'(r) = 80\pi r - \frac{300}{r^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $r = \sqrt[3]{\frac{15}{4\pi}}$, where $C'(r) = 0$. Note that $r = 0$ is not in the domain of C. Using the Second Derivative Test,

$$C''(r) = 80\pi + \frac{600}{r^3} \quad \text{so} \quad C''\left(\sqrt[3]{\frac{15}{4\pi}}\right) = 240\pi > 0,$$

and C has a local minimum value at $\sqrt[3]{\frac{15}{4\pi}}$. Because $C''(r) > 0$ for all $r > 0$, the local minimum is also an absolute minimum. Therefore, the minimum cost of material is achieved when $r = \sqrt[3]{\frac{15}{4\pi}}$ and

$$h = \frac{10}{\pi \sqrt[3]{\frac{225}{16\pi^2}}} = \frac{8}{3}\sqrt[3]{\frac{15}{4\pi}}.$$

The can should have a $\boxed{\text{radius of } \sqrt[3]{\frac{15}{4\pi}} \approx 1.061 \text{ m}}$ and a $\boxed{\text{height of } \frac{8}{3}\sqrt[3]{\frac{15}{4\pi}} \approx 2.829 \text{ m}}$.

13. Let x denote the number of \$1 increases in the rental price above the original \$18 per day price. The rental price is then $18 + x$ and the number of cars rented is $24 - x$ so the rental income I is

$$I = (18 + x)(24 - x) = 432 + 6x - x^2.$$

The domain of I is the closed interval $[0, 24]$. The function I is differentiable on the open interval $(0, 24)$, so the critical numbers occur where $I'(x) = 0$. Now,

$$I'(x) = 6 - 2x,$$

so $I'(x) = 0$ when $x = 3$. Evaluating I at the endpoints of the interval $[0, 24]$ and at the critical number 3 yields

$$I(0) = 432, \quad I(3) = 21^2 = 441, \quad \text{and} \quad I(24) = 0.$$

Rental income is therefore maximized when $x = 3$, meaning that the agency should charge $\boxed{18 + 3 = \$21 \text{ per day}}$ to maximize income.

15. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point on the graph of the parabola $y = x^2$ and the point $\left(2, \frac{1}{2}\right)$. Then

$$\begin{aligned} D &= (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2 \\ &= x^2 - 4x + 4 + x^4 - x^2 + \frac{1}{4} \\ &= x^4 - 4x + \frac{17}{4}. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 4x^3 - 4 = 4(x-1)(x^2 + x + 1),$$

so $x = 1$ is the only critical number. Using the Second Derivative Test,

$$D''(x) = 12x^2 \quad \text{so} \quad D''(1) = 12 > 0$$

and D has a local minimum value at 1. Because $D''(x) \geq 0$ for all x, the local minimum is also an absolute minimum. The point $\boxed{(1, 1^2) = (1, 1)}$ on the graph of the parabola $y = x^2$ is closest to the point $\left(2, \frac{1}{2}\right)$.

17. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point on the graph of the parabola $y = 4 - x^2$ and the point $(6, 2)$. Then

$$\begin{aligned} D &= (x-6)^2 + (y-2)^2 = (x-6)^2 + (4 - x^2 - 2)^2 \\ &= x^2 - 12x + 36 + 4 - 4x^2 + x^4 \\ &= x^4 - 3x^2 - 12x + 40. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 4x^3 - 6x - 12.$$

Using the computer algebra system *Maple*, the only critical number is $x \approx 1.784$. Using the Second Derivative Test,

$$D''(x) = 12x^2 - 6 \quad \text{so} \quad D''(1.784) \approx 32.192 > 0$$

and D has a local minimum value at 1.784. As

$$\lim_{x \to \pm\infty} D(x) = \infty,$$

it follows that the local minimum is also an absolute minimum. The point $\boxed{(1.784, 4 - 1.784^2) \approx (1.784, 0.817)}$ on the graph of the parabola $y = 4 - x^2$ is closest to the point $(6, 2)$.

19. Let x denote the traveling speed of a truck and let

$$C(x) = \left(\frac{1600}{x} + x\right)a + \frac{200}{x}b + c$$

be the cost associated with a 200-mile trip where a is the cost per gallon for gasoline, b is the hourly rate paid to the driver, and c is a commission paid to the driver. With a top speed of 75 mph, the domain of C is the interval $(0, 75]$. The critical numbers of C occur where $C'(x) = 0$ or where $C'(x)$ does not exist. Now,

$$C'(x) = \left(1 - \frac{1600}{x^2}\right)a - \frac{200}{x^2}b,$$

so $C'(x)$ is equal to zero when

$$x = \pm\sqrt{\frac{1600a + 200b}{a}}$$

and does not exist when $x = 0$. Of these numbers, only $\sqrt{\frac{1600a + 200b}{a}}$ lies inside the interval $(0, 75)$. Using the Second Derivative Test,

$$C''(x) = \frac{3200}{x^3}a + \frac{400}{x^3}b > 0$$

for all x in $(0, 75)$, so C has a local and an absolute minimum at

$$x = \sqrt{\frac{1600a + 200b}{a}}.$$

(a) With $a = \$3.50$, $b = 0$, and $c = 0$, the speed that minimizes cost is

$$x = \sqrt{\frac{1600(3.50) + 200(0)}{3.50}} = \sqrt{\frac{5600}{3.50}} = \sqrt{1600} = \boxed{40 \text{ miles per hour}}.$$

(b) With $a = \$3.50$, $b = \$10.00$, and $c = \$500$, the speed that minimizes cost is

$$x = \sqrt{\frac{1600(3.50) + 200(10.00)}{3.50}} = \sqrt{\frac{7600}{3.50}} \approx \boxed{46.6 \text{ miles per hour}}.$$

(c) With $a = \$4.00$, $b = \$20.00$, and $c = 0$, the speed that minimizes cost is

$$x = \sqrt{\frac{1600(4.00) + 200(20.00)}{4.00}} = \sqrt{\frac{10400}{4.00}} = \sqrt{2600} \approx \boxed{51.0 \text{ miles per hour}}.$$

21. (a) Let x denote the distance from the point on the road closest to house A to the point where the path from house A to house B meets the road. Then the length L of the path from house A to house B is

$$L = \sqrt{q^2 + x^2} + \sqrt{(p - x)^2 + r^2},$$

where the first term accounts for the distance from house A to the road and the second term accounts for the distance from the road to house B. The domain of this function is the closed interval $[0, p]$, and the critical numbers occur where $L'(x) = 0$. Now,

$$L'(x) = \frac{x}{\sqrt{q^2 + x^2}} - \frac{p - x}{\sqrt{(p - x)^2 + r^2}},$$

so $L'(x) = 0$ when

$$\begin{aligned}
\frac{x}{\sqrt{q^2 + x^2}} &= \frac{p - x}{\sqrt{(p - x)^2 + r^2}} \\
x^2[(p - x)^2 + r^2] &= (p - x)^2(q^2 + x^2) \\
x^2 r^2 &= (p - x)^2 q^2 \\
xr &= |p - x|q = (p - x)q \quad \text{because } p \geq x \\
x &= \frac{pq}{q + r}.
\end{aligned}$$

Evaulating L at the endpoints of the interval $[0, p]$ and at the critical number $\dfrac{pq}{q + r}$ yields

$$\begin{aligned}
L(0) &= q + \sqrt{r^2 + p^2}, \\
L\left(\frac{pq}{q + r}\right) &= \sqrt{q^2 + \left(\frac{pq}{q + r}\right)^2} + \sqrt{\left(p - \frac{pq}{q + r}\right)^2 + r^2} \\
&= \sqrt{q^2 + \left(\frac{pq}{q + r}\right)^2} + \sqrt{\left(\frac{pr}{q + r}\right)^2 + r^2} \\
&= \sqrt{\frac{q^2}{(q + r)^2}[(q + r)^2 + p^2]} + \sqrt{\frac{r^2}{(q + r)^2}[p^2 + (q + r)^2]} \\
&= \frac{q}{q + r}\sqrt{p^2 + (q + r)^2} + \frac{r}{q + r}\sqrt{p^2 + (q + r)^2} = \sqrt{p^2 + (q + r)^2}, \quad \text{and} \\
L(p) &= \sqrt{q^2 + p^2} + r.
\end{aligned}$$

Because p, q, and r are all positive numbers,

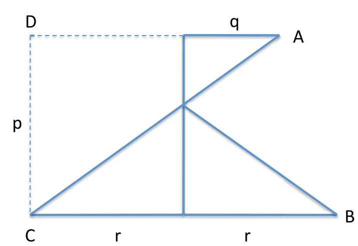
$$q+\sqrt{r^2+p^2} = \sqrt{q^2+2q\sqrt{r^2+p^2}+r^2+p^2} > \sqrt{q^2+2qr+r^2+p^2} = \sqrt{(q+r)^2+p^2},$$

and

$$\sqrt{q^2+p^2}+r = \sqrt{q^2+p^2+2r\sqrt{q^2+p^2}+r^2} > \sqrt{q^2+2qr+r^2+p^2} = \sqrt{(q+r)^2+p^2}.$$

Therefore, the length of the shortest path that goes from house A to the road and then on to house B is $\boxed{\sqrt{(q+r)^2+p^2}}$.

(b) Following the hint, reflect the point B across the road to the point C (see the diagram below). Note that any path from A to C is equivalent to (has the same distance as) the path from A to B obtained by reflecting the portion of the path to the left of the road across to the right side of the road. Additionally, the shortest path from A to C is a straight line, and therefore has a distance equal to the hypotenuse of triangle ADC: $\sqrt{(q+r)^2+p^2}$. Therefore, the length of the shortest path from A to the road and then on to B is $\boxed{\sqrt{(q+r)^2+p^2}}$, matching the result from part (a).



23. The shortest beam will simultaneously be in contact with the wall that is being supported, the 2-m high wall in the middle, and the ground. Let x and h be as shown in the diagram below. The length of the ladder is then

$$L = \sqrt{(x+5)^2+h^2}.$$

By similar triangles,

$$\frac{x}{2} = \frac{x+5}{h} \quad \text{so} \quad h = \frac{2(x+5)}{x}.$$

Substituting for h in the length formula yields

$$L = \sqrt{(x+5)^2+\left(\frac{2(x+5)}{x}\right)^2} = \frac{x+5}{x}\sqrt{x^2+4} = \left(1+\frac{5}{x}\right)\sqrt{x^2+4}.$$

The domain of L is the interval $(0,\infty)$. Now,

$$L'(x) = \left(1+\frac{5}{x}\right)\cdot\frac{x}{\sqrt{x^2+4}} + \sqrt{x^2+4}\cdot\left(-\frac{5}{x^2}\right) = \frac{x^3+5x^2-5x^2-20}{x^2\sqrt{x^2+4}} = \frac{x^3-20}{x^2\sqrt{x^2+4}},$$

so the only critical number inside the open interval $(0,\infty)$ is $x=\sqrt[3]{20}$, where $L'(x)=0$. Note that $x=0$ is not in the domain of L. Using the Second Derivative Test,

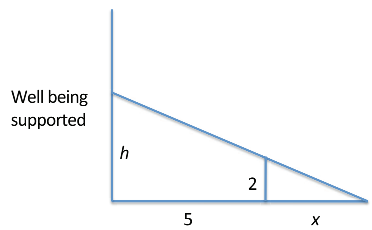
$$\begin{aligned} L''(x) &= \frac{x^2\sqrt{x^2+4}\cdot 3x^2-(x^3-20)\cdot\left(\frac{x^3}{\sqrt{x^2+4}}+2x\sqrt{x^2+4}\right)}{x^4(x^2+4)} \\ &= \frac{3x^4(x^2+4)-x^3(x^3-20)-2x(x^3-20)(x^2+4)}{x^4(x^2+4)^{3/2}} \\ &= \frac{3x^6+12x^4-x^6+20x^3-2x^6-8x^4+40x^3+160x}{x^4(x^2+4)^{3/2}} \\ &= \frac{4x^4+60x^3+160x}{x^4(x^2+4)^{3/2}} = \frac{4x^3+60x^2+160}{x^3(x^2+4)^{3/2}}, \end{aligned}$$

so $L''(x) > 0$ for all $x > 0$ and L has a local and absolute minimum at $\sqrt[3]{20}$. The length of the shortest beam that can be used to brace the wall is therefore

$$L = \left(1 + \frac{5}{\sqrt[3]{20}}\right)\sqrt{\sqrt[3]{400} + 4} \approx \boxed{9.582 \text{ meters}}.$$

The angle of elevation of the beam is

$$\tan^{-1}\frac{2}{\sqrt[3]{20}} \approx 0.635 \text{ rad} \approx \boxed{36.383°}.$$



25. Using the hint (that is, choosing the width of the beam to be $2x$ and the depth of the beam to be $2y$) gives

$$S = k(2x)(2y)^2 = 8kxy^2,$$

where k is a positive constant of proportionality. Solve the equation of the cross section, $10x^2 + 9y^2 = 90$, for $y^2 = 10 - \frac{10}{9}x^2$ and then substitute this expression into the formula for S, to obtain

$$S = 8kx\left(10 - \frac{10}{9}x^2\right) = 80kx - \frac{80}{9}kx^3.$$

The domain of this function is the closed interval $[0, 3]$, and the critical numbers occur where $S'(x) = 0$. Now,

$$S'(x) = 80k - \frac{80}{3}kx^2,$$

so the only critical number inside the open interval $(0, 3)$ is $x = \sqrt{3}$, where $S'(x) = 0$. Note that $x = -\sqrt{3}$ is not in the domain of S. Evaluating S at the endpoints of the interval $[0, 3]$ and at the critical number $\sqrt{3}$ yields

$$S(0) = 0, \quad S(\sqrt{3}) = 80k\sqrt{3} - 80k\frac{\sqrt{3}}{3} = \frac{160k\sqrt{3}}{3}, \quad \text{and} \quad S(3) = 0.$$

Therefore, S achieves its absolute maximum value when $x = \sqrt{3}$ and $y = \sqrt{10 - \frac{10}{3}} = \frac{2\sqrt{15}}{3}$. The strongest beam that can be cut from a log whose cross section has the form of the ellipse $10x^2 + 9y^2 = 90$ has a $\boxed{\text{width of } 2\sqrt{3}}$ and a $\boxed{\text{depth of } \frac{4\sqrt{15}}{3}}$.

27. Let x denote the number of cases produced and p denote the price of a case. Solve the equation

$$x = 1430 - \frac{11}{6}p \quad \text{for} \quad p = 780 - \frac{6}{11}x,$$

and then substitute for p in the formula

$$P = xp - 132x = x\left(780 - \frac{6}{11}x\right) - 132x = 648x - \frac{6}{11}x^2.$$

The domain for P is the closed interval $[0, 1100]$, and the critical numbers occur where $P'(x) = 0$. Now,

$$P'(x) = 648 - \frac{12}{11}x,$$

so $x = 594$ is the only critical number. Evaluating P at the endpoints of the interval $[0, 1100]$ and at the critical number 594 yields

$$P(0) = 0, \quad P(594) = 192,456, \quad \text{and} \quad P(1100) = 52,800.$$

Therefore, P achieves its absolute maximum when $x = 594$ and

$$p = 780 - \frac{6}{11}(594) = 456.$$

To maximize profit, the winemaker should produce $\boxed{\text{594 cases of wine}}$ and sell them for $\boxed{\text{\$456 per case}}$.

29. Maximizing the capacity of the trough will be accomplished by maximizing the cross-sectional area of the trough. Let θ denote the angle between the two sides of the V-shaped trough. The cross-sectional area A of the trough is then

$$A = \frac{1}{2}28^2 \sin\theta = 392 \sin\theta.$$

The domain of A is the closed interval $[0, \pi]$, and the critical numbers occur where $A'(\theta) = 0$. Now,

$$A'(\theta) = 392 \cos\theta,$$

so $\theta = \dfrac{\pi}{2}$ is the only critical number. Evaluating A at the endpoints of the interval $[0, \pi]$ and at the critical number $\dfrac{\pi}{2}$ yields

$$A(0) = 0, \quad A\left(\frac{\pi}{2}\right) = 392, \quad \text{and} \quad A(\pi) = 0.$$

Therefore, A achieves its absolute maximum when $\theta = \dfrac{\pi}{2}$, so the capacity of the trough is maximum when the angle between the sides is $\boxed{\dfrac{\pi}{2}}$.

31. Let r denote the radius of the semicircular ceiling (so that the width of the floor is $2r$) and h denote the height of the vertical walls. Without loss of generality suppose that the floor and vertical walls cost one unit per square meter so that the ceiling costs three units per square meter. The cost C for each one-meter long section of the tunnel is then

$$C = 2h(1) + 2r(1) + \pi r(3) = 2h + 2r + 3\pi r,$$

where the first term accounts for the cost of the vertical walls, the second term accounts for the cost of the floor, and the final term accounts for the cost of the ceiling. Let A denote the fixed cross-sectional area of the tunnel, so that

$$A = 2rh + \frac{1}{2}\pi r^2 \quad \text{or} \quad h = \frac{A - \frac{1}{2}\pi r^2}{2r} = \frac{A}{2r} - \frac{\pi r}{4}.$$

Substituting for h in the cost formula yields

$$C = 2\left(\frac{A}{2r} - \frac{\pi r}{4}\right) + 2r + 3\pi r = \frac{A}{r} + \left(\frac{5\pi}{2} + 2\right)r.$$

In order for h to be non-negative, we must have

$$\frac{A}{2r} - \frac{\pi r}{4} \geq 0 \quad \text{or} \quad r \leq \sqrt{\frac{2A}{\pi}}.$$

The domain of C is therefore the interval $\left(0, \sqrt{\frac{2A}{\pi}}\right]$. Now,

$$C'(r) = -\frac{A}{r^2} + \left(\frac{5\pi}{2} + 2\right),$$

so the only critical number inside the open interval $\left(0, \sqrt{\frac{2A}{\pi}}\right)$ is $r = \sqrt{\frac{2A}{5\pi+4}}$, where $C'(r) = 0$. Note that $r = 0$ is not in the domain of C. Using the Second Derivative Test,

$$C''(r) = \frac{2A}{r^3} > 0$$

for all $r > 0$. Therefore, C has both a local and an absolute minimum when $r = \sqrt{\frac{2A}{5\pi+4}}$. Finally, the most economical ratio of the diameter of the semicircular cylinder to the height of the vertical walls is

$$\frac{2r}{h} = \frac{4r^2}{A - \frac{1}{2}\pi r^2} = \frac{4 \cdot \frac{2A}{5\pi+4}}{A - \frac{1}{2}\pi \cdot \frac{2A}{5\pi+4}} = \frac{\frac{8}{5\pi+4}}{1 - \frac{\pi}{5\pi+4}} = \frac{8}{4\pi+4} = \boxed{\frac{2}{\pi+1}}.$$

33. Let the weaker light source have intensity I^* and be located at $x = 0$. The other light source has intensity $8I^*$ and is located at $x = 6$. For $0 < x < 6$, the intensity of light I at location x is

$$I = \frac{kI^*}{x^2} + \frac{8kI^*}{(6-x)^2},$$

where k is the constant of proportionality in the inverse square law, the first term accounts for illumination from the weaker light source and the second term accounts for illumination from the stronger source. Now,

$$I'(x) = -\frac{2kI^*}{x^3} + \frac{16kI^*}{(6-x)^3},$$

so $I'(x)$ is equal to zero when

$$\begin{aligned} \frac{2kI^*}{x^3} &= \frac{16kI^*}{(6-x)^3} \\ (6-x)^3 &= 8x^3 \\ 6-x &= 2x \\ x &= 2. \end{aligned}$$

Note that neither $x = 0$ nor $x = 6$ are in the domain of I, so 2 is the only critical number. Using the Second Derivative Test,

$$I''(x) = \frac{6kI^*}{x^4} + \frac{48kI^*}{(6-x)^4} > 0$$

for all x inside the interval $(0, 6)$. Therefore, I has both a local and an absolute minimum when $x = 2$, so the point between the two light sources at which illumination is minimum is $\boxed{\text{2 meters from the weaker light source}}$.

35. Let x denote the side length of the square and r denote the radius of the circle made from the two pieces of wire. The length L of the wire is then

$$L = 4x + 2\pi r.$$

Given that the area enclosed by the two pieces of wire is to be 64 cm^2, x and r are related by the equation

$$x^2 + \pi r^2 = 64 \quad \text{or} \quad x = \sqrt{64 - \pi r^2}.$$

In order to have x defined, we must have

$$\pi r^2 \leq 64 \quad \text{or} \quad r \leq \frac{8}{\sqrt{\pi}}.$$

Substituting for x in the length formula yields

$$L = 4\sqrt{64 - \pi r^2} + 2\pi r,$$

where the domain is the closed interval $\left[0, \frac{8}{\sqrt{\pi}}\right]$. Now,

$$L'(r) = 4\frac{-\pi r}{\sqrt{64 - \pi r^2}} + 2\pi,$$

so the only critical number inside the open interval $\left(0, \frac{8}{\sqrt{\pi}}\right)$ occurs when

$$\begin{aligned} \frac{4\pi r}{\sqrt{64 - \pi r^2}} &= 2\pi \\ 2r &= \sqrt{64 - \pi r^2} \\ 4r^2 &= 64 - \pi r^2 \\ (4 + \pi)r^2 &= 64 \\ r &= \frac{8}{\sqrt{4 + \pi}}. \end{aligned}$$

Note that neither $r = -\frac{8}{\sqrt{\pi}}$ nor $r = -\frac{8}{\sqrt{4 + \pi}}$ is in the domain of L, and $r = \frac{8}{\sqrt{\pi}}$ is not inside the open interval $\left(0, \frac{8}{\sqrt{\pi}}\right)$. Evaluating L at the endpoints of the interval $\left[0, \frac{8}{\sqrt{\pi}}\right]$ and at the critical number $\frac{8}{\sqrt{4 + \pi}}$ yields

$$L(0) = 32, \quad L\left(\frac{8}{\sqrt{4 + \pi}}\right) = \frac{64 + 16\pi}{\sqrt{4 + \pi}} \approx 42.758, \quad \text{and} \quad L\left(\frac{8}{\sqrt{\pi}}\right) = 16\sqrt{\pi} \approx 28.359.$$

Therefore, the

$$\boxed{\text{minimum length of wire that can be used is approximately 28.359 cm}},$$

and the

$$\boxed{\text{maximum length wire that can be used is approximately 42.758 cm}}.$$

37. Let x denote the length of wire used to make the square. Then the side length of the square is $x/4$, the length of wire used to make the circle is $35 - x$ and the radius of the resulting circle is $\frac{35 - x}{2\pi}$. The area A enclosed by the two figures is then

$$A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{35 - x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(35 - x)^2}{4\pi}.$$

The domain of this function is the closed interval $[0, 35]$, and the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = \frac{x}{8} - \frac{35 - x}{2\pi},$$

so $A'(x) = 0$ when

$$\begin{aligned} \frac{x}{8} &= \frac{35 - x}{2\pi} \\ 2\pi x &= 280 - 8x \\ x &= \frac{140}{4 + \pi}. \end{aligned}$$

Evaluating A at the endpoints of the interval $[0, 35]$ and at the critical number $\frac{140}{4+\pi}$ yields

$$\begin{aligned} A(0) &= \frac{35^2}{4\pi} \approx 97.482; \\ A\left(\frac{140}{4+\pi}\right) &= \frac{1}{16}\left(\frac{140}{4+\pi}\right)^2 + \frac{1}{4\pi}\left(35 - \frac{140}{4+\pi}\right)^2 \\ &= \frac{70^2}{4(4+\pi)^2} + \frac{35^2\pi^2}{4\pi(4+\pi)^2} = \frac{70^2 + 35^2\pi}{4(4+\pi)^2} \approx 42.883; \text{ and} \\ A(35) &= \frac{35^2}{16} = 76.5625. \end{aligned}$$

(a) To enclose the minimum area, the wire should be cut so that

$$\frac{140}{4+\pi} \approx \boxed{19.603 \text{ cm}}$$

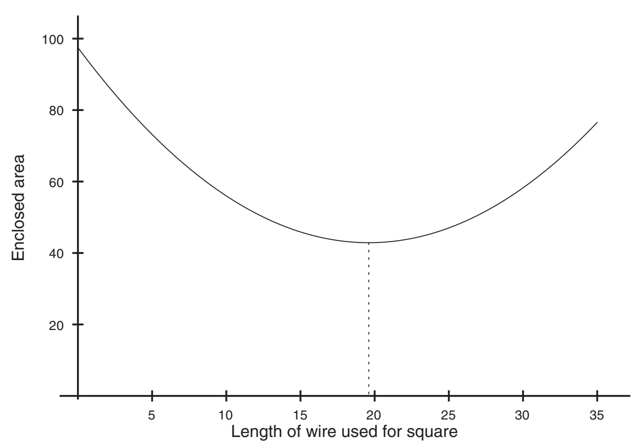
are used for the square and the remaining

$$35 - \frac{140}{4+\pi} = \frac{35\pi}{4+\pi} \approx \boxed{15.397 \text{ cm}}$$

are used for the circle.

(b) To enclose the maximum area, the **entire length of wire should be formed into a circle** since this corresponds to choosing $x = 0$.

(c) The figure below displays the graph of the area enclosed as a function of the length of the wire used to make the square. The absolute minimum occurs for x a little less than 20 cm, confirming the result of part (a); the absolute maximum occurs for $x = 0$, confirming the result of part (b).



39. Let r denote the radius and h the height of the cylindrical can. The surface area A is then

$$A = 2\pi r^2 + 2\pi rh,$$

where the first term accounts for the area of the top and bottom of the can and the second term accounts for the lateral surface area. Given that the volume of the can is fixed to be V, r and h are related by the equation

$$V = \pi r^2 h \quad \text{or} \quad h = \frac{V}{\pi r^2}.$$

Substituting for h in the area formula yields

$$A = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

The domain for this function is the interval $(0, \infty)$. Now,

$$A'(r) = 4\pi r - \frac{2V}{r^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $r = \sqrt[3]{\frac{V}{2\pi}}$, where $A'(r) = 0$. Note that $r = 0$ is not in the domain of A. Using the Second Derivative Test,

$$A''(r) = 4\pi + \frac{4V}{r^3} > 0$$

for all $r > 0$. Therefore, A has a local and an absolute minimum at $\sqrt[3]{\frac{V}{2\pi}}$. When $r = \sqrt[3]{\frac{V}{2\pi}}$,

$$h = \frac{V}{\pi \sqrt[3]{\left(\frac{V}{2\pi}\right)^2}} = \frac{V \sqrt[3]{\frac{V}{2\pi}}}{\pi \frac{V}{2\pi}} = 2\sqrt[3]{\frac{V}{2\pi}} = 2r,$$

so the cylindrical container of fixed volume which uses the least material has a height that is twice its radius.

41. Let $(x, y) = (x, 9 - x^2)$ be the coordinates of the vertex of the rectangle that lies on the graph of the parabola. Then the width of the rectangle is x, the height is $9 - x^2$, and the area is

$$A = x(9 - x^2) = 9x - x^3.$$

The domain of this function is the closed interval $[0, 3]$, and the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 9 - 3x^2,$$

so the only critical number inside the open interval $(0, 3)$ is $x = \sqrt{3}$, where $A'(x) = 0$. Note that $x = -\sqrt{3}$ is not in the domain of A. Evaluating A at the endpoints of the interval $[0, 3]$ and at the critical number $\sqrt{3}$ yields

$$A(0) = 0, \quad A(\sqrt{3}) = 6\sqrt{3}, \quad \text{and} \quad A(3) = 0.$$

Therefore, A has an absolute maximum value of $6\sqrt{3}$; in other words, the largest area of a rectangle with one vertex on the parabola $y = 9 - x^2$, another at the origin, and the remaining two on the positive coordinate axes is $\boxed{6\sqrt{3} \text{ square units}}$.

43. Let a and b be positive real numbers and let m denote the slope of a line through the point (a, b).

- Case I: $m > 0$.
  Let $m = b/a$. The line $\boxed{y = mx = bx/a}$ passes through the origin, so $x_0 = y_0 = 0$ and the distance between the points $(x_0, 0)$ and $(0, y_0)$ is zero, an absolute minimum.
- Case II: $m < 0$ so that $x_0 > 0$ and $y_0 > 0$.
  The equation of the line of slope m which passes through the point (a, b) is

$$y - b = m(x - a) \quad \text{or} \quad y = mx + b - ma.$$

The y-intercept of this line is $y_0 = b - ma$ and the x-intercept is $x_0 = \dfrac{am - b}{m}$, so the distance D between the points $(x_0, 0)$ and $(0, y_0)$ is

$$D = \sqrt{\left(\frac{am - b}{m}\right)^2 + (b - ma)^2} = |b - ma|\sqrt{1 + \frac{1}{m^2}} = (b - ma)\sqrt{1 + \frac{1}{m^2}},$$

where the absolute value could be removed because $a > 0$, $b > 0$, and $m < 0$ so that $b - ma > 0$ The critical numbers of D occur where $D'(m) = 0$. Now,

$$D'(m) = (b - ma) \cdot \frac{-\frac{2}{m^3}}{2\sqrt{1 + \frac{1}{m^2}}} - a\sqrt{1 + \frac{1}{m^2}} = -\frac{1}{\sqrt{1 + \frac{1}{m^2}}}\left(\frac{b - ma}{m^3} + a + \frac{a}{m^2}\right),$$

so $D'(m) = 0$ when

$$\begin{aligned} \frac{b - ma}{m^3} &= -\left(a + \frac{a}{m^2}\right) \\ b - ma &= -am^3 - ma \\ m &= -\sqrt[3]{\frac{b}{a}}. \end{aligned}$$

As there is just the one critical number and

$$\lim_{m \to 0^-} D(m) = \infty \quad \text{and} \quad \lim_{m \to -\infty} D(m) = \infty,$$

D has an absolute minimum at $-\sqrt[3]{\dfrac{b}{a}}$. The line

$$\boxed{y = -\sqrt[3]{\frac{b}{a}}x + b + a\sqrt[3]{\frac{b}{a}}}$$

produces the minimum distance between the points $(x_0, 0)$ and $(0, y_0)$, where x_0 and y_0 are the x- and y-intercepts, respectively, of the line.

45. Let

$$F = \frac{cmg}{c\sin\theta + \cos\theta},$$

where the domain is the closed interval $[0, \pi/2]$. The critical numbers of F occur where $F'(\theta) = 0$ or where $F'(\theta)$ does not exist. Now,

$$F'(\theta) = \frac{-cmg}{(c\sin\theta + \cos\theta)^2} \cdot (c\cos\theta - \sin\theta).$$

$F'(\theta)$ exists for each θ in the open interval $(0, \pi/2)$ and is equal to zero when

$$c\cos\theta - \sin\theta = 0 \quad \text{or} \quad \tan\theta = c.$$

Based on the diagram below, when $\tan\theta = c$,

$$\sin\theta = \frac{c}{\sqrt{c^2+1}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{c^2+1}},$$

so, evaluating F at the endpoints of the interval $[0, \pi/2]$ and at the critical number associated with $\tan\theta = c$ yields

$$F(0) = cmg, \quad F(\tan^{-1} c) = \frac{cmg}{\sqrt{c^2+1}}, \quad \text{and} \quad F\left(\frac{\pi}{2}\right) = mg.$$

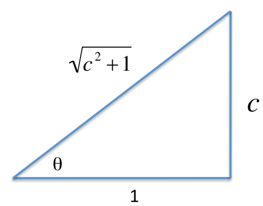
Because

$$\frac{c}{\sqrt{c^2+1}} < c \quad \text{its follows that} \quad F(\tan^{-1} c) = mg \cdot \frac{c}{\sqrt{c^2+1}} < cmg = F(0),$$

and because

$$\frac{c}{\sqrt{c^2+1}} < 1 \quad \text{its follows that} \quad F(\tan^{-1} c) = mg \cdot \frac{c}{\sqrt{c^2+1}} < mg = F\left(\frac{\pi}{2}\right),$$

so F has an absolute minimum when $\tan\theta = c$.



47. Let x denote the distance from the observer to the wall, and let the angles θ and ψ be as noted in the diagram below. Then

$$\tan\theta = \tan\left((\theta+\psi) - \psi\right) = \frac{\tan(\theta+\psi) - \tan\psi}{1 + \tan(\theta+\psi)\tan\psi} = \frac{\frac{7}{x} - \frac{3}{x}}{1 + \frac{7}{x}\cdot\frac{3}{x}} = \frac{4x}{x^2+21}$$

and

$$\theta = \tan^{-1}\left(\frac{4x}{x^2+21}\right).$$

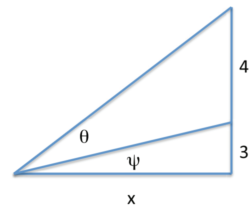
The domain of this function is the interval $(0, \infty)$, and the critical numbers occur where $\theta'(x) = 0$. Now,

$$\begin{aligned} \theta'(x) &= \frac{1}{1 + \left(\frac{4x}{x^2+21}\right)^2} \cdot \frac{(x^2+21)\cdot 4 - 4x\cdot 2x}{(x^2+21)^2} \\ &= \frac{4x^2 + 84 - 8x^2}{(x^2+21)^2 + 16x^2} = 4\frac{21 - x^2}{(x^2+21)^2 + 16x^2}, \end{aligned}$$

so the only critical number inside the open interval $(0, \infty)$ is $x = \sqrt{21}$, where $\theta'(x) = 0$. Note that $x = -\sqrt{21}$ is not in the domain of θ. As there is just this one critical number and

$$\lim_{x\to 0^+} \theta(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} \theta(x) = 0,$$

θ has an absolute maximum at $\sqrt{21}$. Therefore, to obtain the most favorable view of the picture, the observer should stand $\boxed{\sqrt{21} \approx 4.583 \text{ meters}}$ from the wall.



49. Let a, b, and c be positive constants and consider the function $f(x) = ae^{cx} + be^{-cx}$. Then

$$f'(x) = ace^{cx} - bce^{-cx},$$

and the only critical number occurs when

$$\begin{aligned} ace^{cx} &= bce^{-cx} \\ e^{2cx} &= \frac{b}{a} \\ x &= \frac{1}{2c}\ln\frac{b}{a}. \end{aligned}$$

Next,

$$f''(x) = ac^2e^{cx} + bc^2e^{-cx} \geq 0$$

for all x. Therefore, f achieves an absolute minimum value when $x = \dfrac{1}{2c}\ln\dfrac{b}{a}$, and that absolute minimum value is

$$f\left(\frac{1}{2c}\ln\frac{b}{a}\right) = ae^{\frac{1}{2}\ln\frac{b}{a}} + be^{-\frac{1}{2}\ln\frac{b}{a}} = a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} = 2\sqrt{ab}.$$

Challenge Problems

51. Based on the diagram in the text,

$$\cos\theta = \frac{20}{s} \quad \text{so} \quad s = \frac{20}{\cos\theta}$$

and

$$I = \frac{\sin\theta}{s} = \frac{\sin\theta\cos\theta}{20} = \frac{\sin(2\theta)}{40}.$$

The domain of this function is the interval $[0, \pi/2)$. Note that s is undefined for $\theta = \pi/2$, so this number cannot be included in the domain. Now,

$$I'(\theta) = \frac{1}{20}\cos(2\theta),$$

so $I'(\theta) = 0$ on the open interval $(0, \pi/2)$ when $\theta = \pi/4$. Because

$$I(0) = 0, \quad I\left(\frac{\pi}{4}\right) = \frac{1}{40}, \quad \text{and} \quad \lim_{\theta\to\pi/2^-} I(\theta) = 0,$$

it follows that I has an absolute maximum when $\theta = \pi/4$. If h is the height of the lamp, then

$$\tan\theta = \frac{h}{20} \quad \text{so} \quad h = 20\tan\theta.$$

For maximum illumination on the walk, the height of the lamp should be

$$h = 20\tan\frac{\pi}{4} = \boxed{20 \text{ ft}}.$$

53. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point (x, y) on the graph of $y = e^{-x/2}$ and the point $(1, 8)$. Then

$$D = (x-1)^2 + (y-8)^2 = (x-1)^2 + (e^{-x/2} - 8)^2.$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 2(x-1) + 2(e^{-x/2} - 8) \cdot \left(-\frac{1}{2}e^{-x/2}\right) = 2(x-1) - e^{-x/2}(e^{-x/2} - 8).$$

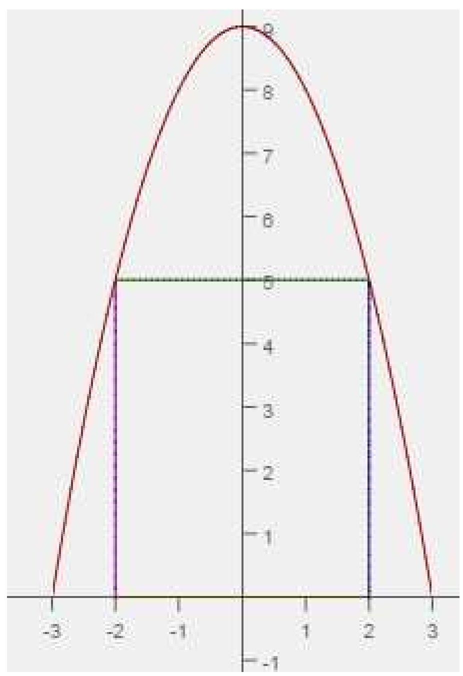
Using the computer algebra system *Maple*, the only critical number is $x \approx -3.758$. As

$$\lim_{x \to \pm\infty} D(x) = \infty,$$

it follows that D has an absolute minimum at approximately -3.758. The point $\boxed{(-3.758, e^{1.879}) \approx (-3.758, 6.547)}$ on the graph of $y = e^{-x/2}$ is closest to the point $(1, 8)$.

AP® Practice Problems

1.



Let y denote the height of the rectangle and $2x$ be the width of the base of the rectangle.

$A = wh = 2xy$ with $y = 9 - x^2$

Substituting for y in the area formula yields

$$\begin{aligned} A &= 2x(9 - x^2) \\ &= 18x - 2x^3 \end{aligned}$$

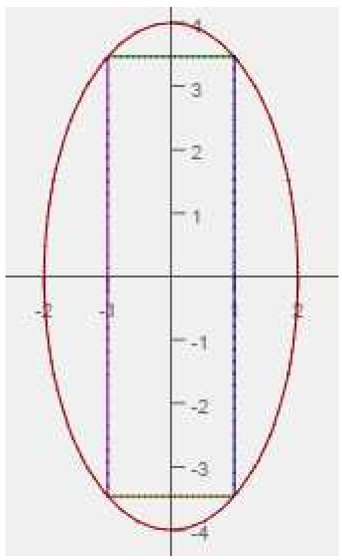
The domain of this function is the open interval $(-3, 3)$. The function A is differentiable on the open interval $(-3, 3)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$\begin{aligned} A'(x) &= 18 - 6x^2 \\ \text{Set } A'(x) &= 18 - 6x^2 = 0 \\ 6(3 - x^2) &= 0 \\ x &= \pm\sqrt{3} \end{aligned}$$

So the base of the rectangle is $2x = \boxed{2\sqrt{3}}$ and the height is $y = 9 - \left(\sqrt{3}\right)^2 = 9 - 3 = \boxed{6}$.

CHOICE D

3.



For the x-coordinate and the y-coordinate on the Cartesian coordinate system, the width of the rectangle is $2x$ and the height is $2y$.

$$\begin{aligned} A &= (2x)(2y) = 4xy \\ \text{Given } 4x^2 + y^2 &= 16 \\ y^2 &= 16 - 4x^2 \\ y &= (16 - 4x^2)^{\frac{1}{2}} \end{aligned}$$

Substituting into A yields

$$\begin{aligned} A &= 4x\left(16 - 4x^2\right)^{\frac{1}{2}} \\ A' &= 4\left[1\left(16 - 4x^2\right)^{\frac{1}{2}} + \frac{1}{2}\left(16 - 4x^2\right)^{\frac{-1}{2}}(-8x)(x)\right] \\ &= 4\left[\left(16 - 4x^2\right)^{\frac{1}{2}} - \frac{4x^2}{(16 - 4x^2)^{\frac{1}{2}}}\right] \end{aligned}$$

$$\text{Let } A' = 0$$

$$4\left[\left(16-4x^2\right)^{\frac{1}{2}} - \frac{4x^2}{\left(16-4x^2\right)^{\frac{1}{2}}}\right] = 0$$

$$16 - 4x^2 - 4x^2 = 0$$

$$16 - 8x^2 = 0$$

$$8x^2 = 16$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

$$y = \sqrt{16 - 4\left(\sqrt{2}\right)^2}$$

$$= \sqrt{16-8}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

$$A = 4xy = 4\left(\sqrt{2}\right)\left(2\sqrt{2}\right) = \boxed{16}.$$

CHOICE D

5. The minimum distance between $(1,0)$ and $(x-1)y = 4$ is to be determined by using the distance formula between $(1,0)$ and a point on $(x-1)y = 4$.

$$D = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (y)^2}$$

Substituting $y = \frac{4}{x-1}$ yields

$$D = \sqrt{(x-1)^2 + \left(\frac{4}{x-1}\right)^2}$$

$$D^2 = (x-1)^2 + \left(\frac{4}{x-1}\right)^2$$

$$= (x-1)^2 + 16(x-1)^{-2}$$

$$2DD' = 2(x-1) + 16(-2)(x-1)^{-3}$$

$$\text{Let } 2DD' = 0$$

$$2x - 2 - \frac{32}{(x-1)^3} = 0$$

$$2(x-1)^4 - 32 = 0$$

$$(x-1)^4 = 16$$

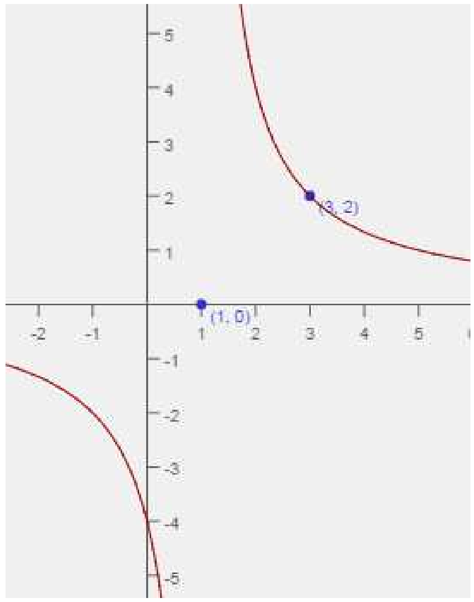
$$x - 1 = 2$$

$$x = 3$$

$$y = \frac{4}{x-1} = \frac{4}{3-1} = 2$$

The point on the graph of $(x-1)y = 4$ closest to $(1,0)$ is $\boxed{(3,2)}$.

CHOICE C



4.8 Antiderivatives; Differential Equations

Concepts and Vocabulary

1. A function F is called an **antiderivative** of a function f if $F' = f$.

3. All the antiderivatives of $y = x^{-1}$ are $\boxed{\ln|x| + C, \text{ where } C \text{ is a constant}}$.

5. False. The general solution of a differential equation $\dfrac{dy}{dx} = f(x)$ consists of all the antiderivatives
of $f(x)$.

7. True. To find a particular solution of a differential equation $\dfrac{dy}{dy} = f(x)$, we need a boundary condition.

Skill Building

9. All antiderivatives of the function $f(x) = 2$ are $F(x) = \boxed{2x + C}$, where C is a constant.

11. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4x^5$ are

$$F(x) = 4 \cdot \frac{x^{5+1}}{5+1} + C = \boxed{\frac{2}{3}x^6 + C},$$

where C is a constant.

13. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 5x^{3/2}$ are

$$F(x) = 5 \cdot \frac{x^{3/2+1}}{3/2+1} + C = \boxed{2x^{5/2} + C},$$

where C is a constant.

15. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 2x^{-2}$ are

$$F(x) = 2 \cdot \frac{x^{-2+1}}{-2+1} + C = -2x^{-1} + C = \boxed{-\frac{2}{x} + C},$$

where C is a constant.

17. All antiderivatives of the function $f(x) = \sqrt{x} = x^{1/2}$ are

$$F(x) = \frac{x^{1/2+1}}{1/2+1} + C = \boxed{\frac{2}{3}x^{3/2} + C},$$

where C is a constant.

19. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4x^3 - 3x^2 + 1$ are

$$F(x) = 4 \cdot \frac{x^{3+1}}{3+1} - 3 \cdot \frac{x^{2+1}}{2+1} + x + C = \boxed{x^4 - x^3 + x + C},$$

where C is a constant.

21. Expand

$$(2-3x)^2 \quad \text{as} \quad 9x^2 - 12x + 4.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function
$f(x) = (2-3x)^2 = 9x^2 - 12x + 4$ are

$$F(x) = 9 \cdot \frac{x^{2+1}}{2+1} - 12 \cdot \frac{x^{1+1}}{1+1} + 4x + C = \boxed{3x^3 - 6x^2 + 4x + C},$$

where C is a constant.

23. Rewrite

$$\frac{3x-2}{x} \quad \text{as} \quad \frac{3x}{x} - \frac{2}{x} = 3 - \frac{2}{x} = 3 - 2x^{-1}.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function
$f(x) = \dfrac{3x-2}{x} = 3 - 2x^{-1}$ are

$$F(x) = \boxed{3x - 2\ln|x| + C},$$

where C is a constant.

25. Rewrite

$$f(x) = \frac{3x^{1/2}-4}{x} \quad \text{as} \quad \frac{3x^{1/2}}{x} - \frac{4}{x} = 3x^{-1/2} - 4x^{-1}.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function
$f(x) = \dfrac{3x^{1/2}-4}{x} = 3x^{-1/2} - 4x^{-1}$ are

$$F(x) = \frac{3x^{-1/2+1}}{-1/2+1} - 4\ln|x| = \boxed{6x^{1/2} - 4\ln|x| + C},$$

where C is a constant.

27. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 2x - 3\cos x$ are

$$F(x) = 2 \cdot \frac{x^{1+1}}{1+1} - 3 \cdot \sin x + C = \boxed{x^2 - 3\sin x + C},$$

where C is a constant.

29. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4e^x + x$ are

$$F(x) = 4 \cdot e^x + \frac{x^{1+1}}{1+1} + C = \boxed{4e^x + \frac{1}{2}x^2 + C},$$

where C is a constant.

31. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = \dfrac{7}{1+x^2}$ are

$$F(x) = 7 \cdot \tan^{-1} x + C = \boxed{7\tan^{-1} x + C},$$

where C is a constant.

33. Using the Sum Rule, all antiderivatives of the function $f(x) = e^x + \dfrac{1}{x\sqrt{x^2-1}}$ are

$$\boxed{F(x) = e^x + \sec^{-1} x + C},$$

where C is a constant.

35. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 3\sinh x$ are

$$\boxed{F(x) = 3\cosh x + C},$$

where C is a constant.

37. The general solution of the differential equation $\dfrac{dy}{dx} = 3x^2 - 2x + 1$ is

$$y = 3 \cdot \frac{x^{2+1}}{2+1} - 2 \cdot \frac{x^{1+1}}{1+1} + x + C = x^3 - x^2 + x + C,$$

where C is a constant. Applying the boundary condition that when $x = 2$, then $y = 1$ yields

$$1 = 2^3 - 2^2 + 2 + C = 6 + C,$$

so that $C = -5$. The particular solution of the differential equation with the boundary condition that when $x = 2$, then $y = 1$, is therefore

$$\boxed{y = x^3 - x^2 + x - 5}.$$

39. The general solution of the differential equation

$$\frac{dy}{dx} = x^{1/3} + x\sqrt{x} - 2 = x^{1/3} + x^{3/2} - 2$$

is

$$y = \frac{x^{1/3+1}}{1/3+1} + \frac{x^{3/2+1}}{3/2+1} - 2x + C = \frac{3}{4}x^{4/3} + \frac{2}{5}x^{5/2} - 2x + C,$$

where C is a constant. Applying the boundary condition that when $x = 1$, then $y = 2$ yields

$$2 = \frac{3}{4}1^{4/3} + \frac{2}{5}1^{5/2} - 2(1) + C = \frac{3}{4} + \frac{2}{5} - 2 + C = -\frac{17}{20} + C,$$

so that $C = \frac{57}{20}$. The particular solution of the differential equation with the boundary condition that when $x = 1$, then $y = 2$, is therefore

$$\boxed{y = \frac{3}{4}x^{4/3} + \frac{2}{5}x^{5/2} - 2x + \frac{57}{20}}.$$

41. The general solution of the differential equation

$$\frac{ds}{dt} = t^3 + \frac{1}{t^2} = t^3 + t^{-2}$$

is

$$s = \frac{t^{3+1}}{3+1} + \frac{t^{-2+1}}{-2+1} + C = \frac{1}{4}t^4 - t^{-1} + C = \frac{1}{4}t^4 - \frac{1}{t} + C,$$

where C is a constant. Applying the boundary condition that when $t = 1$, then $s = 2$ yields

$$2 = \frac{1}{4}1^4 - \frac{1}{1} + C = \frac{1}{4} - 1 + C = -\frac{3}{4} + C,$$

so that $C = \frac{11}{4}$. The particular solution of the differential equation with the boundary condition that when $t = 1$, then $s = 2$, is therefore

$$\boxed{s = \frac{1}{4}t^4 - \frac{1}{t} + \frac{11}{4}}.$$

43. The general solution of the differential equation $f'(x) = x - 2\sin x$ is

$$f(x) = \frac{x^{1+1}}{1+1} - 2 \cdot (-\cos x) + C = \frac{1}{2}x^2 + 2\cos x + C,$$

where C is a constant. Applying the boundary condition that when $x = \pi$, then $f(\pi) = 0$ yields

$$0 = \frac{1}{2}\pi^2 + 2\cos\pi + C = \frac{1}{2}\pi^2 - 2 + C,$$

so that $C = 2 - \frac{1}{2}\pi^2$. The particular solution of the differential equation with the boundary condition that when $x = \pi$, then $f(\pi) = 0$, is therefore

$$\boxed{f(x) = \frac{1}{2}x^2 + 2\cos x + 2 - \frac{1}{2}\pi^2}.$$

45. All the antiderivatives of $\frac{d^2y}{dx^2} = e^x$ are

$$\frac{dy}{dx} = e^x + C_1,$$

and all the antiderivatives of $\frac{dy}{dx} = e^x + C_1$ are

$$y = e^x + C_1 x + C_2,$$

where C_1 and C_2 are constants. Applying the boundary condition that when $x = 0$, then $y = 2$ yields

$$2 = e^0 + C_1(0) + C_2 = 1 + C_2,$$

so that $C_2 = 1$. Next, applying the boundary condition that when $x = 1$, then $y = e$ yields

$$e = e^1 + C_1(1) + 1 = e + 1 + C_1,$$

so that $C_1 = -1$. The particular solution of the differential equation with the given boundary conditions is therefore

$$\boxed{y = e^x - x + 1}.$$

47. The general solution of the differential equation $\frac{dv}{dt} = a(t) = -32$ is

$$v(t) = -32t + C_1,$$

where C_1 is a constant. Applying the initial condition $v(0) = 128$ yields

$$128 = -32(0) + C_1 = C_1,$$

so that $v(t) = -32t + 128$. Next, the general solution of the differential equation $\frac{ds}{dt} = v(t) = -32t + 128$ is

$$s(t) = -32 \cdot \frac{t^{1+1}}{1+1} + 128t + C_2 = -16t^2 + 128t + C_2.$$

Applying the initial condition $s(0) = 0$ yields

$$0 = -16(0)^2 + 128(0) + C_2 = C_2,$$

so that $\boxed{s(t) = -16t^2 + 128t}$.

49. The general solution of the differential equation $\frac{dv}{dt} = a(t) = 3t$ is

$$v(t) = 3 \cdot \frac{t^{1+1}}{1+1} + C_1 = \frac{3}{2}t^2 + C_1,$$

where C_1 is a constant. Applying the initial condition $v(0) = 18$ yields

$$18 = \frac{3}{2}0^2 + C_1 = C_1,$$

so that $v(t) = \frac{3}{2}t^2 + 18$. Next, the general solution of the differential equation $\frac{ds}{dt} = v(t) = \frac{3}{2}t^2 + 18$ is

$$s(t) = \frac{3}{2} \cdot \frac{t^{2+1}}{2+1} + 18t + C_2 = \frac{1}{2}t^3 + 18t + C_2.$$

Applying the initial condition $s(0) = 2$ yields

$$2 = \frac{1}{2}(0)^3 + 18(0) + C_2 = C_2,$$

so that $\boxed{s(t) = \frac{1}{2}t^3 + 18t + 2}$.

51. The general solution of the differential equation $\dfrac{dv}{dt} = a(t) = \sin t \, \text{ft/s}^2$ is

$$v(t) = -\cos t + C_1 \, \text{ft/s},$$

where C_1 is a constant. Applying the initial condition $v(0) = 5 \, \text{ft/s}$ yields

$$5 \, \text{ft/s} = -\cos(0) + C_1 = -1 + C_1,$$

so that $C_1 = 5 + 1 = 6 \, \text{ft/s}$ and $v(t) = -\cos t + 6 \, \text{ft/s}$. Next, the general solution of the differential equation $\dfrac{ds}{dt} = v(t) = -\cos t + 6 \, \text{ft/s}$ is

$$s(t) = -\sin t + 6t + C_2 \, \text{ft}.$$

Applying the initial condition $s(0) = 0$ ft yields

$$0 \, \text{ft} = -\sin(0) + 6(0) + C_2 = C_2,$$

so that $C_2 = 0 \, \text{ft}$ and $\boxed{s(t) = -\sin t + 6t \, \text{ft}}$.

Applications and Extensions

53. Because $u^2 + 10u + 21 = (u+3)(u+7)$ and $3u + 9 = 3(u+3)$,

$$f(u) = \frac{u^2 + 10u + 21}{3u + 9} = \frac{(u+3)(u+7)}{3(u+3)} = \frac{u+7}{3} = \frac{1}{3}u + \frac{7}{3},$$

for $u \neq -3$. Then, for $u \neq -3$, all of the antiderivatives of f are

$$F(u) = \frac{1}{3} \cdot \frac{u^2}{2} + \frac{7}{3}u + C = \boxed{\frac{1}{6}u^2 + \frac{7}{3}u + C},$$

where C is a constant.

55. Write

$$\frac{t^4 + 3t - 1}{t} \quad \text{as} \quad \frac{t^4}{t} + \frac{3t}{t} - \frac{1}{t} = t^3 + 3 - \frac{1}{t}.$$

The general solution of the differential equation

$$f'(t) = \frac{t^4 + 3t - 1}{t} = t^3 + 3 - \frac{1}{t}$$

is then

$$f(t) = \frac{1}{4}t^4 + 3t - \ln|t| + C,$$

where C is a constant. Applying the boundary condition that when $t = 1$, then $f(1) = \dfrac{1}{4}$ yields

$$\frac{1}{4} = \frac{1}{4}1^4 + 3(1) - \ln 1 + C = \frac{13}{4} + C,$$

so that $C = -3$. The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{f(t) = \frac{1}{4}t^4 + 3t - \ln|t| - 3}.$$

57. Given
$$\frac{d}{dx}(x\cos x + \sin x) = -x\sin x + 2\cos x,$$
it follows that $x\cos x + \sin x$ is an antiderivative of the function $-x\sin x + 2\cos x$. The general solution of the differential equation
$$\frac{dF}{dx} = -x\sin x + 2\cos x$$
is $F(x) = x\cos x + \sin x + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $F(0) = 1$ yields
$$1 = 0 \cdot \cos 0 + \sin 0 + C = C.$$
The particular solution of the differential equation with the given boundary condition is therefore
$$\boxed{F(x) = x\cos x + \sin x + 1}.$$

59. Let $t = 0$ denote the time the brakes are applied, let v_0 denote the speed of the car when the brakes are applied, and let $s(t)$ represent the distance the car has traveled t seconds after the brakes have been applied. Because the car decelerates at the rate of 10 m/s^2, its acceleration a is given by
$$a(t) = \frac{dv}{dt} = -10.$$
Solve this differential equation for $v(t)$ to obtain $v(t) = -10t + C_1$, where C_1 is a constant. Applying the initial condition $v(0) = v_0$ yields
$$v_0 = -10(0) + C_1 = C_1,$$
so that $C_1 = v_0$ and $v(t) = -10t + v_0$. Next, solve the differential equation
$$\frac{ds}{dt} = v(t) = -10t + v_0$$
to obtain
$$s(t) = -10 \cdot \frac{t^2}{2} + v_0 t + C_2 = -5t^2 + v_0 t + C_2,$$
where C_2 is a constant. Applying the initial condition $s(0) = 0$ gives
$$0 = -5(0)^2 + v_0(0) + C_2 = C_2,$$
so that $C_2 = 0$ and $s(t) = -5t^2 + v_0 t$. Now, the car comes to a stop when its speed is equal to zero; that is when
$$-10t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{10}.$$
By the time the car comes to rest, it has traveled
$$s\left(\frac{v_0}{10}\right) = -5\left(\frac{v_0}{10}\right)^2 + v_0 \cdot \frac{v_0}{10} = \frac{v_0^2}{20} \text{ meters}$$
from the moment the brakes were applied. If the car is to stop within 15 meters, then
$$s\left(\frac{v_0}{10}\right) \le 15 \quad \text{or} \quad \frac{v_0^2}{20} \le 15.$$
The maximum possible velocity for the car is therefore $\boxed{v_0 = 10\sqrt{3}\ \text{m/s} \approx 17.32\ \text{m/s}}$. In miles per hour, this is approximately
$$10\sqrt{3}\ \text{m/s} \cdot \frac{3600\ \text{s}}{1\ \text{h}} \cdot \frac{1\ \text{km}}{1000\ \text{m}} \cdot \frac{1\ \text{mi}}{1.60934\ \text{km}} \approx \boxed{38.74\ \text{mi/h}}.$$

61. Let $t = 0$ denote the time that the car begins to accelerate, and let $s(t)$ represent the distance the car has traveled after t seconds. Because the car accelerates at a constant rate from 0 to 60 mph in 5 s, its acceleration a is given by

$$a = \frac{60 \text{ mph}}{5 \text{ s}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = 17.6 \text{ ft/s}^2.$$

Solve the differential equation

$$\frac{dv}{dt} = a(t) = 17.6$$

for $v(t)$ to obtain $v(t) = 17.6t + C_1$, where C_1 is a constant. Apply the initial condition $v(0) = 0$ to determine

$$0 = 17.6(0) + C_1 = C_1,$$

so that $v(t) = 17.6t$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = 17.6t$$

for $s(t)$ to obtain

$$s(t) = 17.6 \cdot \frac{t^2}{2} + C_2 = 8.8t^2 + C_2.$$

The initial condition $s(0) = 0$ determines

$$0 = 8.8(0)^2 + C_2 = C_2,$$

so $s(t) = 8.8t^2$. Therefore, after 5 seconds, the car has traveled

$$s(5) = 8.8(5)^2 = \boxed{220 \text{ ft}}.$$

63. Let v_0 denote the initial upward velocity of the ball, let $t = 0$ denote the time at which the ball is released, and let $v(t)$ and $s(t)$ represent the upward speed of the ball and the height of the ball above ground level, respectively, t seconds after release. Given that the ball is released at an initial height of 1 m, it follows that $s(0) = 1$. Now, solve the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

to obtain $v(t) = -9.8t + C_1$, where C_1 is a constant. The initial condition $v(0) = v_0$ determines

$$v_0 = -9.8(0) + C_1 = C_1,$$

so $v(t) = -9.8t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0$$

for $s(t)$ to obtain

$$s(t) = -9.8\frac{t^2}{2} + v_0 t + C_2 = -4.9t^2 + v_0 t + C_2.$$

Use the initial condition $s(0) = 1$ to determine

$$1 = -4.9(0)^2 + v_0(0) + C_2 = C_2,$$

so $s(t) = -4.9t^2 + v_0 t + 1$. The ball reaches it maximum height when $v(t) = 0$; that is, when

$$-9.8t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{9.8} \text{ s}.$$

The maximum height is therefore

$$s\left(\frac{v_0}{9.8}\right) = -4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 1 = \frac{v_0^2}{19.6} + 1 \text{ m}.$$

To achieve a maximum height of at least 9.8 m requires

$$\frac{v_0^2}{19.6} + 1 \geq 9.8 \quad \text{or} \quad v_0 \geq \sqrt{19.6(8.8)} \approx \boxed{13.133 \text{ m/s}}.$$

65. Note that constant force implies constant acceleration. Given that the object is initially at rest and has a velocity of 12 m/s after 6 s, the acceleration a of the object is

$$a = \frac{12 - 0}{6} = 2 \text{ m/s}^2.$$

By Newton's Second Law, $F = ma$, so the force applied to the object is

$$F = ma = 4(2) = \boxed{8 \text{ N}}.$$

67. Let $a = g \sin 20^\circ = 9.8 \sin 20^\circ$. The general solution of the differential equation

$$\frac{dv}{dt} = a = 9.8 \sin 20^\circ$$

is

$$v(t) = (9.8 \sin 20^\circ)t + C,$$

where C is a constant. The initial condition $v(0) = 0$ (the skier starts from rest) determines

$$0 = 9.8 \sin 20^\circ(0) + C = C,$$

so

$$v(t) = (9.8 \sin 20^\circ)t.$$

After 5 seconds,

$$v(5) = 9.8 \sin 20^\circ(5) \approx \boxed{16.759 \text{ m/s}}.$$

69. First consider Javier Sotomayor jumping on Earth. Let $t = 0$ denote the time Sotomayor loses contact with the ground, let v_0 denote his initial upward speed, and let $s(t)$ represent his height t seconds after initiating the jump. The general solution of the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

is $v(t) = -9.8t + C_1$, where C_1 is a constant. Applying the initial condition $v(0) = v_0$ yields

$$v_0 = -9.8(0) + C_1 = C_1,$$

so that $C_1 = v_0$ and $v(t) = -9.8t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0$$

to obtain

$$s(t) = -9.8 \cdot \frac{t^2}{2} + v_0 t + C_2 = -4.9t^2 + v_0 t + C_2,$$

where C_2 is a constant. Applying the initial condition $s(0) = 0$ gives

$$0 = -4.9(0)^2 + v_0(0) + C_2 = C_2,$$

so that $C_2 = 0$ and $s(t) = -4.9t^2 + v_0 t$. Now, Sotomayor achieves his maximum height when $v(t)$ is equal to zero; that is when

$$-9.8t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{9.8}.$$

Substituting this time into $s(t)$ and setting the resulting expression equal to 2.45 gives

$$2.45 = -\frac{v_0^2}{19.6} + \frac{v_0^2}{9.8} = \frac{v_0^2}{19.6},$$

so that

$$v_0 = \sqrt{48.02} \text{ m/s}.$$

Now, consider Javier Sotomayor jumping on the moon. Assume the condition that "he propels himself with the same force on the moon as on Earth" means that Sotomayor produces the same initial velocity, $v_0 = \sqrt{48.02}$ m/s, as on Earth. Following the procedure used above, with an acceleration due to gravity on the moon of 1.6 m/s^2, Sotomayor's velocity during the jump on the moon is

$$v(t) = -1.6t + \sqrt{48.02},$$

so that he achieves maximum height when

$$-1.6t + \sqrt{48.02} = 0 \quad \text{or} \quad t = \frac{\sqrt{48.02}}{1.6} \text{ s}.$$

Sotomayor's height t seconds after initiating the jump is

$$s(t) = -1.6\frac{t^2}{2} + \sqrt{48.02}t = -0.8t^2 + \sqrt{48.02}t,$$

so

$$s\left(\frac{\sqrt{48.02}}{1.6}\right) = -\frac{48.02}{3.2} + \frac{48.02}{1.6} = \frac{48.02}{3.2} = \boxed{15.00625 \text{ m}}.$$

On the moon, Sotomayor would attain a height of a little more than 15 m.

71. Divide $2x^3 + 2x + 3$ by $1 + x^2$ to find the quotient and remainder and rewrite

$$\frac{2x^3 + 2x + 3}{1 + x^2} = 2x + \frac{3}{1 + x^2}.$$

Then, using the Sum Rule and the Constant Multiple Rule, the antiderivative is

$$\boxed{x^2 + 3\tan^{-1} x + C},$$

where C is a constant.

Challenge Problems

73. (a) Let $I(x)$ denote the intensity of the radiation at the depth x into the tissue. The rate of change of the intensity of the radiation with respect to the depth x is then $\frac{dI}{dx}$. Given that intensity of the radiation decreases with increasing depth (because radiation is absorbed as it passes through the tissue) and that the rate of change is proportional to the intensity with positive constant of proportionality k, it follows that

$$\frac{dI}{dx} = -kI.$$

(b) Because the constant of proportionality k has been designated to be positive, the negative sign is needed in the differential equation in part (a) to ensure that the $\boxed{\text{intensity of the radiation decreases with increasing depth}}$.

(c) To solve the differential equation from part (a), we need to find the most general function whose derivative is $-k$ times itself. Note that e^{-kx} is one such function whose derivative is $-k$ times itself; that is

$$\frac{d}{dx}e^{-kx} = e^{-kx} \cdot (-k) = -ke^{-kx}.$$

Now, suppose that f is *any* function for which $f'(x) = -kf(x)$ and consider the function $f(x)e^{kx}$. Because

$$\frac{d}{dx}[f(x)e^{kx}] = f(x) \cdot ke^{kx} + e^{kx} \cdot f'(x) = kf(x)e^{kx} - kf(x)e^{kx} = 0,$$

it follows that $f(x)e^{kx}$ is equal to some constant C. In other words, if f is a function for which $f'(x) = -kf(x)$, then $f(x) = Ce^{-kx}$ for some constant C. It follows that

$$I(x) = Ce^{-kx}$$

for some constant C. Applying the initial condition $I(0) = I_0$ yields

$$I_0 = Ce^{-k(0)} = C;$$

therefore, $\boxed{I(x) = I_0e^{-kx}}$.

(d) If the intensity is reduced by 90% at the depth of 2.0 cm, then $I(2.0) = 0.1I_0$. Using the formula for $I(x)$ from part (c),

$$0.1I_0 = I_0e^{-2k} \quad \text{so that} \quad k = \boxed{-\frac{1}{2}\ln 0.1 \text{ cm}^{-1}}.$$

AP® Practice Problems

1. $s'(t) = v(t) = t^2 + t$

$$\text{so } s(t) = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$s(0) = \frac{(0)^3}{3} + \frac{(0)^2}{2} + C = -1$$

$$C = -1$$

$$s(t) = \frac{t^3}{3} + \frac{t^2}{2} - 1$$

$$s(3) = \frac{(3)^3}{3} + \frac{(3)^2}{2} - 1 = \boxed{\frac{25}{2}}$$

CHOICE C

3. $f(x) = 2x - \cos x$

$F'(x) = f(x)$, where $F(x)$ is the antiderivative

$F'(x) = 2x - \cos x$

so $y = F(x) = \boxed{x^2 - \sin x + C}$

CHOICE C

5.
$$\begin{aligned}
v'(t) &= a(t) = 2 + 12t \\
\text{so } v(t) &= 2t + 6t^2 + C_1 \\
v(0) &= 2(0) + 6(0)^2 + C_1 = 5 \\
C_1 &= 5 \\
v(t) &= 6t^2 + 2t + 5 \\
s'(t) &= v(t) = 6t^2 + 2t + 5 \\
\text{so } s(t) &= 2t^3 + t^2 + 5t + C_2 \\
s(3) &= 2(3)^3 + (3)^2 + 5(3) + C_2 = 78 + C_2 \\
s(1) &= 2(1)^3 + (1)^2 + 5(1) + C_2 = 8 + C_2 \\
s(3) - s(1) &= (78 + C_2) - (8 + C_2) = \boxed{70}
\end{aligned}$$

CHOICE B

Chapter 4 Review Exercises

1. The volume and surface area of a sphere of radius r are

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2,$$

respectively. Differentiating with respect to time,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Solving the first equation for $\frac{dr}{dt}$ and substituting into the second equation yields

$$\frac{dS}{dt} = \frac{8\pi r}{4\pi r^2}\frac{dV}{dt} = \frac{2}{r}\frac{dV}{dt}.$$

Given that the snowball is melting at the rate of 2 cm^3/min,

$$\frac{dV}{dt} = -2.$$

When $r = 5$ cm,

$$\frac{dS}{dt} = \frac{2}{5}(-2) = \boxed{-\frac{4}{5}\ \text{cm}^2/\text{min}}.$$

3. Let $x(t)$ denote the distance between the plane approaching the airport from the west and the airport, and let $y(t)$ denote the distance between the plane approaching the airport from the north and the airport. The distance D between the two planes is then

$$D = \sqrt{x^2 + y^2}.$$

Differentiating with respect to time yields

$$\frac{dD}{dt} = \frac{x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

Given that the plane to the west is approaching the airport at 200 mph and the plane to the north is approaching the airport at 250 mph,

$$\frac{dx}{dt} = -200 \quad \text{and} \quad \frac{dy}{dt} = -250.$$

When $x = 20$ miles and $y = 30$ miles,

$$\frac{dD}{dt} = \frac{20 \cdot -200 + 30 \cdot -250}{\sqrt{20^2 + 30^2}} = \frac{-11500}{\sqrt{1300}} \approx -318.953 \text{ mph}.$$

The two planes are approaching one another at $\boxed{\text{approximately } 318.953 \text{ mph}}$.

5. For the function given in the graph,

$(-8, -9)$:	$\boxed{\text{absolute minimum}}$
$(-5, 0)$:	$\boxed{\text{neither}}$
$(-2, 9)$:	$\boxed{\text{local maximum and absolute maximum}}$
$(1, 0)$:	$\boxed{\text{local minimum}}$
$(3, 4)$:	$\boxed{\text{local maximum}}$
$(5, 0)$:	$\boxed{\text{neither}}$

7. Let $f(x) = \cos(2x)$ and consider the closed interval $[0, \pi]$. Because the trigonometric function f is differentiable everywhere, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = -2\sin(2x),$$

so $f'(x) = 0$ when $2x = n\pi$ for any integer n. It follows that $x = \dfrac{n\pi}{2}$ is a critical number for any integer n. Among these numbers, only 0, $\dfrac{\pi}{2}$, and π lie on the closed interval $[0, \pi]$. Therefore, $\boxed{0, \frac{\pi}{2}, \text{ and } \pi}$ are the critical numbers of f on the closed interval $[0, \pi]$.

9. Let $f(x) = \dfrac{3}{2}x^4 - 2x^3 - 6x^2 + 5$, and consider the closed interval $[-2, 3]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-2, 3]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the interval. Now, the polynomial function f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 6x^3 - 6x^2 - 12x = 6x(x^2 - x - 2) = 6x(x-2)(x+1) = 0$$

when $x = -1$, $x = 0$, and $x = 2$. Evaluating f at the endpoints of the interval $[-2, 3]$ and at the three critical numbers yields

$$f(-2) = 21, \quad f(-1) = \frac{5}{2}, \quad f(0) = 5, \quad f(2) = -11, \quad f(3) = \frac{37}{2}.$$

Therefore, the absolute maximum value of f on the interval $[-2, 3]$ is $\boxed{21}$ (and this occurs at the endpoint $x = -2$), while the aboslute minimum value of f is $\boxed{-11}$ (and this occurs at the critical number $x = 2$).

11. Let $f(x) = \dfrac{2x-1}{x} = 2 - \dfrac{1}{x}$. The function f is continuous and differentiable on the set $\{x | x \neq 0\}$, so it is continuous on the closed interval $[1, 4]$ and differentiable on the open interval $(1, 4)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 4]$. Now,

$$f'(x) = \frac{1}{x^2} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{\frac{7}{4} - 1}{3} = \frac{1}{4},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{1}{c^2} = \frac{1}{4}.$$

Therefore, $c = \pm 2$. Of these numbers, only $c = 2$ is in the interval $(1, 4)$. Finally, at the point $\boxed{\left(2, \frac{3}{2}\right)}$, the graph of f has a tangent line with a slope equal to that of the secant line joining $(1, 1)$ and $\left(4, \frac{7}{4}\right)$.

13. Let $f(x) = x^3 - x^2 - 8x + 1$. The polynomial function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 - 2x - 8 = (3x + 4)(x - 2),$$

so 2 and $-\frac{4}{3}$ are the critical numbers of f.

(a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 2 and $-\frac{4}{3}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3x + 4$	Sign of $x - 2$	Sign of $f'(x)$	Conclusion
$(-\infty, -\frac{4}{3})$	$-$	$-$	$+$	f is increasing
$(-\frac{4}{3}, 2)$	$+$	$-$	$-$	f is decreasing
$(2, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $\left(-\infty, -\frac{4}{3}\right)$ and $(2, \infty)$ and decreasing on the interval $\left(-\frac{4}{3}, 2\right)$. By the First Derivative Test, it follows that f has a local maximum value at $-\frac{4}{3}$ and a local minimum value at 2. The $\boxed{\text{local maximum value is } f\left(-\frac{4}{3}\right) = \frac{203}{27}}$, and the $\boxed{\text{local minimum value is } f(2) = -11}$.

(b) The second derivative of f is

$$f''(x) = 6x - 2.$$

Evaluating the second derivative at the critical numbers yields

$$f''\left(-\frac{4}{3}\right) = -10 < 0 \quad \text{and} \quad f''(2) = 10 > 0.$$

By the Second Derivative Test, it follows that f has a local maximum value at $-\frac{4}{3}$ and a local minimum value at 2. The $\boxed{\text{local maximum value is } f\left(-\frac{4}{3}\right) = \frac{203}{27}}$, and the $\boxed{\text{local minimum value is } f(2) = -11}$.

15. Let $f(x) = x^4 e^{-2x}$. The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = x^4 \cdot -2e^{-2x} + 4x^3 e^{-2x} = (4x^3 - 2x^4)e^{-2x} = 2x^3(2 - x)e^{-2x},$$

so 0 and 2 are the critical numbers of f

(a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and 2 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2x^3$	Sign of $(2-x)e^{-2x}$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$-$	$+$	$-$	f is decreasing
$(0, 2)$	$+$	$+$	$+$	f is increasing
$(2, \infty)$	$+$	$-$	$-$	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and increasing on the interval $(0, 2)$. By the First Derivative Test, it follows that f has a local minimum value at 0 and a local maximum value at 2. The $\boxed{\text{local minimum value is } f(0) = 0}$, and the $\boxed{\text{local maximum value is } f(2) = 16e^{-4}}$.

(b) The second derivative of f is

$$f''(x) = -2(4x^3 - 2x^4)e^{-2x} + (12x^2 - 8x^3)e^{-2x} = (4x^4 - 16x^3 + 12x^2)e^{-2x}.$$

Evaluating the second derivative at the critical numbers yields

$$f''(0) = 0 \quad \text{and} \quad f''(2) = -16e^{-4} < 0.$$

By the Second Derivative Test, it follows that f has a local maximum value at 2; the $\boxed{\text{local maximum value is } f(2) = 16e^{-4}}$. The Second Derivative Test is inconclusive at 0.

17. Let $y = f(x) = -x^3 - x^2 + 2x$.

Step 1 The polynomial function f has a domain of $\boxed{\text{all real numbers}}$. $f(0) = 0$, so the $\boxed{y\text{-intercept is } 0}$. To find the x-intercepts, solve the equation $f(x) = 0$. Because

$$-x^3 - x^2 + 2x = -x(x^2 + x - 2) = -x(x-1)(x+2),$$

it follows the graph of f has $\boxed{\text{three } x\text{-intercepts: } -2, 0, \text{ and } 1}$.

Step 2 The $\boxed{\text{graphs of polynomial functions do not have asymptotes}}$, but the end behavior of the graph of f will resemble the power function $y = -x^3$.

Step 3 Now

$$\begin{aligned} f'(x) &= -3x^2 - 2x + 2; \text{ and} \\ f''(x) &= -6x - 2 = -2(3x+1). \end{aligned}$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)(2)}}{-6} = \frac{2 \pm 2\sqrt{7}}{-6} = \boxed{-\frac{1}{3} \pm \frac{1}{3}\sqrt{7}}$$

are the critical numbers. At the points

$$\left(-\frac{1}{3} - \frac{1}{3}\sqrt{7}, -\frac{20}{27} - \frac{14}{27}\sqrt{7}\right) \approx (-1.215, -2.113)$$

and

$$\left(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, -\frac{20}{27} + \frac{14}{27}\sqrt{7}\right) \approx (0.549, 0.631),$$

the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers $-\frac{1}{3} \pm \frac{1}{3}\sqrt{7}$ to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\frac{1}{3} - \frac{1}{3}\sqrt{7})$	$-$	f is decreasing on $(-\infty, -\frac{1}{3} - \frac{1}{3}\sqrt{7})$
$(-\frac{1}{3} - \frac{1}{3}\sqrt{7}, -\frac{1}{3} + \frac{1}{3}\sqrt{7})$	$+$	f is increasing on $(-\frac{1}{3} - \frac{1}{3}\sqrt{7}, -\frac{1}{3} + \frac{1}{3}\sqrt{7})$
$(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, \infty)$	$-$	f is decreasing on $(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\frac{1}{3} - \frac{1}{3}\sqrt{7}$ and a local maximum value at $-\frac{1}{3} + \frac{1}{3}\sqrt{7}$. The

$$\boxed{\text{local minimum value is } f\left(-\frac{1}{3} - \frac{1}{3}\sqrt{7}\right) = -\frac{20}{27} - \frac{14}{27}\sqrt{7}},$$

while the

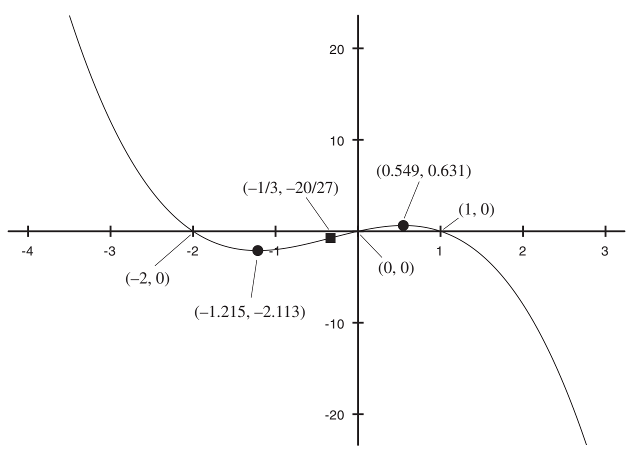
$$\boxed{\text{local maximum value is } f\left(-\frac{1}{3} + \frac{1}{3}\sqrt{7}\right) = -\frac{20}{27} + \frac{14}{27}\sqrt{7}}.$$

Step 6 The second derivative is equal to zero at $x = -\frac{1}{3}$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{1}{3})$	$+$	f is concave up on $(-\infty, -\frac{1}{3})$
$(-\frac{1}{3}, \infty)$	$-$	f is concave down on $(-\frac{1}{3}, \infty)$

The concavity of f changes at $-\frac{1}{3}$, so the point $\boxed{\left(-\frac{1}{3}, -\frac{20}{27}\right)}$ is a point of inflection.

Step 7 The figure below displays the graph of f. Local extreme values are highlighted by closed circles, and the point of inflection is highlighted by a closed square.



19. Let $y = f(x) = xe^x$.

Step 1 The domain of f is the set of $\boxed{\text{all real numbers}}$. The $\boxed{x\text{-intercept is } 0}$, and the $\boxed{y\text{-intercept is } f(0) = 0}$.

Step 2 Because the domain of f is the set of all real numbers, the graph of f has $\boxed{\text{no vertical asymptotes}}$. To determine if there is a horizontal asymptote, consider the limits at infinity and at negative infinity:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$$

and

$$\lim_{x\to\infty} xe^x = \infty,$$

where L'Hôpital's Rule was used in the first limit. Therefore, the graph of f has the line $\boxed{y = 0 \text{ as a horizontal asymptote}}$ as $x \to -\infty$ and no horizontal asymptote as $x \to \infty$.

Step 3 Now,

$$\begin{aligned} f'(x) &= xe^x + e^x = e^x(x+1); \text{ and} \\ f''(x) &= e^x + e^x(x+1) = e^x(x+2). \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $\boxed{x = -1}$. At the point $\left(-1, -\frac{1}{e}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number -1 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is decreasing on $(-\infty, -1)$
$(-1, \infty)$	$+$	f is increasing on $(-1, \infty)$

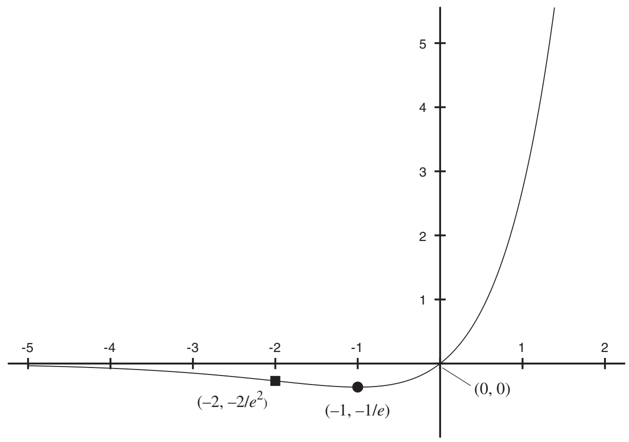
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -1. The $\boxed{\text{local minimum value is } f(-1) = -\frac{1}{e}}$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = -2$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	$-$	f is concave down on $(-\infty, -2)$
$(-2, \infty)$	$+$	f is concave up on $(-2, \infty)$

The concavity of f changes at -2, so the point $\boxed{\left(-2, -\frac{2}{e^2}\right)}$ is a point of inflection.

Step 7 The figure below displays the graph of f. The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



21. Let $y = f(x) = x\sqrt{x-3}$.

Step 1 The domain of f is given by the solution to the inequality $x - 3 \geq 0$, that is, the set $\boxed{\{x|x \geq 3\}}$. The $\boxed{x\text{-intercept is } 3}$, and there is $\boxed{\text{no } y\text{-intercept}}$ because 0 is not in the domain of f.

Step 2 Because

$$\lim_{x\to\infty} x\sqrt{x-3} = \infty,$$

the graph of f $\boxed{\text{does not have a horizontal asymptote}}$. The graph also $\boxed{\text{does not have any vertical asymptotes}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x\sqrt{x-3}) = x \cdot \frac{1}{2\sqrt{x-3}} + \sqrt{x-3} = \frac{3x-6}{2\sqrt{x-3}}; \text{ and} \\ f''(x) &= \frac{d}{dx}\left(\frac{3x-6}{2\sqrt{x-3}}\right) = \frac{2\sqrt{x-3}\cdot 3 - (3x-6)\cdot(x-3)^{-1/2}}{4(x-3)} \\ &= \frac{6(x-3)-(3x-6)}{4(x-3)^{3/2}} = \frac{3x-12}{4(x-3)^{3/2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 2$ and does not exist when $x = 3$; however, 2 is not in the domain of f, so 2 is not a critical number. Therefore, $\boxed{3}$ is the only critical number of f. At the point $(3, 0)$, the tangent line is vertical.

Step 4 Because $f'(x) > 0$ for all $x > 3$, f is $\boxed{\text{increasing on the interval } (3, \infty)}$.

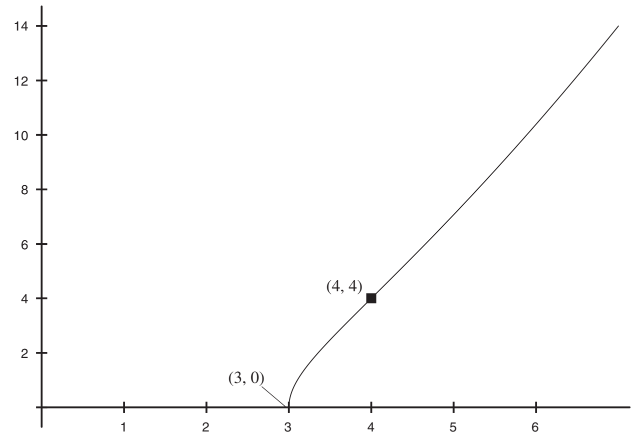
Step 5 Because the only critical number of f is an endpoint of the domain of f, f has $\boxed{\text{no local extreme values}}$.

Step 6 The second derivative exists for $x > 3$ and is equal to zero when $x = 4$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(3, 4)$	$-$	f is concave down on $(3, 4)$
$(4, \infty)$	$+$	f is concave up on $(4, \infty)$

The concavity of f changes at 4, so the point $\boxed{(4, 4)}$ is a point of inflection.

Step 7 The figure below displays the graph of f. The point of inflection is highlighted by a closed square.



23. Let $f(x) = x^4 + 12x^2 + 36x - 11$. Then

$$\begin{aligned} f'(x) &= 4x^3 + 24x + 36; \text{ and} \\ f''(x) &= 12x^2 + 24. \end{aligned}$$

(a) Using the computer algebra system *Maple*, it is found that $f'(x) = 0$ for $x \approx -1.207$. For $x < -1.207$, $f'(x) < 0$, while for $x > -1.207$, $f'(x) > 0$. Therefore, f is

$$\boxed{\text{decreasing on the approximate interval } (-\infty, -1.207)}$$

and

$$\boxed{\text{increasing on the approximate interval } (-1.207, \infty)}.$$

(b) Because $f''(x) \geq 24 > 0$ for all x, f is $\boxed{\text{concave up on the interval } (-\infty, \infty)}$.

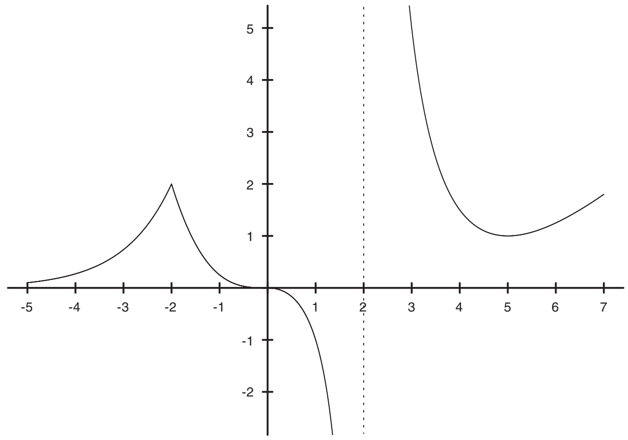
(c) Because the concavity of f does not change, f has $\boxed{\text{no points of inflection}}$.

25. Because $y' > 0$ and $y'' < 0$ for all x, the graph of $y = f(x)$ is always increasing and concave down.

(A) This cannot be part of the graph of $y = f(x)$ because this graph segment is concave up.
(B) This could be part of the graph of $y = f(x)$ because this graph segment is increasing and concave down.
(C) This cannot be part of the graph of $y = f(x)$ because this graph segment is decreasing.
(D) This cannot be part of the graph of $y = f(x)$ because this graph segment is decreasing.

Therefore, $\boxed{\text{(B)}}$ could be a part of the graph of f.

27. Answers will vary. The figure below displays the graph of a function f with the following properties: $f(-2) = 2$; $f(5) = 1$; $f(0) = 0$; $f'(x) > 0$ if $x < -2$ or $5 < x$ and $f'(x) < 0$ if $-2 < x < 2$ or $2 < x < 5$; $f''(x) > 0$ if $x < 0$ or $2 < x$ and $f''(x) < 0$ if $0 < x < 2$; $\lim\limits_{x \to 2^-} f(x) = -\infty$; $\lim\limits_{x \to 2^+} f(x) = \infty$.



29. Let x denote the side length in inches of the square cut from each corner of the base (see the diagram below, where all lengths are given in inches). When the sides are turned up, the resulting box will have a rectangular base measuring $24 - 2x$ inches by $36 - 2x$ inches and a height of x inches; the volume will therefore be

$$V = x(24 - 2x)(36 - 2x) = 4x^3 - 120x^2 + 864x.$$

To determine the domain of V, note that the length of each of the three sides of the box must be non-negative. This requires $x \geq 0$, $24 - 2x \geq 0$, and $36 - 2x \geq 0$. The solution of this set of three inequalities is $0 \leq x \leq 12$; it follows that the domain of V is the closed interval $[0, 12]$. The function V is differentiable on the open interval $(0, 12)$, so the critical numbers occur where $V'(x) = 0$. Now,

$$V'(x) = 12x^2 - 240x + 864 = 12(x^2 - 20x + 72),$$

so the only critical number inside the open interval $(0, 12)$ is

$$x = \frac{20 - \sqrt{20^2 - 4(1)(72)}}{2} = \frac{20 - 4\sqrt{7}}{2} = 10 - 2\sqrt{7} \approx 4.708.$$

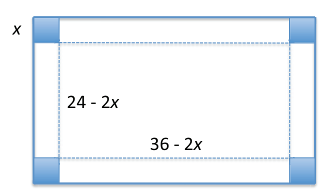
Note that

$$x = \frac{20 + \sqrt{20^2 - 4(1)(72)}}{2} = \frac{20 + 4\sqrt{7}}{2} = 10 + 2\sqrt{7} \approx 15.292$$

is not in the domain of V. Evaluating V at the endpoints of the interval $[0, 12]$ and at the critical number 4.708 yields

$$V(0) = 0, \quad V(4.708) \approx 1825.297, \quad \text{and} \quad V(12) = 0.$$

The largest volume is therefore achieved when $x = 10 - 2\sqrt{7} \approx 4.708$. Squares with a side length of $\boxed{10 - 2\sqrt{7} \approx 4.708 \text{ inches}}$ should be cut out to produce a box with maximum volume.



31. All antiderivatives of the function $f(x) = 0$ are $F(x) = \boxed{C}$, where C is a constant.

33. All antiderivatives of the function $f(x) = \cos x$ are $F(x) = \boxed{\sin x + C}$, where C is a constant.

35. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = \dfrac{2}{x}$ are

$$F(x) = 2 \cdot \ln|x| + C = \boxed{2\ln|x| + C},$$

where C is a constant.

37. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4x^3 - 9x^2 + 10x - 3$ are

$$F(x) = 4 \cdot \frac{x^{3+1}}{3+1} - 9 \cdot \frac{x^{2+1}}{2+1} + 10 \cdot \frac{x^{1+1}}{1+1} - 3x + C = \boxed{x^4 - 3x^3 + 5x^2 - 3x + C},$$

where C is a constant.

39. Let $t = 0$ denote the time the box begins to move with a velocity v_0, and let $v(t)$ and $s(t)$ denote the velocity and distance traveled by the box, respectively, t seconds after it begins to move. The general solution of the differential equation

$$\frac{dv}{dt} = a(t) = t^2(t-3) = t^3 - 3t$$

is

$$v(t) = \frac{t^4}{4} - 3\frac{t^2}{2} + C_1 = \frac{1}{4}t^4 - \frac{3}{2}t^2 + C_1,$$

where C_1 is a constant. The initial condition $v(0) = v_0$ determines

$$v_0 = \frac{1}{4}0^4 - \frac{3}{2}0^2 + C_1 = C_1.$$

Next, the general solution of the differential equation

$$\frac{ds}{dt} = v(t) = \frac{1}{4}t^4 - \frac{3}{2}t^2 + v_0$$

is

$$s(t) = \frac{1}{4} \cdot \frac{t^5}{5} - \frac{3}{2} \cdot \frac{t^3}{3} + v_0 t + C_2 = \frac{1}{20}t^5 - \frac{1}{2}t^3 + v_0 t + C_2,$$

where C_2 is a constant. The initial condition $s(0) = 0$ determines

$$0 = \frac{1}{20}0^5 - \frac{1}{2}0^3 + v_0(0) + C_2 = C_2.$$

The box travels 10 cm in 2 s, so $s(2) = 10$ and

$$10 = \frac{1}{20}2^5 - \frac{1}{2}2^3 + 2v_0 = \frac{8}{5} - 4 + 2v_0.$$

Therefore,

$$v_0 = 7 - \frac{4}{5} = \boxed{\frac{31}{5}\ \text{cm/s}}.$$

41. Because

$$\lim_{x\to 0}(xe^{3x} - x) = 0 \quad \text{and} \quad \lim_{x\to 0}[1 - \cos(2x)] = 0,$$

the expression $\dfrac{xe^{3x} - x}{1 - \cos(2x)}$ is an $\boxed{\text{indeterminate form at 0 of the type } \frac{0}{0}}$.

43. Because

$$\lim_{x\to 0}\frac{1}{x^2}=\infty \quad\text{and}\quad \lim_{x\to 0}\frac{1}{x^2\sec x}=\infty,$$

the expression $\dfrac{1}{x^2}-\dfrac{1}{x^2\sec x}$ is an $\boxed{\text{indeterminate form at 0 of the type } \infty-\infty}$.

45. Rewrite

$$\frac{\sec^2 x}{\sec^2(3x)} \quad\text{as}\quad \frac{\cos^2(3x)}{\cos^2 x}.$$

Because

$$\lim_{x\to\pi/2}[\cos^2(3x)]=0 \quad\text{and}\quad \lim_{x\to\pi/2}\cos^2 x=0,$$

the expression $\dfrac{\cos^2(3x)}{\cos^2 x}$ is an indeterminate form at $\pi/2$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to\pi/2}\frac{\cos^2(3x)}{\cos^2 x}=\lim_{x\to\pi/2}\frac{\frac{d}{dx}\cos^2(3x)}{\frac{d}{dx}\cos^2 x}=\lim_{x\to\pi/2}\frac{-6\cos(3x)\sin(3x)}{-2\cos x\sin x}=\lim_{x\to\pi/2}\frac{3\sin(6x)}{\sin(2x)}.$$

Now,

$$\lim_{x\to\pi/2}[3\sin(6x)]=0 \quad\text{and}\quad \lim_{x\to\pi/2}\sin(2x)=0,$$

so the expression $\dfrac{3\sin(6x)}{\sin(2x)}$ is an indeterminate form at $\pi/2$ of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x\to\pi/2}\frac{\cos^2(3x)}{\cos^2 x}=\lim_{x\to\pi/2}\frac{3\sin(6x)}{\sin(2x)}=\lim_{x\to\pi/2}\frac{\frac{d}{dx}[3\sin(6x)]}{\frac{d}{dx}\sin(2x)}=\lim_{x\to\pi/2}\frac{18\cos(6x)}{2\cos(2x)}=\frac{18(-1)}{2(-1)}=9.$$

Therefore,

$$\lim_{x\to\pi/2}\frac{\sec^2 x}{\sec^2(3x)}=\lim_{x\to\pi/2}\frac{\cos^2(3x)}{\cos^2 x}=\boxed{9}.$$

47. Because

$$\lim_{x\to 0}(e^x-e^{-x})=0 \quad\text{and}\quad \lim_{x\to 0}\sin x=0,$$

the expression $\dfrac{e^x-e^{-x}}{\sin x}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{\frac{d}{dx}(e^x-e^{-x})}{\frac{d}{dx}\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{1+1}{1}=\boxed{2}.$$

49. Because

$$\lim_{x\to 0}(\tan x+\sec x-1)=0 \quad\text{and}\quad \lim_{x\to 0}(\tan x-\sec x+1)=0,$$

the expression $\dfrac{\tan x+\sec x-1}{\tan x-\sec x+1}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\tan x+\sec x-1}{\tan x-\sec x+1}=\lim_{x\to 0}\frac{\frac{d}{dx}(\tan x+\sec x-1)}{\frac{d}{dx}(\tan x-\sec x+1)}=\lim_{x\to 0}\frac{\sec^2 x+\sec x\tan x}{\sec^2 x-\sec x\tan x}=\frac{1+0}{1-0}=\boxed{1}.$$

51. Because

$$\lim_{x \to 0} (x - \sin x) = 0 \quad \text{and} \quad \lim_{x \to 0} x^3 = 0,$$

the expression $\dfrac{x - \sin x}{x^3}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx} x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2}.$$

Now,

$$\lim_{x \to 0} (1 - \cos x) = 0 \quad \text{and} \quad \lim_{x \to 0} (3x^2) = 0,$$

so the expression $\dfrac{1 - \cos x}{3x^2}$ is also an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(3x^2)} = \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{6} \cdot 1 = \boxed{\frac{1}{6}}.$$

53. Because

$$\lim_{x \to \infty} (1 + 4x) = \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{2}{x} = 0,$$

the expression $(1+4x)^{2/x}$ is an indeterminate form at ∞ of the type ∞^0. Let $y = (1+4x)^{2/x}$. Then

$$\ln y = \ln(1 + 4x)^{2/x} = \frac{2}{x} \ln(1 + 4x) = \frac{2\ln(1 + 4x)}{x},$$

which is an indeterminate form at ∞ of the type $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{2\ln(1 + 4x)}{x} = \lim_{x \to \infty} \frac{\frac{d}{dx}[2\ln(1 + 4x)]}{\frac{d}{dx} x} = \lim_{x \to \infty} \frac{\frac{8}{1+4x}}{1} = 0.$$

Because $\lim_{x \to \infty} \ln y = 0$, it follows that

$$\lim_{x \to \infty} y = \lim_{x \to \infty} (1 + 4x)^{2/x} = e^0 = \boxed{1}.$$

55. Because

$$\lim_{x \to 4} (x^2 - 16) = 0 \quad \text{and} \quad \lim_{x \to 4} (x^2 + x - 20) = 0,$$

the expression $\dfrac{x^2 - 16}{x^2 + x - 20}$ is an indeterminate form at 0 of the type $\dfrac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \to 4} \frac{x^2 - 16}{x^2 + x - 20} = \lim_{x \to 4} \frac{\frac{d}{dx}(x^2 - 16)}{\frac{d}{dx}(x^2 + x - 20)} = \lim_{x \to 4} \frac{2x}{2x + 1} = \boxed{\frac{8}{9}}.$$

Alternately,

$$\lim_{x \to 4} \frac{x^2 - 16}{x^2 + x - 20} = \lim_{x \to 4} \frac{(x + 4)(x - 4)}{(x - 4)(x + 5)} = \lim_{x \to 4} \frac{x + 4}{x + 5} = \boxed{\frac{8}{9}}.$$

57. The general solution of the differential equation

$$\frac{dy}{dx} = e^x$$

is $y = e^x + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $y = 2$ yields

$$2 = e^0 + C = 1 + C \quad \text{so that} \quad C = 1.$$

The particular solution of the differential equation with the given boundary condition is therefore $\boxed{y = e^x + 1}$.

59. The general solution of the differential equation

$$\frac{dy}{dx} = \frac{2}{x}$$

is $y = 2\ln|x| + C$, where C is a constant. Applying the boundary condition that when $x = 1$, then $y = 4$ yields

$$4 = 2\ln 1 + C = C.$$

The particular solution of the differential equation with the given boundary condition is therefore $\boxed{y = 2\ln|x| + 4}$.

61. The cost of producing x items is $C(x) = 200 + 35x + 0.02x^2$ and each item can be sold for \$78, so the revenue produced by selling x items is $R(x) = 78x$. The profit generated by producing and selling x items is then

$$P(x) = R(x) - C(x) = 78x - (200 + 35x + 0.02x^2) = -200 + 43x - 0.02x^2.$$

The polynomial function P is differentiable everywhere, so the critical numbers of P occur where $P'(x) = 0$. Now,

$$P'(x) = 43 - 0.04x,$$

so $x = 1075$ is the only critical number. Using the Second Derivative Test,

$$P''(x) = -0.04 < 0$$

for all x, so P has both a local maximum and an absolute maximum at 1075. Therefore, $\boxed{\text{1075 items}}$ should be produced and sold to maximize profit.

63. Let $(x, \ln x)$ be the coordinates of the vertex of the rectangle on the graph of $y = \ln x$ with $0 < x < 1$. The area A of the rectangle is

$$A = -x\ln x,$$

where the negative sign is included because $\ln x < 0$ for $0 < x < 1$, and

$$A'(x) = -\left(x \cdot \frac{1}{x} + \ln x\right) = -(1 + \ln x).$$

$A'(x)$ exists everywhere on the interval $(0, 1)$ and is equal to zero when $x = e^{-1}$. Using the Second Derivative Test,

$$A''(x) = -\frac{1}{x} \quad \text{so} \quad A''(e^{-1}) = -e < 0,$$

and A has a local maximum at e^{-1}. Because $A''(x) < 0$ for all $0 < x < 1$, the local maximum is also an absolute maximum. Therefore, the area of the largest rectangle in the fourth quadrant that has three vertices on the coordinate axes and the fourth vertex on the graph of $y = \ln x$ is

$$A = -e^{-1}\ln e^{-1} = e^{-1} = \boxed{\frac{1}{e}}.$$

AP® Review Problems

1. From the graph of $f(x)$ it appears that for $x<0$, $f'(x)<0$; for $0<x<1.4$, $f'(x)>0$; and for $x>1.4$, $f'(x)<0$. $f'(x)=0$ at $x=0$ and $x\approx 1.4$.

   $\boxed{\text{The only graph of } f \text{ to satisfy these conditions is graph B}}$.

   CHOICE B

3. $y(t)=te^{-t^2}$

   $y'(t)=e^{-t^2}+t(e^{-t^2}(-2t))=e^{-t^2}(1-2t^2)$

   The object is at rest when $y'(t)=0$

   $$\begin{aligned} y'(t)=e^{-t^2}(1-2t^2)&=0\\ 1-2t^2&=0\\ t&=\pm\frac{\sqrt{2}}{2} \end{aligned}$$

   For the domain of $t\geq 0$ the sole value for t is $\boxed{t=\frac{\sqrt{2}}{2}}$.

   CHOICE A

5. The Intermediate Value Theorem as applied to this problem provides that since f is continuous for all real numbers and since 0 is a y-coordinate between $f(-4)=3$ and $f(1)=-8$ there is at least one number c in the domain such that $f(c)=0$.

   CHOICE C

7. $\lim\limits_{x\to\infty}\dfrac{x}{\ln x}$

   Since $\lim\limits_{x\to\infty}x=\infty$ and $\lim\limits_{x\to\infty}(\ln x)=\infty$, $\lim\limits_{x\to\infty}\frac{x}{\ln x}$ is an indeterminate form of the type $\frac{\infty}{\infty}$ and L'Hôpital's Rule is applicable as follows.

   $$\lim_{x\to\infty}\frac{x}{\ln x}=\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\lim_{x\to\infty}x=\boxed{\infty}.$$

   CHOICE D

9. Consider each choice in turn.

   I. With the given information, f could be an even function greater than 2 and $f(c)$ is not necessarily equal to 0.

   II. Rolle's Theorem, a specific application of the Mean Value Theorem, provides that if f is a continuous function, as it is here being a polynomial function, and differentiable on the open interval (a,b) then if $f(a)=f(b)$, then there is at least one number c in (a,b) such that $f'(c)=0$. So,

   $\boxed{f'(c)=0 \text{ is true.}}$

   III. $f''(c)=0$ is not true as there can be a function f with the specified requirements without a change of concavity. For instance, for

   $$\begin{aligned} f(x)&=x^4+x^2\\ f'(x)&=4x^3+2x\\ f''(x)&=12x^2+2 \end{aligned}$$

   there is no c in (a,b) for which $f''(c)=12x^2+2=0$.

   CHOICE B

11.
$$\begin{aligned}
V &= \pi r^2 h \\
h + 2\pi r &= 300 \\
h &= 300 - 2\pi r \\
V &= \pi r^2(300 - 2\pi r) \\
V &= 300\pi r^2 - 2\pi^2 r^3 \\
V' &= 600\pi r - 6\pi^2 r^2 \\
\text{Let } V' &= 600\pi r - 6\pi^2 r^2 = 0 \\
6\pi r(100 - \pi r) &= 0 \\
100 - \pi r &= 0 \\
r &= \boxed{\frac{100}{\pi}\text{ m}} \\
h &= 300 - 2\pi r \\
&= 300 - 2\pi\left(\frac{100}{\pi}\right) \\
h &= \boxed{100\,\text{m}}.
\end{aligned}$$

CHOICE B

13. (a) For f, a polynomial function, the critical numbers will be determined at

$$\begin{aligned}
f'(x) &= 0 \\
\text{For } f(x) &= x^3 + 3x^2 + 2 \\
f'(x) &= 3x^2 + 6x \\
\text{Let } f'(x) &= 3x^2 + 6x = 0 \\
3x(x+2) &= 0 \\
x &= 0 \quad x = -2
\end{aligned}$$

The critical numbers for f are both $\boxed{x = 0 \text{ and } x = -2}$

(b)

Interval	Sign of x	Sign of $x+2$	Sign of $f'(x) = 3x^2 + 6x$	Conclusion
$(-\infty, -2)$	$-$	$-$	$+$	Increasing
$(-2, 0)$	$-$	$+$	$-$	Decreasing
$(0, \infty)$	$+$	$+$	$+$	Increasing

f is increasing on $\boxed{(-\infty, -2] \cup [0, \infty)}$

(c) From the chart in (b) above, the local extreme points occur at $x = -2$ and $x = 0$

$$f(-2) = (-2)^3 + 3(-2)^2 + 2 = 6$$
$$f(0) = 0^3 + 3(0)^2 + 2 = 2$$

By the first derivative test, there is a $\boxed{\text{local maximum at } (-2, 6)}$ since f changes from increasing to decreasing from left to right about the critical number, $x = -2$.

By the first derivative test, there is a $\boxed{\text{local minimum at } (0, 2)}$ since f changes from decreasing to increasing from left to right about the critical number, $x = 0$.

(d) For $f(x) = x^3 + 3x^2 + 2$

$$f'(x) = 3x^2 + 6x$$
$$f''(x) = 6x + 6$$

Let $f''(x) = 6x + 6 = 0$

$$x = -1$$

Interval	Sign of $f''(x)$	Conclusion
$(-\infty, -1)$	$-$	Concave Down
$(-1, \infty)$	$+$	Concave Up

(e) Referring to the chart in (d) above, there is a Point of Inflection at $x = -1$ since f changes concavity, from concave down to concave up, at $x = -1$.

$$f(-1) = (-1)^3 + 3(-1)^2 + 2 = 4$$

The point of inflection is at $\boxed{(-1, 4)}$.

(f) $y - 4 = f'(-1)(x - (-1))$

$$y - 4 = -3(x + 1)$$
$$y - 4 = -3x - 3$$
$$\boxed{y = -3x + 1}.$$

15. (a) $\dfrac{d^2y}{dx^2} = 3x^2 - 6x$

$$\frac{dy}{dx} = x^3 - 3x^2 + C_1$$

$$\boxed{y = \frac{x^4}{4} - x^3 + C_1 x + C_2}.$$

(b)
$$y = \frac{x^4}{4} - x^3 + C_1 x + C_2$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^4}{4}\right) - \frac{d}{dx}(x^3) + \frac{d}{dx}(C_1 x) + \frac{d}{dx}(C_2)$$
$$= x^3 - 3x^2 + C_1$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + \frac{d}{dx}(C_1)$$
$$= 3x^2 - 6x.$$

(c) $y = \dfrac{x^4}{4} - x^3 + C_1 x + C_2$

For $(0, 2), 2 = C_2$

For $(1, 3), 3 = \dfrac{1^4}{4} - 1^3 + C_1(1) + 2$

$$3 = \frac{1}{4} - 1 + C_1 + 2$$
$$C_1 = \frac{7}{4}$$
$$\boxed{y = \frac{x^4}{4} - x^3 + \frac{7}{4}x + 2}.$$

AP® Practice Exam
Big Ideas 1 and 2: Limits and Derivatives

Section 1: Multiple Choice

1. $$\lim_{x\to\infty}\left(\frac{3x^2-5x-2}{x^2-4}\right)\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)=\lim_{x\to\infty}\left(\frac{\frac{3x^2}{x^2}-\frac{5x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}}\right)=\lim_{x\to\infty}\frac{3-\frac{5}{x}-\frac{2}{x^2}}{1-\frac{4}{x^2}}=\boxed{3}.$$

An alternative approach is to examine the applicability of L'Hôpital's Rule.

$$\lim_{x\to\infty}\left(3x^2-5x-2\right)=\infty$$
$$\lim_{x\to\infty}\left(x^2-4\right)=\infty$$

Since $\lim_{x\to\infty}\left(3x^2-5x-2\right)=\infty$ and $\lim_{x\to\infty}\left(x^2-4\right)=\infty$

$\lim_{x\to\infty}\left(\frac{3x^2-5x-2}{x^2-4}\right)$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$ and L'Hôpital's Rule is applicable as follows:

$$\lim_{x\to\infty}\left(\frac{3x^2-5x-2}{x^2-4}\right)=\lim_{x\to\infty}\frac{6x-5}{2x}$$

Since $\lim_{x\to\infty}(6x-5)=\infty$ and $\lim_{x\to\infty}(2x)=\infty$, $\lim_{x\to\infty}\frac{6x-5}{2x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$ and L'Hôpital's Rule once again is applicable as follows:

$$\lim_{x\to\infty}\left(\frac{3x^2-5x-2}{x^2-4}\right)=\lim_{x\to\infty}\frac{6x-5}{2x}=\frac{6}{2}=\boxed{3}.$$

CHOICE C

2. $$\frac{d}{dx}\left(\tan^{-1}x^2\right)=\frac{1}{1+\left(x^2\right)^2}(2x)=\frac{2x}{1+x^4}.$$

CHOICE A

3. $$\lim_{x\to\infty}\left(\frac{\cos x}{x^2+4x}\right)\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)=\lim_{x\to\infty}\frac{\frac{\cos x}{x^2}}{1+\frac{4}{x}}$$

$$\lim_{x\to\infty}\frac{-1}{x^2}\le\lim_{x\to\infty}\frac{\cos x}{x^2}\le\lim_{x\to\infty}\frac{1}{x^2}$$
$$0\le\lim_{x\to\infty}\frac{\cos x}{x^2}\le 0$$

By the Squeeze Theorem $\lim_{x\to\infty}\frac{\cos x}{x^2}=0$

Therefore, $\lim_{x\to\infty}\frac{\frac{\cos x}{x^2}}{1+\frac{4}{x}}=\frac{0}{1}=\boxed{0}.$

L'Hôpital's Rule would not have been applicable since $\lim_{x\to\infty}(\cos x)\ne\infty$ while $\lim_{x\to\infty}\left(x^2+4x\right)=\infty$ and $\lim_{x\to\infty}\left(\frac{\cos x}{x^2+4x}\right)$ is not indeterminate form.

CHOICE B

4. $f(x) = \dfrac{x^2+2x-ax-2a}{x-a} = \dfrac{x(x+2)-a(x+2)}{x-a} = \dfrac{(x-a)(x+2)}{x-a} = x+2$

$f(a) = \boxed{a+2}$.

CHOICE C

5. $\lim\limits_{h\to 0} \dfrac{\cos(2\pi+h)-1}{h} = \lim\limits_{h\to 0} \dfrac{\cos 2\pi\cos(h) - \sin 2\pi \sin(h) - 1}{h} = \lim\limits_{h\to 0} \dfrac{1\cos(h) - 0\sin(h) - 1}{h} =$

$\lim\limits_{h\to 0} \dfrac{\cos(h)-1}{h}\left(\dfrac{\cos(h)+1}{\cos(h)+1}\right) = \lim\limits_{h\to 0} \dfrac{\cos^2 h - 1}{h(\cos(h)+1)} = \lim\limits_{h\to 0} \dfrac{-\sin^2 h}{h(\cos(h)+1)} =$

$\lim\limits_{h\to 0}\left[-\left(\dfrac{\sin(h)}{h}\right)\left(\dfrac{\sin(h)}{\cos(h)+1}\right)\right] = -\lim\limits_{h\to 0}\left(\dfrac{\sin(h)}{h}\right)\lim\limits_{h\to 0}\left(\dfrac{\sin(h)}{\cos(h)+1}\right) = (1)(0) = \boxed{0}$.

An alternate approach is to apply L'Hôpital's Rule.

For $\lim\limits_{h\to 0} \frac{\cos(2\pi+h)-1}{h}$, $\lim\limits_{h\to 0}(\cos(2\pi+h)-1) = 0$ and $\lim\limits_{h\to 0}(h) = 0$ so $\lim\limits_{h\to 0} \frac{\cos(2\pi+h)-1}{h}$ is an indeterminate form at 0 of the type $\frac{0}{0}$ and L'Hôpital's Rule is applicable as follows: $\lim\limits_{h\to 0} \frac{\cos(2\pi+h)-1}{h} = \lim\limits_{h\to 0} \frac{-\sin(2\pi+h)}{1} = -\sin 2\pi = \boxed{0}$.

CHOICE B

6. Since f is differentiable, $2ax^2 + bx - 1 = bx^2 + bx - a$ at $x = 3$

$$\begin{aligned}(2a-b)x^2 &= 1-a\\ x^2 &= \frac{1-a}{2a-b}\\ \text{at } x=3,\quad 9 &= \frac{1-a}{2a-b}\end{aligned}$$

$$f'(x) = \begin{cases} 4ax+b & x \le 3\\ 2bx+b & x > 3\end{cases}$$

Since f is differentiable, $4ax+b = 2bx+b$ at $x=3$

$$2a = b$$

Solve the system: $\begin{cases} 9 = \frac{1-a}{2a-b}\\ 2a = b\end{cases}$

$$\begin{aligned}18a - 9b &= 1-a\\ 19a - 9b &= 1\\ 19a - 9(2a) &= 1\\ a &= 1\\ b &= 2\end{aligned}$$

$\boxed{a+b = 1+2 = 3}$.

CHOICE C

7. $s(t) = \frac{1}{15}t^3 - \frac{1}{2}t^2 + 5t^{-1}$

$$v(t) = s'(t) = \frac{3}{15}t^2 - \frac{2}{2}t - 5t^{-2}$$
$$= \frac{t^2}{5} - t - 5t^{-2}$$
$$a(t) = v'(t) = s''(t) = \frac{2t}{5} - 1 + 10t^{-3}$$
$$a(5) = \frac{2(5)}{5} - 1 + \frac{10}{(5)^3} = \boxed{\frac{27}{25}}.$$

CHOICE D

8. For $\lim_{x\to 0} \frac{x^2e^x}{\cos x - 1}$

$\lim_{x\to 0} x^2e^x = 0$ and $\lim_{x\to 0} (\cos x - 1) = 0$

Therefore $\lim_{x\to 0} \frac{x^2e^x}{\cos x - 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$ and L'Hôpital's Rule is applicable as follows:

$$\lim_{x\to 0} \frac{x^2e^x}{\cos x - 1} = \lim_{x\to 0} \frac{2xe^x + x^2e^x}{-\sin x}$$

For $\lim_{x\to 0} \frac{2xe^x + x^2e^x}{-\sin x}$

$\lim_{x\to 0} \left(2xe^x + x^2e^x\right) = 0$ and $\lim_{x\to 0} (-\sin x) = 0$

Therefore $\lim_{x\to 0} \frac{2xe^x + x^2e^x}{-\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$ and L'Hôpital's Rule is again applicable as follows:

$$\lim_{x\to 0} \frac{2xe^x + x^2e^x}{-\sin x} = \lim_{x\to 0} \frac{2e^x + 2xe^x + 2xe^x + x^2e^x}{-\cos x} = \lim_{x\to 0} \frac{2\left(e^x\left(1 + 2x + x^2\right)\right)}{-\cos x} = \boxed{-2}.$$

CHOICE B

9. The slope of the tangent line is $f'(x)$.

$$f'(x) = e^x + xe^x$$
$$f'(1) = e^{(1)} + (1)e^{(1)} = \boxed{2e}.$$

CHOICE C

10. $f(x) = \sqrt{2x+4} = (2x+4)^{1/2}$

$$f'(x) = \frac{1}{2}(2x+4)^{-1/2}(2) = (2x+4)^{-1/2}$$
$$f''(x) = \frac{-1}{2}(2x+4)^{-3/2}(2) = \frac{-1}{(2x+4)^{3/2}}$$
$$f''(-1) = \frac{-1}{(-2+4)^{3/2}} = \frac{-1}{2^{3/2}} = \frac{-1}{2\sqrt{2}} = \boxed{\frac{-\sqrt{2}}{4}}.$$

CHOICE B

11. $\dfrac{d}{dx}\left(\dfrac{\ln x}{x^2}\right) = \dfrac{\frac{d}{dx}(\ln x)\left(x^2\right) - (\ln x)\left(\frac{d}{dx}\left(x^2\right)\right)}{x^4} = \dfrac{\frac{x^2}{x} - (\ln x)(2x)}{x^4} =$

$\dfrac{x - 2x\ln x}{x^4} = \dfrac{x(1 - 2\ln x)}{x^4} = \boxed{\dfrac{1 - 2\ln x}{x^3}}.$

CHOICE C

12. $f(x) = \tan\left(x^2\right)$

$f'(x) = \left(\sec^2\left(x^2\right)\right)(2x) = 2x\sec^2\left(x^2\right)$

$f'\left(\sqrt{\dfrac{\pi}{3}}\right) = 2\left(\sqrt{\dfrac{\pi}{3}}\right)\sec^2\left(\sqrt{\dfrac{\pi}{3}}\right)^2 = \dfrac{2\sqrt{3\pi}}{3}\left(\dfrac{1}{\cos^2\left(\frac{\pi}{3}\right)}\right) =$

$\dfrac{2\sqrt{3\pi}}{3}\left(\dfrac{1}{\frac{1}{4}}\right) = \boxed{\dfrac{8\sqrt{3\pi}}{3}}.$

CHOICE C

13. $f(x) = 3x\sqrt{\cos(3x)} = 3x(\cos(3x))^{1/2}$

$f'(x) = 3\Big[1(\cos(3x))^{1/2} + \dfrac{x}{2}(\cos(3x))^{-1/2}(-\sin(3x))(3)\Big] = 3\left[(\cos(3x))^{1/2} - \dfrac{3x\sin(3x)}{2(\cos(3x))^{1/2}}\right]$

$f'(0) = 3\left[(\cos(3(0)))^{1/2} - \dfrac{3(0)\sin(3(0))}{2(\cos(3(0)))^{1/2}}\right] = 3\Big[(\cos 0)^{1/2} - 0\Big] = \boxed{3}.$

CHOICE D

14. $y = \sin(2x + 1)$

$y' = 2\cos(2x + 1)$

$y\left(-\dfrac{1}{2}\right) = \sin\left(2\left(-\dfrac{1}{2}\right) + 1\right) = \sin(0) = 0$

The point of tangency is $\left(\dfrac{-1}{2}, 0\right)$

$y'\left(-\dfrac{1}{2}\right) = 2\cos\left(2\left(-\dfrac{1}{2}\right) + 1\right) = 2\cos 0 = 2$

$y - 0 = 2\left(x - \left(-\dfrac{1}{2}\right)\right)$

$\boxed{y = 2x + 1}.$

CHOICE B

15. $x^2 - 2xy - y^2 = -1$

$$\frac{d}{dx}\left(x^2 - 2xy - y^2\right) = \frac{d}{dx}(-1)$$

$$\frac{d}{dx}\left(x^2\right) - 2\left(\left(\frac{d}{dx}(x)\right)y + x\left(\frac{dy}{dx}\right)\right) - 2(y)\left(\frac{dy}{dx}\right) = 0$$

$$2x - 2\left(y + x\left(\frac{dy}{dx}\right)\right) - 2y\left(\frac{dy}{dx}\right) = 0$$

$$2x - 2y - 2x\left(\frac{dy}{dx}\right) - 2y\left(\frac{dy}{dx}\right) = 0$$

$$x - y - x\left(\frac{dy}{dx}\right) - y\left(\frac{dy}{dx}\right) = 0$$

$$(-x - y)\left(\frac{dy}{dx}\right) = -x + y$$

$$\frac{dy}{dx} = \frac{-x + y}{-x - y} = \frac{-1(x - y)}{-1(x + y)} = \boxed{\frac{x - y}{x + y}}.$$

CHOICE D

16.
$$x + \sin(xy) + y = 1$$

$$1 + \cos(xy)\left(1y + x\frac{dy}{dx}\right) + \frac{dy}{dx} = 0$$

$$1 + y\cos(xy) + x\cos(xy)\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$(x\cos(xy) + 1)\frac{dy}{dx} = -1 - y\cos(xy)$$

$$\frac{dy}{dx} = \frac{-1 - y\cos(xy)}{x\cos(xy) + 1}$$

Solve for y at $x = 0$:

$$(0) + \sin(0) + y = 1$$
$$y = 1$$

at $(0, 1)$ $\frac{dy}{dx} = \frac{-1 - (1)\cos(0)}{(0)\cos(0) + 1}$

$$= \frac{-1 - (1)(1)}{0\cos 0 + 1}$$

$$= \frac{-2}{1}$$

$$= \boxed{-2}.$$

CHOICE A

17. $y = \frac{1}{2^{3x}} = 2^{-3x}$

$$y' = \left(2^{-3x}\right)(-3)(\ln 2) = (-1)\left(2^{-3x}\right)(3\ln 2) = (-1)\left(2^{-3x}\right)\left(\ln 2^3\right) = \boxed{-2^{-3x}(\ln 8)}.$$

CHOICE A

18. The critical number(s) can be determined where $f'(x) = 0$ or where $f'(x)$ does not exist.

    Here, the critical number(s) are determined solely at $f'(x) = 0$ as follows:

$$
\begin{aligned}
f(x) &= 2xe^{-x^2} \\
f'(x) &= 2\Big(1\Big(e^{-x^2}\Big) + x\Big(e^{-x^2}\Big)(-2x)\Big) \\
&= 2\Big(e^{-x^2}(1-2x^2)\Big)
\end{aligned}
$$

Let

$$
\begin{aligned}
f'(x) = 2\Big(e^{-x^2}(1-2x^2)\Big) &= 0 \\
1 - 2x^2 &= 0 \\
2x^2 &= 1 \\
x^2 &= \frac{1}{2} \\
x &= \boxed{\pm\frac{\sqrt{2}}{2}}.
\end{aligned}
$$

CHOICE C

19. $f(0) = \ln 3 \quad f'(0) = 2 \quad g(0) = -\frac{1}{2} \quad g'(0) = \frac{1}{2}$

$$
\begin{aligned}
h(x) &= \frac{e^{f(x)}}{(g(x))^2} \\
h'(x) &= \frac{e^{f(x)}(f'(x))(g(x))^2 - 2g(x)g'(x)e^{f(x)}}{(g(x))^4} \\
h'(0) &= \frac{e^{f(0)}(f'(0))(g(0))^2 - 2g(0)g'(0)e^{f(0)}}{(g(0))^4} \\
&= \frac{e^{\ln 3}(2)\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)e^{\ln 3}}{\left(-\frac{1}{2}\right)^4} \\
&= \frac{3(2)\left(\frac{1}{4}\right) + \frac{1}{2}(3)}{\frac{1}{16}} \\
&= \frac{\frac{3}{2} + \frac{3}{2}}{\frac{1}{16}} \\
&= \boxed{48}.
\end{aligned}
$$

CHOICE D

20. The function f has no derivative at $x = -2$ because there is a cusp at $x = -2$. f has no derivative at $x = 0$ since it has a vertical tangent at $x = 0$.

    CHOICE B

21. $f'(x) = x^2e^x - 5xe^x + 6e^x$

Set $f'(x) = x^2e^x - 5xe^x + 6e^x = 0$ to determine the critical number(s), if any, to then define the interval(s) where f is increasing.

$$\begin{aligned} f'(x) = x^2e^x - 5xe^x + 6e^x &= 0 \\ e^x\left(x^2 - 5x + 6\right) &= 0 \\ e^x(x-3)(x-2) &= 0 \\ x = 3 \quad x &= 2 \end{aligned}$$

Interval	Sign of e^x	Sign of $x-3$	Sign of $x-2$	Sign of $f'(x) = 3x^2 + 6x$	Conclusion
$(-\infty, 2)$	$+$	$-$	$-$	$+$	Increasing
$(2, 3)$	$+$	$-$	$+$	$-$	Decreasing
$(3, \infty)$	$+$	$+$	$+$	$+$	Increasing

$\boxed{f \text{ is increasing for } x \leq 2 \text{ and } x \geq 3}$.

22. Summarizing in chart form the pertinent information from the graph and the given for determining where the function f is decreasing, which is where $f'(x) < 0$, yields

Interval	Sign of $f'(x)$	Conclusion
$(-\infty, -2)$	$+$	Increasing
$(-2, 0)$	$+$	Increasing
$(0, 1)$	$-$	Decreasing
$(1, \infty)$	$+$	Increasing

The function f is decreasing on $\boxed{0, 1}$.

CHOICE C

23. The function f is concave down where $f''(x) < 0$ determined on the graph where the tangent line(s) to the graph of f' have a negative slope. Summarizing the pertinent information from the graph along with the given information yields the following chart:

Interval	Sign of $f''(x)$	Conclusion
$(-\infty, -2)$	$-$	Concave Down
$(-2, -0.843)$	$+$	Concave Up
$(-0.843, 0.593)$	$-$	Concave Down
$(0.593, \infty)$	$+$	Concave Up

$\boxed{f \text{ is concave down on } (-\infty, -2) \text{ and } (-0.843, 0.593).}$

CHOICE A

24. $f(x) = \ln\left(2x^2\right)$ is differentiable on the interval $\left[\frac{1}{\sqrt{2}}, \sqrt{\frac{e}{2}}\right]$ and is therefore also continuous on $\left[\frac{1}{\sqrt{2}}, \sqrt{\frac{e}{2}}\right]$, thereby satisfying the conditions of the Mean Value Theorem.

$$f\left(\frac{\sqrt{2}}{2}\right) = \ln\left(2\left(\frac{\sqrt{2}}{2}\right)^2\right) = \ln\left(2\left(\frac{1}{2}\right)\right) = \ln(1) = 0$$

$$f\left(\frac{\sqrt{2e}}{2}\right) = \ln\left(2\left(\frac{\sqrt{2e}}{2}\right)^2\right) = \ln\left(2\left(\frac{2e}{4}\right)\right) = \ln e = 1$$

The endpoints on the interval are $\left(\frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2e}}{2}, 1\right)$

The number(s) c in the open interval $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2e}}{2}\right)$ guaranteed by the Mean Value Theorem satisfy the equation $f'(c) = \dfrac{1-0}{\frac{\sqrt{2e}}{2} - \frac{\sqrt{2}}{2}} = \dfrac{2}{\sqrt{2e} - \sqrt{2}}$

$$\begin{aligned} f(x) &= \ln\left(2x^2\right) = \ln 2 + 2\ln x \\ f'(x) &= \frac{d}{dx}(\ln 2 + 2\ln x) = \frac{d}{dx}(\ln 2) + 2\frac{d}{dx}(\ln x) = \frac{2}{x} \\ f'(c) &= \frac{2}{c} = \frac{2}{\sqrt{2e} - \sqrt{2}} \\ c &= \sqrt{2e} - \sqrt{2} = \boxed{\sqrt{2}(\sqrt{e} - 1)}. \end{aligned}$$

CHOICE D

25. $\dfrac{db}{dt} = 2 \quad \dfrac{dh}{dt} = -1$

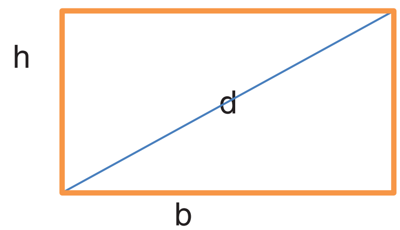
$$A = \frac{1}{2}bh$$

at $h = 5 \quad A = 5$

$$\begin{aligned} 5 &= \frac{1}{2}(b)(5) \\ b &= 2 \\ \frac{dA}{dt} &= \frac{1}{2}\left(b\left(\frac{dh}{dt}\right) + h\frac{db}{dt}\right) = \frac{1}{2}(2(-1) + 5(2)) = \boxed{4}. \end{aligned}$$

CHOICE B

26.



$$\frac{db}{dt} = \frac{dh}{dt} = 3$$
$$b^2 + h^2 = d^2$$
$$2b\left(\frac{db}{dt}\right) + 2h\left(\frac{dh}{dt}\right) = 2d\left(\frac{dd}{dt}\right)$$
$$b\left(\frac{db}{dt}\right) + h\left(\frac{dh}{dt}\right) = d\left(\frac{dd}{dt}\right)$$
$$4(3) + 3(3) = 5\left(\frac{dd}{dt}\right)$$
$$\left(\frac{dd}{dt}\right) = \frac{21}{5} = \boxed{4.2}.$$

CHOICE B

27. $s(t) = 3t^4 - 8t^3 - 6t^2 + 24t$

$$\begin{aligned} v(t) = s'(t) = 12t^3 - 24t^2 - 12t + 24 &= 0 \\ t^3 - 2t^2 - t + 2 &= 0 \\ t^2(t-2) - 1(t-2) &= 0 \\ \left(t^2 - 1\right)(t-2) &= 0 \\ (t-1)(t+1)(t-2) &= 0 \\ t = 1 \quad t = -1 \quad t &= 2 \end{aligned}$$

$\boxed{\text{The toddler is at rest at } t = 1 \text{ and } t = 2.}$

CHOICE D

28. $f(x) = xe^{1-x}$

$$\begin{aligned} f'(x) &= 1\left(e^{1-x}\right) + x\left(e^{1-x}\right)(-1) \\ &= e^{1-x}(1-x) \end{aligned}$$

Solve $f'(x) = e^{1-x}(1-x) = 0$ to determine the critical number(s).

$$1 - x = 0$$
$$x = 1$$

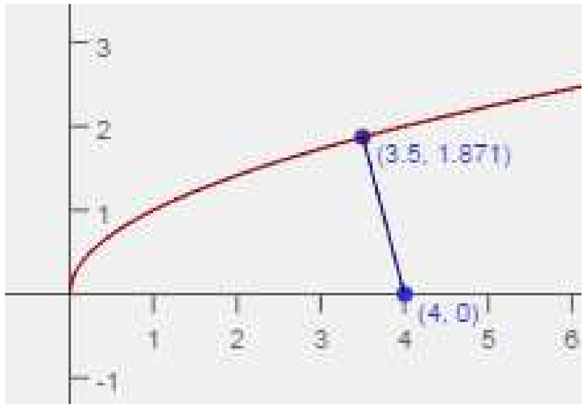
The absolute maximum of f on $[-5, 5]$ occurs at $f(-5)$, $f(5)$, or $f(1)$

$$f(-5) = -5e^{1+5} = -5e^6$$
$$f(5) = 5e^{1-5} = \frac{5}{e^4}$$
$$f(1) = 1e^0 = 1$$

Since $-5e^6 < \frac{5}{e^4} < 1$, $\boxed{\text{the absolute maximum of } f \text{ on } [-5, 5] \text{ is } 1.}$

CHOICE C

29.



$$
\begin{aligned}
y &= \sqrt{x} \\
D &= \sqrt{(x-4)^2 + (y-0)^2} = \sqrt{(x-4)^2 + (y)^2} = \sqrt{(x-4)^2 + \left(\sqrt{x}\right)^2} \\
&= \sqrt{x^2 - 8x + 16 + x} = \sqrt{x^2 - 7x + 16} \\
D^2 &= x^2 - 7x + 16 \\
2DD' &= 2x - 7
\end{aligned}
$$

Let

$$
\begin{aligned}
2DD' &= 2x - 7 = 0 \\
x &= \frac{7}{2} = \boxed{3.5}.
\end{aligned}
$$

CHOICE B

30. $f''(x) = 2xe^x(x-1)$

Let

$$
\begin{aligned}
f''(x) &= 2xe^x(x-1) = 0 \\
x &= 0 \quad x = 1
\end{aligned}
$$

Interval	Sign of x	Sign of e^x	Sign of $x-1$	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	$-$	$+$	$-$	$+$	Concave Up
$(0, 1)$	$+$	$+$	$-$	$-$	Concave Down
$(1, \infty)$	$+$	$+$	$+$	$+$	Concave Up

The inflection point(s) of f occur where the concavity of f changes. A review of the conclusions summarized in the chart above show a change of concavity at $x = 0$ and $x = 1$. $\boxed{\text{Therefore, } f \text{ has inflection points at 0 and 1.}}$

CHOICE D

Section 2: Free Response

1. (a)

$$
\begin{aligned}
(x-1)^2+(y+1)^2 &= 2\\
\frac{d}{dx}\left[(x-1)^2+(y+1)^2\right] &= \frac{d}{dx}(2)\\
2(x-1)+2(y+1)\frac{dy}{dx} &= 0\\
(x-1)+(y+1)\frac{dy}{dx} &= 0\\
(y+1)\frac{dy}{dx} &= -x+1\\
\boxed{\frac{dy}{dx} = \frac{-x+1}{y+1}}.
\end{aligned}
$$

(b) A horizontal tangent line to the graph will exist where $\frac{dy}{dx}=0$
$\frac{dy}{dx}=0$ where

$$
\begin{aligned}
-x+1 &= 0\\
x &= 1
\end{aligned}
$$

Determine the point of tangency for each horizontal tangent by determining y for $x=1$ as follows:

$$
\begin{aligned}
(1-1)^2+(y+1)^2 &= 2\\
(y+1)^2 &= 2\\
y+1 &= \pm\sqrt{2}\\
y &= -1\pm\sqrt{2}
\end{aligned}
$$

Two points with horizontal tangents: $\left(1,-1+\sqrt{2}\right)$ and $\left(1,\ -1-\sqrt{2}\right)$.

Equations of horizontal tangents: $\boxed{y=-1+\sqrt{2}\text{ and }y=-1-\sqrt{2}}$.

(c) For the equation of the normal line, $y=x+b,\ m=1$.
Consequently, the slope of the tangent line, $\frac{dy}{dx}=-1=\frac{-x+1}{y+1}$

$$
\begin{aligned}
-y-1 &= -x+1\\
x-y &= 2\\
y &= x-2\\
(x-1)^2+(y+1)^2 &= 2\\
(x-1)^2+(x-2+1)^2 &= 2\\
(x-1)^2+(x-1)^2 &= 2\\
2(x-1)^2 &= 2\\
(x-1)^2 &= 1\\
x-1 &= \pm 1\\
x &= 1\pm 1\\
x &= 0 \quad x=2
\end{aligned}
$$

Substituting $x=0$ $x=2$ into $y=x-2$ yields the points $(2,0)$ and $(0,-2)$
For $y=x+b$,
Substituting $(2,0)$ yields $0=2+b \quad b=-2$
Substituting $(0,-2)$ yields $-2=0+b \quad b=-2$

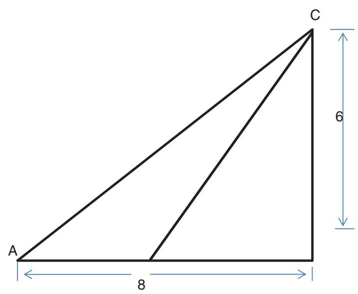
(d) at $(2, 0)$ with $\frac{dy}{dx} = -1$

$y - 0 = -1(x - 2)$ $\boxed{y = -x + 2}$

at $(0, -2)$ with $\frac{dy}{dx} = -1$

$y - (-2) = -1(x - 0)$ $y + 2 = -x$ $\boxed{y = -x - 2}$.

2.



(a) Let z be the distance that the dog swims whether it be a direct swim from A to C or a partial swim from D to C after a partial run. Then $\frac{dz}{dt} = 2$. The dog runs at $\frac{dx}{dt} = 8$. The distance the dog swims directly from A to C is
$AC = \sqrt{6^2 + 8^2} = 10$.

$$\text{Time} = \frac{\text{Distance}}{\text{Rate}} = \frac{10}{\frac{dx}{dt}} = \frac{10}{2} = \boxed{5s}.$$

(b) $$\frac{\text{Distance Run}}{\frac{dx}{dt}} + \frac{\text{Swimming Distance}}{\frac{dz}{dt}} = \frac{8}{8} + \frac{6}{2} = \boxed{4s}.$$

(c) $$T(x) = \frac{8 - x}{8} + \frac{\sqrt{x^2 + 36}}{2}.$$

(d)
$$\begin{aligned} T'(x) &= \frac{-1}{8} + \frac{1}{2}\left(\frac{1}{2}\left(x^2 + 36\right)^{-1/2}\right)(2x) \\ &= \frac{-1}{8} + \frac{x}{2(x^2 + 36)^{1/2}} \end{aligned}$$

$$\text{Let } T'(x) = \frac{-1}{8} + \frac{x}{2(x^2 + 36)^{1/2}} = 0$$

$$\begin{aligned} \frac{1}{8} &= \frac{x}{2(x^2 + 36)^{1/2}} \\ 8x &= 2\left(x^2 + 36\right)^{1/2} \\ 4x &= \left(x^2 + 36\right)^{1/2} \\ 16x^2 &= x^2 + 36 \\ 15x^2 &= 36 \\ x^2 &= \frac{36}{15} \\ x^2 &= \frac{12}{5} \\ x &= \frac{2\sqrt{15}}{5} \approx \boxed{1.549}. \end{aligned}$$

(e) $$T(1.549) = \frac{8 - 1.549}{8} + \frac{\sqrt{1.549^2 + 36}}{2} \approx \boxed{3.904s}.$$

3. (a) The object comes to rest when $v(t) = 0$ which is at $t = 0,\ t = 2,\ t = 5$ and $t = 7$ The object changes direction when, from the left to the right of the t value for which $v(t) = 0$, $v(t)$ changes from $v(t) > 0$ to $v(t) < 0$ or $v(t) < 0$ to $v(t) > 0$. $v(t) > 0$ is shown on the graph with positive y coordinates while $v(t) < 0$ is shown on the graph with negative y coordinates. Therefore, the object changes directions at $t = 5$ and $t = 7$.

   (b) The object is moving left on the interval $(5, 7)$ since $v(t) < 0$ on $(5, 7)$ as shown the graph with negative y coordinates on $(5, 7)$.

   (c) The speed of the object is increasing on the interval $[2, 3.949]$ because $a(t) = v'(t)$ is positive on the interval as evidenced by the slope of the line tangent to $v(t)$ at all points on the interval $[2, 3.949]$.

   (d) The acceleration, $a(t)$, is negative on the intervals $(0.564, 2)$ and $(3.949, 6.287)$ because the slope of the line tangent to each of the points on the respective intervals is negative.

   (e) $a(2) = 0$ because the slope of the line tangent to $v(t)$ at $t = 2$ is 0.

4. (a) $f'(x) = \cos^2(e^x) - \sin^2(e^x)$

$$\begin{aligned} f''(x) &= 2(\cos e^x)(-\sin e^x)(e^x) - 2(\sin e^x)(\cos e^x)(e^x) \\ &= -2e^x(\cos e^x)(\sin e^x) - 2e^x(\cos e^x)(\sin e^x) \\ &= -4e^x(\cos e^x)(\sin e^x) \end{aligned}$$

Let $f''(x) = -4e^x(\cos e^x)(\sin e^x) = 0$

$\cos e^x = 0 \qquad \sin e^x = 0$

$e^x = \dfrac{\pi}{2} \qquad e^x = 0 \qquad e^x = \pi$

$x = \ln\left(\dfrac{\pi}{2}\right) \qquad \emptyset \qquad x = \ln \pi$

$x = 0.451 \qquad x = 1.144$

Interval	Sign of -4	Sign of e^x	Sign of $\cos e^x$	Sign of $\sin e^x$	Sign of $f''(x)$	Conclusion
$(0, 0.451)$	$-$	$+$	$+$	$+$	$-$	Concave Down
$(0.451, 1.144)$	$-$	$+$	$-$	$+$	$+$	Concave Up
$(1.144, 1.5)$	$-$	$+$	$-$	$-$	$-$	Concave Down

f has an inflection point at $x = 0.451$ and at $x = 1.144$ because, as shown in the chart above, the concavity of f changes at $x = 0.451$ and at $x = 1.144$.

   (b) $$\begin{aligned} f'(x) &= \cos^2(e^x) - \sin^2(e^x) \\ &= \cos^2(e^x) - \big(1 - \cos^2(e^x)\big) \\ &= 2\cos^2(e^x) - 1 \end{aligned}$$

To determine the critical number(s),

set $2\cos^2(e^x) - 1 = 0$

$$\cos^2(e^x) = \frac{1}{2}$$

$$\cos(e^x) = \frac{\pm\sqrt{2}}{2}$$

$e^x = \dfrac{\pi}{4} \qquad e^x = \dfrac{3\pi}{4} \qquad e^x = \dfrac{5\pi}{4} \qquad e^x = \dfrac{7\pi}{4}$

$x = \ln\dfrac{\pi}{4} \qquad x = \ln\dfrac{3\pi}{4} \qquad x = \ln\dfrac{5\pi}{4} \qquad x = \ln\dfrac{7\pi}{4}$

$x = -0.241 \qquad x = 0.857 \qquad x = 1.367 \qquad x = 1.704$

The only values of x in the domain are $x = 0.857$ and $x = 1.367$

Interval	Sign of $f'(x)$	Conclusion
$(0, 0.857)$	$-$	Decreasing
$(0.857, 1.367)$	$+$	Increasing
$(1.367, 1.5)$	$-$	Decreasing

By the first derivative test, f has a local minimum at the critical number, $x = 0.857$, since the graph of f changes from decreasing to increasing at $x = 0.857$ as shown in the table above.

(c) Referring to the table in part (a) above, $f''(x) < 0$ on the two intervals, $(0, 0.451)$ and $(1.144, 1.5)$.

(d)
$$\begin{aligned}
y - f'(1) &= f''(1)(x-1) \\
y - (\cos^2 e - \sin^2 e) &= (-4e \cos e \sin e)(x-1) \\
y - 0.663 &= 4.072(x-1) \\
y &= 4.072x - 4.072 + 0.663 \\
y &= 4.072 - 3.409
\end{aligned}$$