Chapter 7 Techniques of Integration

7.1 Integration by Parts

Concepts and Vocabulary

1. True.

Skill Building

3. We use integration by parts with $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$. We obtain

$$\begin{aligned}
\int xe^{2x}\,dx &= x\left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x}\,dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C \\
&= \boxed{\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C}.
\end{aligned}$$

5. We use integration by parts with $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
\int x\cos x\,dx &= x\sin x - \int \sin x\,dx \\
&= x\sin x - (-\cos x) + C \\
&= \boxed{x\sin x + \cos x + C}.
\end{aligned}$$

7. We use integration by parts with $u = \ln x$ and $dv = \sqrt{x}\,dx = x^{1/2}\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{2}{3}x^{3/2}$. We obtain

$$\begin{aligned}
\int \sqrt{x}\ln x\,dx &= (\ln x)\left(\frac{2}{3}x^{3/2}\right) - \int \left(\frac{2}{3}x^{3/2}\right)\left(\frac{1}{x}\right)dx \\
&= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx \\
&= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right) + C \\
&= \boxed{\tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + C}.
\end{aligned}$$

9. We use integration by parts with $u = \cot^{-1} x$ and $dv = dx$. Then $du = \left(-\frac{1}{1+x^2}\right) dx$ and $v = x$. We obtain

$$\begin{aligned}\int \cot^{-1} x\,dx &= \left(\cot^{-1} x\right)(x) - \int \left(-\frac{1}{1+x^2}\right)(x)\,dx \\ &= x\cot^{-1} x + \int \frac{x}{1+x^2}\,dx.\end{aligned}$$

Let $u = 1 + x^2$, then $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$. We now substitute and obtain

$$\begin{aligned}\int \cot^{-1} x\,dx &= x\cot^{-1} x + \int \frac{1}{u}\frac{du}{2} \\ &= x\cot^{-1} x + \frac{1}{2}\int \frac{1}{u}\,du \\ &= x\cot^{-1} x + \frac{1}{2}\ln|u| + C \\ &= x\cot^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + C \\ &= \boxed{x\cot^{-1} x + \tfrac{1}{2}\ln\left(1+x^2\right) + C}.\end{aligned}$$

11. We use integration by parts with $u = (\ln x)^2$ and $dv = dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^2\,dx &= (\ln x)^2 x - \int x\left(\frac{2\ln x}{x}\right)dx \\ &= x(\ln x)^2 - 2\int \ln x\,dx\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^2\,dx &= x(\ln x)^2 - 2\left[(\ln x)x - \int x\left(\frac{1}{x}\right)dx\right] \\ &= x(\ln x)^2 - 2\left[(\ln x)x - \int 1\,dx\right] \\ &= x(\ln x)^2 - 2\left[(\ln x)x - x\right] + C \\ &= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}.\end{aligned}$$

13. We use integration by parts with $u = x^2$ and $dv = \sin x\,dx$. Then $du = 2x\,dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int x^2 \sin x\,dx &= x^2(-\cos x) - \int (-\cos x)(2x)\,dx \\ &= -x^2\cos x + 2\int x\cos x\,dx\end{aligned}$$

We use integration by parts again with $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
\int x^2 \sin x\,dx &= -x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right] \\
&= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx \\
&= -x^2\cos x + 2x\sin x - 2(-\cos x) + C \\
&= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}.
\end{aligned}$$

15. We use integration by parts with $u = x\cos x$ and $dv = \cos x\,dx$. Then $du = (-x\sin x + \cos x)\,dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
\int x\cos^2 x\,dx &= (x\cos x)(\sin x) - \int (\sin x)(-x\sin x + \cos x)\,dx \\
&= x\cos x\sin x + \int x\sin^2 x\,dx - \int \cos x\sin x\,dx \\
&= x\cos x\sin x + \int x\left(1 - \cos^2 x\right)dx - \int \cos x\sin x\,dx \\
&= x\cos x\sin x + \int x\,dx - \int x\cos^2 x\,dx - \int \cos x\sin x\,dx.
\end{aligned}$$

We add $\int x\cos^2 x\,dx$ to each side, and then let $u = \sin x$, so $du = \cos x\,dx$ in the remaining integral. Upon substitution, we obtain

$$\begin{aligned}
2\int x\cos^2 x\,dx &= x\cos x\sin x + \frac{1}{2}x^2 - \int u\,du \\
&= x\cos x\sin x + \frac{1}{2}x^2 - \frac{1}{2}u^2 + C \\
&= x\cos x\sin x + \frac{1}{2}x^2 - \frac{1}{2}\sin^2 x + C.
\end{aligned}$$

Finally, we divide by 2 to obtain

$$\int x\cos^2 x\,dx = \boxed{\tfrac{1}{2}x\cos x\sin x + \tfrac{1}{4}x^2 - \tfrac{1}{4}\sin^2 x + C}.$$

17. We use integration by parts with $u = x$ and $dv = \sinh x\,dx$. Then $du = dx$ and $v = \cosh x$. We obtain

$$\begin{aligned}
\int x\sinh x\,dx &= x\cosh x - \int \cosh x\,dx \\
&= \boxed{x\cosh x - \sinh x + C}.
\end{aligned}$$

19. We use integration by parts with $u = \cosh^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{x^2-1}}\,dx$ and $v = x$. We obtain

$$\int \cosh^{-1} x\,dx = \left(\cosh^{-1} x\right)x - \int x\left(\frac{1}{\sqrt{x^2-1}}\right)dx.$$

Let $u = x^2 - 1$, then $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned}\int \cosh^{-1} x\,dx &= x\cosh^{-1} x - \int u^{-1/2}\,\frac{du}{2}\\ &= x\cosh^{-1} x - \frac{1}{2}\int u^{-1/2}\,du\\ &= x\cosh^{-1} x - \frac{1}{2}\left(2\sqrt{u}\right) + C\\ &= \boxed{x\cosh^{-1} x - \sqrt{x^2-1} + C}.\end{aligned}$$

21. We use integration by parts with $u = \sin(\ln x)$ and $dv = dx$. Then $du = \frac{\cos(\ln x)}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int \sin(\ln x)\,dx &= (\sin(\ln x))x - \int x\left(\frac{\cos(\ln x)}{x}\right)dx\\ &= x\sin(\ln x) - \int \cos(\ln x)\,dx.\end{aligned}$$

We use integration by parts again with $u = \cos(\ln x)$ and $dv = dx$. Then $du = \frac{-\sin(\ln x)}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int \sin(\ln x)\,dx &= x\sin(\ln x) - \left[(\cos(\ln x))x - \int x\left(\frac{-\sin(\ln x)}{x}\right)dx\right]\\ &= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.\end{aligned}$$

We add $\int \sin(\ln x)\,dx$ to each side, divide by 2, and add the constant of integration to obtain

$$\begin{aligned}2\int \sin(\ln x)\,dx &= x\sin(\ln x) - x\cos(\ln x) + C\\ \int \sin(\ln x)\,dx &= \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C = \boxed{\tfrac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C}.\end{aligned}$$

23. We use integration by parts with $u = (\ln x)^3$ and $dv = dx$. Then $du = 3\frac{(\ln x)^2}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^3\,dx &= (\ln x)^3 x - \int x\left(\frac{3(\ln x)^2}{x}\right)dx\\ &= x(\ln x)^3 - 3\int (\ln x)^2\,dx.\end{aligned}$$

We use integration by parts again with $u = (\ln x)^2$ and $dv = dx$. Then $du = 2\frac{\ln x}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - 3\left[(\ln x)^2(x) - \int x\left(2\frac{\ln x}{x}\right)dx\right]\\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6\int \ln x\,dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[(\ln x)x - \int x\left(\frac{1}{x}\right)dx\right] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int dx \\ &= \boxed{x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C}.\end{aligned}$$

25. We use integration by parts with $u = (\ln x)^2$ and $dv = x^2\,dx$. Then $du = 2\frac{\ln x}{x}\,dx$ and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2(\ln x)^2\,dx &= (\ln x)^2\left(\frac{1}{3}x^3\right) - \int\left(\frac{1}{3}x^3\right)\left(2\frac{\ln x}{x}\right)dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2\ln x\,dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = x^2dx$. Then $du = \frac{dx}{x}$ and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2(\ln x)^2\,dx &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[(\ln x)\left(\frac{1}{3}x^3\right) - \int\left(\frac{1}{3}x^3\right)\frac{dx}{x}\right] \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{9}\int x^2\,dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{9}\left(\frac{1}{3}x^3\right) + C \\ &= \boxed{\tfrac{1}{3}x^3(\ln x)^2 - \tfrac{2}{9}x^3\ln x + \tfrac{2}{27}x^3 + C}.\end{aligned}$$

27. We use integration by parts with $u = \tan^{-1}x$ and $dv = x^2\,dx$. Then $du = \frac{1}{1+x^2}\,dx$ and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2\tan^{-1}x\,dx &= \left(\tan^{-1}x\right)\left(\frac{1}{3}x^3\right) - \int\left(\frac{1}{3}x^3\right)\left(\frac{1}{1+x^2}\right)dx \\ &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}\,dx.\end{aligned}$$

Let $u = 1+x^2$, then $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$. Since $x^2 = u-1$, we have $x^3\,dx = \frac{u-1}{2}\,du$. We substitute and obtain

$$\begin{aligned}\int x^2\tan^{-1}x\,dx &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int\frac{\frac{u-1}{2}}{u}\,du \\ &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}\int\left(1-\frac{1}{u}\right)du \\ &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}(u - \ln|u|) + C \\ &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}\left((1+x^2) - \ln\left|1+x^2\right|\right) + C \\ &= \boxed{\tfrac{1}{3}x^3\tan^{-1}x - \tfrac{1}{6}x^2 + \tfrac{1}{6}\ln\left(1+x^2\right) + C},\end{aligned}$$

where we have absorbed any constants with the constant C of integration.

29. We use integration by parts with $u = x$ and $dv = 7^x\,dx$. Then $du = dx$ and $v = \frac{7^x}{\ln 7}$. We obtain

$$\begin{aligned}\int 7^x\,x\,dx &= (x)\left(\frac{7^x}{\ln 7}\right) - \int \frac{7^x}{\ln 7}\,dx \\ &= \frac{7^x\,x}{\ln 7} - \frac{1}{\ln 7}\int 7^x\,dx \\ &= \frac{7^x\,x}{\ln 7} - \frac{1}{\ln 7}\left(\frac{7^x}{\ln 7}\right) + C \\ &= \boxed{\tfrac{7^x\,x}{\ln 7} - \tfrac{7^x}{(\ln 7)^2} + C}.\end{aligned}$$

31. Evaluate $\int e^{-x}\cos(2x)\,dx$ using integration by parts.

Let $u = e^{-x}$ and $dv = \cos(2x)\,dx$.

Then $du = -e^{-x}\,dx$ and $v = \int \cos(2x)\,dx = \frac{1}{2}\sin(2x)$.

Now

$$\begin{aligned}\int e^{-x}\cos(2x)\,dx &= e^{-x}\left[\frac{1}{2}\sin(2x)\right] - \int\left[\frac{1}{2}\sin(2x)\right]\left(-e^{-x}\,dx\right) \\ &= \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{2}\int e^{-x}\sin(2x)\,dx.\end{aligned}$$

Integrate $\int e^{-x}\sin(2x)\,dx$ using integration by parts again.

Let $u = e^{-x}$ and $dv = \sin(2x)\,dx$.

Then $du = -e^{-x}\,dx$ and $v = \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x)$.

Now

$$\begin{aligned}\int e^{-x}\sin(2x)\,dx &= e^{-x}\left[-\frac{1}{2}\cos(2x)\right] - \int\left[-\frac{1}{2}\cos(2x)\right]\left(-e^{-x}\,dx\right) \\ &= -\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{2}\int e^{-x}\cos(2x)\,dx\end{aligned}$$

and

$$\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{2}\left[-\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{2}\int e^{-x}\cos(2x)\,dx\right].$$

Simplifying, we obtain

$$\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) - \frac{1}{4}e^{-x}\cos(2x) - \frac{1}{4}\int e^{-x}\cos(2x)\,dx.$$

Add $\frac{1}{4}\int e^{-x}\cos(2x)\,dx$ to both sides of the equation to obtain

$$\begin{aligned}\int e^{-x}\cos(2x)\,dx + \frac{1}{4}\int e^{-x}\cos(2x)\,dx &= \frac{1}{2}e^{-x}\sin(2x) - \frac{1}{4}e^{-x}\cos(2x) \\ \frac{5}{4}\int e^{-x}\cos(2x)\,dx &= \frac{1}{4}\left[2e^{-x}\sin(2x) - e^{-x}\cos(2x)\right] \\ \int e^{-x}\cos(2x)\,dx &= \frac{1}{5}\left[2e^{-x}\sin(2x) - e^{-x}\cos(2x)\right] \\ &= \boxed{\tfrac{1}{5}e^{-x}[2\sin(2x) - \cos(2x)] + C}\end{aligned}$$

33. Evaluate $\int e^{2x}\sin x\,dx$ using integration by parts.

Let $u = e^{2x}$ and $dv = \sin x\,dx$.

Then $du = 2e^{2x}\,dx$ and $v = \int \sin x\,dx = -\cos x$.

Now

$$\int e^{2x}\sin x\,dx = e^{2x}(-\cos x) - \int (-\cos x)\big(2e^{2x}\,dx\big) = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx.$$

Integrate $\int e^{2x}\cos x\,dx$ using integration by parts again.

Let $u = e^{2x}$ and $dv = \cos x\,dx$.

Then $du = 2e^{2x}\,dx$ and $v = \int \cos x\,dx = \sin x$.
Now

$$\begin{aligned}\int e^{2x}\cos x\,dx &= e^{2x}\sin x - \int \sin x\left(2e^{2x}\,dx\right)\\ &= e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\end{aligned}$$

and

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right).$$

Simplifying, we obtain

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx.$$

Add $4\int e^{2x}\sin x\,dx$ to both sides of the equation to obtain

$$\begin{aligned}\int e^{2x}\sin x\,dx + 4\int e^{2x}\sin x\,dx &= -e^{2x}\cos x + 2e^{2x}\sin x\\ 5\int e^{2x}\sin x\,dx &= -e^{2x}\cos x + 2e^{2x}\sin x\\ \int e^{2x}\sin x\,dx &= \frac{1}{5}\big(-e^{2x}\cos x + 2e^{2x}\sin x\big)\\ &= \boxed{\tfrac{1}{5}e^{2x}(-\cos x + 2\sin x) + C}\end{aligned}$$

35. We use integration by parts with $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$. We obtain

$$\begin{aligned}\int_0^\pi e^x\cos x\,dx &= [e^x\sin x]_0^\pi - \int_0^\pi e^x\sin x\,dx\\ &= \left[e^\pi\sin\pi - e^0\sin 0\right] - \int_0^\pi e^x\sin x\,dx\\ &= [e^\pi(0) - 1(0)] - \int_0^\pi e^x\sin x\,dx\\ &= -\int_0^\pi e^x\sin x\,dx.\end{aligned}$$

We use integration by parts again with $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int_0^\pi e^x \cos x\,dx &= -\left([e^x(-\cos x)]_0^\pi - \int_0^\pi e^x(-\cos x)\,dx\right)\\ &= -\left(\left[e^\pi(-\cos\pi) - e^0(-\cos 0)\right] + \int_0^\pi e^x\cos x\,dx\right)\\ &= -\left([e^\pi(1) - 1(-1)] + \int_0^\pi e^x\cos x\,dx\right)\\ &= -e^\pi - 1 - \int_0^\pi e^x\cos x\,dx.\end{aligned}$$

We add $\int_0^\pi e^x \cos x\,dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned}2\int_0^\pi e^x\cos x\,dx &= -e^\pi - 1\\ \int_0^\pi e^x\cos x\,dx &= \frac{-e^\pi - 1}{2} = \boxed{-\tfrac{e^\pi+1}{2}}.\end{aligned}$$

37. We use integration by parts with $u = x^2$ and $dv = e^{-3x}\,dx$. Then $du = 2x\,dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}\int_0^2 x^2e^{-3x}\,dx &= \left[x^2\left(-\frac{1}{3}e^{-3x}\right)\right]_0^2 - \int_0^2\left(-\frac{1}{3}e^{-3x}\right)(2x)\,dx\\ &= \left[2^2\left(-\frac{1}{3}e^{-3(2)}\right) - 0^2\left(-\frac{1}{3}e^{-3(0)}\right)\right] + \frac{2}{3}\int_0^2 xe^{-3x}\,dx\\ &= -\frac{4}{3}e^{-6} + \frac{2}{3}\int_0^2 xe^{-3x}\,dx.\end{aligned}$$

We use integration by parts again with $u = x$ and $dv = e^{-3x}dx$. Then $du = dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}\int_0^2 x^2e^{-3x}\,dx &= -\frac{4}{3}e^{-6} + \frac{2}{3}\left(\left[x\left(-\frac{1}{3}e^{-3x}\right)\right]_0^2 - \int_0^2\left(-\frac{1}{3}e^{-3x}\right)dx\right)\\ &= -\frac{4}{3}e^{-6} + \frac{2}{3}\left(\left[2\left(-\frac{1}{3}e^{-3(2)}\right) - 0\left(-\frac{1}{3}e^{-3(0)}\right)\right] + \frac{1}{3}\int_0^2 e^{-3x}\,dx\right)\\ &= -\frac{4}{3}e^{-6} + \frac{2}{3}\left(\left(-\frac{2}{3}e^{-6}\right) + \frac{1}{3}\left[-\frac{1}{3}e^{-3x}\right]_0^2\right)\\ &= -\frac{4}{3}e^{-6} + \frac{2}{3}\left(\left(-\frac{2}{3}e^{-6}\right) + \frac{1}{3}\left[-\frac{1}{3}e^{-3(2)} - \left(-\frac{1}{3}e^{-3(0)}\right)\right]\right)\\ &= -\frac{4}{3}e^{-6} + \frac{2}{3}\left(\left(-\frac{2}{3}e^{-6}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{3}e^{-6}\right)\right)\\ &= \boxed{\tfrac{2}{27} - \tfrac{50}{27}e^{-6}}.\end{aligned}$$

39. Evaluate $\int x\sec x\tan x\,dx$ using integration by parts.

Let $u = x$ and $dv = \sec x\tan x\,dx$.

Then $du = dx$ and $v = \int \sec x \tan x\, dx = \sec x$.
Now

$$\int x \sec x \tan x\, dx = x \sec x - \int \sec x\, dx.$$

To integrate $\int \sec x\, dx$, multiply $\sec x$ by $\sec x + \tan x$:

$$\int \sec x\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.$$

Integrate using u-substitution:

Let $u = \sec x + \tan x$. Then $du = \left(\sec x \tan x + \sec^2 x\right) dx$.
Now

$$\int \sec x\, dx = \int \frac{1}{\sec x + \tan x} \cdot \left(\sec^2 x + \sec x \tan x\right) dx = \int \frac{1}{u}\, du = \ln|u| = \ln|\sec x + \tan x|.$$

(This also could have been found using a Table of Integrals.)
So

$$\int x \sec x \tan x = x \sec x - \ln|\sec x + \tan x| + C.$$

Then

$$\begin{aligned}\int_0^{\pi/4} x \sec x \tan x\, dx &= \left(\frac{\pi}{4} \sec \frac{\pi}{4} - \ln\left|\sec \frac{\pi}{4} + \tan \frac{\pi}{4}\right|\right) - (0 \cdot \sec 0 - \ln|\sec 0 + \tan 0|) \\ &= \left(\frac{\pi}{4} \cdot \sqrt{2} - \ln\left|\sqrt{2} + 1\right|\right) - (0 \cdot 1 - \ln|1 + 0|) \\ &= \boxed{\tfrac{\sqrt{2}}{4}\pi - \ln\left(\sqrt{2} + 1\right)}\end{aligned}$$

41. We rewrite $\int_1^9 \ln\sqrt{x}\, dx = \int_1^9 \ln x^{1/2}\, dx = \frac{1}{2}\int_1^9 \ln x\, dx$. We use integration by parts with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$. We obtain

$$\begin{aligned}\int_1^9 \ln\sqrt{x}\, dx &= \frac{1}{2}\int_1^9 \ln x\, dx \\ &= \frac{1}{2}\left([(\ln x)(x)]_1^9 - \int_1^9 x\left(\frac{1}{x}\right) dx\right) \\ &= \frac{1}{2}\left([(\ln 9)(9) - (\ln 1)(1)] - \int_1^9 dx\right) \\ &= \frac{1}{2}(9 \ln 9 - 8) \\ &= \boxed{9 \ln 3 - 4}.\end{aligned}$$

43. We use integration by parts with $u = (\ln x)^2$ and $dv = dx$. Then $du = 2\frac{\ln x}{x}\, dx$ and $v = x$. We obtain

$$\begin{aligned}\int_1^e (\ln x)^2\, dx &= \left[(\ln x)^2 (x)\right]_1^e - \int_1^e x\left(2\frac{\ln x}{x}\right) dx \\ &= (\ln e)^2 (e) - (\ln 1)^2 (1) - \int_1^e 2 \ln x\, dx \\ &= e - 2\int_1^e \ln x\, dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}
\int_1^e (\ln x)^2\,dx &= e - 2\left([(\ln x)\,x]_1^e - \int_1^e x\frac{1}{x}\,dx\right) \\
&= e - 2\left((\ln e)\,e - (\ln 1)\,(1) - \int_1^e dx\right) \\
&= e - 2\,(e - (e-1)) \\
&= \boxed{e - 2}.
\end{aligned}$$

45. To determine where the graphs intersect, set $3\ln x = x\ln x$, and obtain $x = 3$. Since $\ln x \geq 0$ when $1 \leq x \leq 3$, we have $x\ln x \leq 3\ln x$ on the interval $1 \leq x \leq 3$ and the area enclosed by the graphs of f and g is

$$\int_1^3 (3\ln x - x\ln x)\,dx = \int_1^3 (3-x)\ln x\,dx.$$

We use integration by parts with $u = \ln x$ and $dv = (3-x)\,dx$. Then $du = \frac{1}{x}dx$ and $v = 3x - \frac{1}{2}x^2$. We obtain

$$\begin{aligned}
\int_1^3 (3-x)\ln x\,dx &= \left[(\ln x)\left(3x - \frac{1}{2}x^2\right)\right]_1^3 - \int_1^3 \frac{3x - \frac{1}{2}x^2}{x}\,dx \\
&= \left[(\ln 3)\left(3(3) - \frac{1}{2}3^2\right) - (\ln 1)\left(3(1) - \frac{1}{2}1^2\right)\right] - \int_1^3 \left(3 - \frac{1}{2}x\right)dx \\
&= \frac{9}{2}\ln 3 - \left[3x - \frac{1}{4}x^2\right]_1^3 \\
&= \frac{9}{2}\ln 3 - \left[\left(3(3) - \frac{1}{4}3^2\right) - \left(3(1) - \frac{1}{4}1^2\right)\right] \\
&= \boxed{\tfrac{9}{2}\ln 3 - 4}.
\end{aligned}$$

47. The area under the graph of $y = e^x \sin x$ from 0 to π is given by $\int_0^\pi e^x \sin x\,dx$. We use integration by parts with $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}
\int_0^\pi e^x \sin x\,dx &= [e^x(-\cos x)]_0^\pi - \int_0^\pi e^x(-\cos x)\,dx \\
&= \left[e^\pi(-\cos\pi) - e^0(-\cos 0)\right] + \int_0^\pi e^x \cos x\,dx \\
&= e^\pi + 1 + \int_0^\pi e^x\cos x\,dx.
\end{aligned}$$

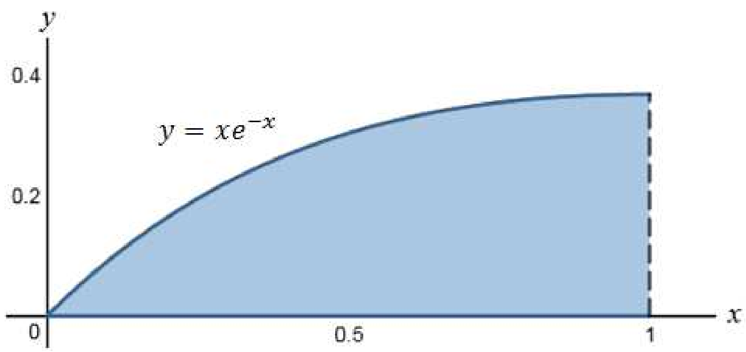
We use integration by parts again with $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
\int_0^\pi e^x \sin x\,dx &= e^\pi + 1 + [e^x \sin x]_0^\pi - \int_0^\pi e^x \sin x\,dx \\
&= e^\pi + 1 + \left(e^\pi \sin\pi - e^0 \sin 0\right) - \int_0^\pi e^x \sin x\,dx \\
&= e^\pi + 1 - \int_0^\pi e^x \sin x\,dx
\end{aligned}$$

We add $\int_0^\pi e^x \sin x\,dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned} 2\int_0^\pi e^x \sin x\,dx &= e^\pi + 1 \\ \int_0^\pi e^x \sin x\,dx &= \boxed{\tfrac{e^\pi+1}{2}}. \end{aligned}$$

49. Since $y = xe^{-x}$ is nonnegative on $[0,1]$, $\int_0^1 xe^{-x}\,dx$ is the area under the graph of $y = xe^{-x}$ from $x = 0$ to $x = 1$.



Evaluate $\int xe^{-x}\,dx$ using integration by parts.

Let $u = x$ and $dv = e^{-x}\,dx$.

Then $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$.

Now

$$\int xe^{-x}\,dx = -xe^{-x} - \int \left(-e^{-x}\right) dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1).$$

So,

$$\int_0^1 xe^{-x}\,dx = \left[-e^{-x}(x+1)\right]_0^1 = -2e^{-1} - (-1) = \boxed{1 - \tfrac{2}{e}}.$$

51. (a) The velocity of the object at time t is given by

$$v(t) = v(0) + \int_0^t a(x)\,dx = 8 + \int_0^t e^{-2x}\sin x\,dx.$$

Integrate $\int e^{-2x}\sin x\,dx$ using integration by parts.

Let $u = e^{-2x}$ and $dv = \sin x\,dx$.

Then $du = -2e^{-2x}\,dx$ and $v = \int \sin x\,dx = -\cos x$.

Now

$$\begin{aligned} \int e^{-2x}\sin x\,dx &= e^{-2x}(-\cos x) - \int (-\cos x)\left(-2e^{-2x}\,dx\right) \\ &= -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,dx. \end{aligned}$$

Integrate $\int e^{-2x}\cos x\,dx$ using integration by parts again.

Let $u = e^{-2x}$ and $dv = \cos x\,dx$.
Then $du = -2e^{-2x}\,dx$ and $v = \int \cos x\,dx = \sin x$.
Now

$$\begin{aligned}\int e^{-2x}\cos x\,dx &= e^{-2x}\sin x - \int \sin x\left(-2e^{-2x}\,dx\right)\\ &= e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx\end{aligned}$$

so

$$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2\left(e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx\right).$$

Simplifying, we obtain

$$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\sin x\,dx.$$

Add $4\int e^{-2x}\sin x\,dx$ to both sides of the equation to obtain

$$\begin{aligned}\int e^{-2x}\sin x\,dx + 4\int e^{-2x}\sin x\,dx &= -e^{-2x}\cos x - 2e^{-2x}\sin x\\ 5\int e^{-2x}\sin x\,dx &= -e^{-2x}\cos x - 2e^{-2x}\sin x\\ \int e^{-2x}\sin x\,dx &= \frac{1}{5}\left(-e^{-2x}\cos x - 2e^{-2x}\sin x\right)\\ &= \frac{1}{5}e^{-2x}(-\cos x - 2\sin x) + C\end{aligned}$$

Therefore,

$$\begin{aligned}v(t) &= 8 + \int_0^t e^{-2x}\sin x\,dx\\ &= 8 + \left[\frac{1}{5}e^{-2x}(-\cos x - 2\sin x)\right]_0^t\\ &= 8 + \left[\frac{1}{5}e^{-2t}(-\cos t - 2\sin t) - \frac{1}{5}e^{0}(-\cos 0 - 2\sin 0)\right]\\ &= 8 + \frac{1}{5}\left[e^{-2t}(-\cos t - 2\sin t) + 1\right]\\ &= 8 - \frac{1}{5}\left[e^{-2t}(\cos t + 2\sin t) - 1\right]\\ &= \boxed{\tfrac{41}{5} - \tfrac{1}{5}[e^{-2t}(\cos t + 2\sin t)]}.\end{aligned}$$

(b) The position of the object at time t is given by

$$s(t) = s(0) + \int_0^t v(x)\,dx = 0 + \int_0^t \left\{\frac{41}{5} - \frac{1}{5}\left[e^{-2x}(\cos x + 2\sin x)\right]\right\}dx$$

Recall $\int e^{-2x}\sin x\,dx = \frac{1}{5}e^{-2x}(-\cos x - 2\sin x) + C$.
Similarly, $\int e^{-2x}\cos x\,dx = \frac{1}{5}e^{-2x}(\sin x - 2\cos x) + C$.

$$\begin{aligned}\int e^{-2x}(\cos x + 2\sin x)\,dx &= \int e^{-2x}\cos x\,dx + 2\int e^{-2x}\sin x\,dx \\ &= \frac{1}{5}e^{-2x}(\sin x - 2\cos x) + 2\left[\frac{1}{5}e^{-2x}(-\cos x - 2\sin x)\right] \\ &= \frac{1}{5}e^{-2x}(\sin x - 2\cos x - 2\cos x - 4\sin x) \\ &= -\frac{1}{5}e^{-2x}(3\sin x + 4\cos x)\end{aligned}$$

So,

$$\begin{aligned}s(t) &= \int_0^t \left\{\frac{41}{5} - \frac{1}{5}\left[e^{-2x}(\cos x + 2\sin x)\right]\right\}\,dx \\ &= \left\{\frac{41}{5}x - \frac{1}{5}\left[-\frac{1}{5}e^{-2x}(3\sin x + 4\cos x)\right]\right\}_0^t \\ &= \left\{\frac{41}{5}t - \frac{1}{5}\left[-\frac{1}{5}e^{-2t}(3\sin t + 4\cos t)\right]\right\} - \frac{4}{25} \\ &= \boxed{\tfrac{41}{5}t - \tfrac{1}{25}e^{-2t}(3\sin t + 4\cos t) - \tfrac{4}{25}}.\end{aligned}$$

53. The volume is given by the integral $\int_0^{\pi/2} 2\pi x\sin x\,dx$. We use integration by parts with $u = 2\pi x$ and $dv = \sin x\,dx$. Then $du = 2\pi\,dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int_0^{\pi/2} 2\pi x\sin x\,dx &= [(2\pi x)(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2}(-\cos x)(2\pi)\,dx \\ &= \left[\left(2\pi\left(\frac{\pi}{2}\right)\right)\left(-\cos\frac{\pi}{2}\right) - (2\pi(0))(-\cos 0)\right] + 2\pi\int_0^{\pi/2}\cos x\,dx \\ &= 2\pi\int_0^{\pi/2}\cos x\,dx \\ &= 2\pi\left[\sin x\right]_0^{\pi/2} \\ &= 2\pi\left[\sin\frac{\pi}{2} - \sin 0\right] \\ &= \boxed{2\pi}.\end{aligned}$$

55. The volume is given by the integral $\int_1^e \pi(\ln x)^2\,dx$. We use integration by parts with $u = (\ln x)^2$ and $dv = \pi\,dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v = \pi x$. We obtain

$$\begin{aligned}\int_1^e \pi(\ln x)^2\,dx &= \left[(\ln x)^2(\pi x)\right]_1^e - \int_1^e(\pi x)\left(\frac{2\ln x}{x}\right)\,dx \\ &= \left[(\ln e)^2(\pi e) - (\ln 1)^2(\pi 1)\right] - 2\pi\int_1^e \ln x\,dx \\ &= \pi e - 2\pi\int_1^e \ln x\,dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$. We obtain

$$\begin{aligned}\int_1^e \pi(\ln x)^2\,dx &= \pi e - 2\pi\left([(\ln x)x]_1^e - \int_1^e x\left(\frac{1}{x}\right)dx\right)\\ &= \pi e - 2\pi\left([(\ln e)e - (\ln 1)1] - \int_1^e dx\right)\\ &= \pi e - 2\pi(e - (e-1))\\ &= \boxed{\pi e - 2\pi}.\end{aligned}$$

57. Using the method of disks, the volume is given by

$$V = \pi\int_1^{e^2}[f(x)]^2\,dx = \pi\int_1^{e^2}\left(x\sqrt{\ln x}\right)^2dx = \pi\int_1^{e^2}x^2\ln x\,dx.$$

Integrate $\int x^2\ln x\,dx$ using integration by parts.

Let $u = \ln x$ and $dv = x^2\,dx$.

Then $du = \frac{1}{x}\,dx$ and $v = \int x^2\,dx = \frac{1}{3}x^3$.
Now

$$\begin{aligned}\int x^2\ln x\,dx = \ln x\left(\frac{1}{3}x^3\right) - \int\frac{1}{3}x^3\cdot\frac{1}{x}\,dx &= \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx\\ &= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C = \frac{1}{9}x^3(3\ln x - 1) + C.\end{aligned}$$

The volume of the solid is

$$\begin{aligned}V &= \pi\int_1^{e^2}x^2\ln x\,dx\\ &= \pi\cdot\frac{1}{9}\left[x^3(3\ln x - 1)\right]_1^{e^2}\\ &= \frac{\pi}{9}\left[5e^6 - (-1)\right]\\ &= \boxed{\tfrac{\pi}{9}\left(5e^6+1\right)}.\end{aligned}$$

59. Integrate $\int xf'(x)\,dx$ using integration by parts.

Let $u = x$ and $dv = f'(x)\,dx$.

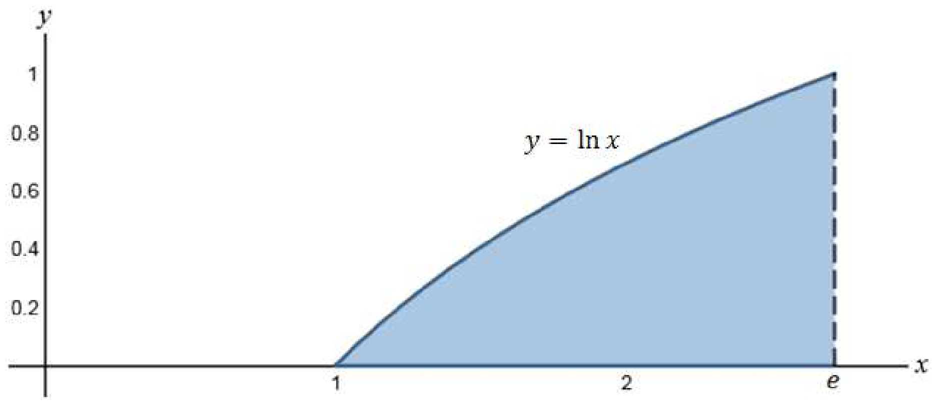
Then $du = dx$ and $v = \int f'(x)\,dx = f(x)$.
Now

$$\int xf'(x)\,dx = u\cdot v - \int v\cdot\,du = x\,f(x) - \int f(x)\,dx.$$

So,

$$\int_1^5 x\,f'(x)\,dx = [x\,f(x)]_1^5 - \int_1^5 f(x)\,dx = 5f(5) - f(1) - 10 = 5(-5) - 2 - 10 = \boxed{-37}.$$

61. (a) Since $y = \ln x$ is nonnegative on $[1, e]$, $A = \int_1^e \ln x\, dx$ is the area under the graph of $y = \ln x$ from $x = 1$ to $x = e$.



Evaluate $\int \ln x\, dx$ using integration by parts.
Let $u = \ln x$ and $dv = dx$.
Then $du = \frac{1}{x}\, dx$ and $v = \int dx = x$.
Now

$$\int \ln x\, dx = (\ln x)(x) - \int x\left(\frac{1}{x}\, dx\right) = x \ln x - \int dx = x \ln x - x = x\,(\ln x - 1) + C.$$

So,

$$A = \int_1^e \ln x\, dx = \left[x\,(\ln x - 1)\right]_1^e = e(1-1) - 1(0-1) = \boxed{1}.$$

(b) Using the method of shells, the volume is given by

$$V = 2\pi \int_1^e x\, f(x)\, dx = 2\pi \int_1^e x \ln x\, dx.$$

Integrate $\int x \ln x\, dx$ using integration by parts.
Let $u = \ln x$ and $dv = x\, dx$.
Then $du = \frac{1}{x}\, dx$ and $v = \int x\, dx = \frac{1}{2}x^2$.
Now

$$\begin{aligned}\int x \ln x\, dx &= \ln x\left(\frac{1}{2}x^2\right) - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\, dx\right) = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \\ &= \frac{1}{4}x^2(2\ln x - 1) + C.\end{aligned}$$

The volume of the solid is

$$\begin{aligned}V &= 2\pi \int_1^e x^2 \ln x\, dx = 2\pi \cdot \frac{1}{4}\left[x^2(2\ln x - 1)\right]_1^e \\ &= \frac{\pi}{2}\left[e^2 - (-1)\right] \\ &= \boxed{\tfrac{\pi}{2}\left(e^2 + 1\right)}.\end{aligned}$$

63. Let $w = \sqrt{x}$, then $dw = \frac{1}{2\sqrt{x}}\,dx$, so $dx = 2w\,dw$. We substitute and obtain

$$\int \sin\sqrt{x}\,dx = \int 2w \sin w\,dw.$$

We use integration by parts, with $u = 2w$ and $dv = \sin w\,dw$. Then $du = 2\,dw$ and $v = -\cos w$. We obtain

$$\begin{aligned}\int 2w\sin w\,dw &= (2w)(-\cos w) - \int(-\cos w)(2)\,dw \\ &= -2w\cos w + 2\int \cos w\,dw \\ &= -2w\cos w + 2(\sin w) + C \\ &= \boxed{-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C}.\end{aligned}$$

65. Let $w = \sin x$, then $dw = \cos x\,dx$. We substitute and obtain

$$\int \cos x \ln(\sin x)\,dx = \int \ln w\,dw.$$

We use integration by parts, with $u = \ln w$ and $dv = dw$. Then $du = \frac{1}{w}\,dw$ and $v = w$. We obtain

$$\begin{aligned}\int \ln w\,dw &= (\ln w)(w) - \int (w)\left(\frac{1}{w}\right)dw \\ &= w\ln w - \int dw \\ &= w\ln w - w + C \\ &= \boxed{\sin x \ln(\sin x) - \sin x + C}.\end{aligned}$$

67. Let $w = e^{2x}$, then $dw = 2e^{2x}\,dx$, so $e^{2x}\,dx = \frac{1}{2}\,dw$. We substitute and obtain

$$\int e^{4x}\cos\left(e^{2x}\right)dx = \int e^{2x}\cos\left(e^{2x}\right)e^{2x}\,dx = \int w\cos w\left(\frac{1}{2}\,dw\right) = \int \frac{1}{2}w\cos w\,dw.$$

We use integration by parts, with $u = \frac{1}{2}w$ and $dv = \cos w\,dw$. Then $du = \frac{1}{2}\,dw$ and $v = \sin w$. We obtain

$$\begin{aligned}\int \frac{1}{2}w\cos w\,dw &= \left(\frac{1}{2}w\right)(\sin w) - \int(\sin w)\left(\frac{1}{2}\right)dw \\ &= \frac{1}{2}w\sin w - \frac{1}{2}\int \sin w\,dw \\ &= \frac{1}{2}w\sin w - \frac{1}{2}(-\cos w) + C \\ &= \frac{1}{2}w\sin w + \frac{1}{2}\cos w + C \\ &= \boxed{\tfrac{1}{2}e^{2x}\sin\left(e^{2x}\right) + \tfrac{1}{2}\cos\left(e^{2x}\right) + C}.\end{aligned}$$

69. We use integration by parts with $u = x^2$ and $dv = xe^{x^2}\,dx$. Then $du = 2x\,dx$ and $v = \frac{1}{2}e^{x^2}$. We obtain

$$\begin{aligned} \int x^3 e^{x^2}\,dx &= (x^2)\left(\frac{1}{2}e^{x^2}\right) - \int \left(\frac{1}{2}e^{x^2}\right)(2x)\,dx \\ &= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2}\,dx \\ &= \boxed{\tfrac{1}{2}x^2 e^{x^2} - \tfrac{1}{2}e^{x^2} + C}. \end{aligned}$$

71. We use integration by parts with $u = x\cos x$ and $dv = e^x\,dx$. Then $du = (-x\sin x + \cos x)\,dx$ and $v = e^x$. We obtain

$$\int xe^x \cos x\,dx = (x\cos x)e^x - \int (-x\sin x + \cos x)e^x\,dx.$$

We use integration by parts again with $u = -x\sin x + \cos x$ and $dv = e^x\,dx$. Then $du = (-2\sin x - x\cos x)\,dx$ and $v = e^x$. We obtain

$$\begin{aligned} \int xe^x \cos x\,dx &= (x\cos x)e^x - \left((-x\sin x + \cos x)e^x - \int (-2\sin x - x\cos x)e^x\,dx\right) \\ &= x\cos x\,e^x + x\sin x\,e^x - \cos x\,e^x - 2\int \sin x\,e^x\,dx - \int x\cos x\,e^x\,dx. \end{aligned}$$

We add $\int xe^x\cos x\,dx$ to each side and divide by 2 to obtain

$$\begin{aligned} 2\int xe^x \cos x\,dx &= x\cos x\,e^x + x\sin x\,e^x - \cos x\,e^x - 2\int \sin x\,e^x\,dx \\ \int xe^x \cos x\,dx &= \frac{1}{2}(x\cos x\,e^x + x\sin x\,e^x - \cos x\,e^x) - \int \sin x\,e^x\,dx. \end{aligned}$$

We determine $\int \sin x\,e^x\,dx$ using integration by parts. Let $u = \sin x$ and $dv = e^x\,dx$. Then $du = \cos x\,dx$ and $v = e^x$. We obtain

$$\int \sin x\,e^x\,dx = (\sin x)e^x - \int (\cos x)e^x\,dx.$$

We integrate by parts again with $u = \cos x$ and $dv = e^x\,dx$. Then $du = -\sin x\,dx$ and $v = e^x$. We obtain

$$\begin{aligned} \int \sin x\,e^x\,dx &= \sin x\,e^x - \left[(\cos x)e^x - \int (-\sin x)e^x\,dx\right] \\ &= \sin x\,e^x - \cos x\,e^x - \int \sin xe^x\,dx. \end{aligned}$$

We add $\int (\sin x)e^x\,dx$ to each side and divide by 2 to obtain

$$\begin{aligned} 2\int \sin x\,e^x\,dx &= \sin x\,e^x - \cos x\,e^x \\ \int \sin x\,e^x\,dx &= \frac{1}{2}\sin x\,e^x - \frac{1}{2}\cos x\,e^x + C. \end{aligned}$$

We substitute into the above to obtain

$$\begin{aligned} \int xe^x \cos x\,dx &= \frac{1}{2}(x\cos x\,e^x + x\sin x\,e^x - \cos x\,e^x) - \left(\frac{1}{2}\sin x\,e^x - \frac{1}{2}\cos x\,e^x\right) + C \\ &= \frac{1}{2}x\cos x\,e^x + \frac{1}{2}x\sin x\,e^x - \frac{1}{2}\sin x\,e^x + C \\ &= \boxed{\tfrac{1}{2}e^x\,(x\sin x\, + x\cos x - \sin x\,) + C}. \end{aligned}$$

73. We use integration by parts with $u = \sin^{-1} x$ and $dv = x^n\,dx$. Then $du = \frac{1}{\sqrt{1-x^2}}\,dx$ and $v = \frac{x^{n+1}}{n+1}$. We obtain

$$\int x^n \sin^{-1} x\,dx = \frac{x^{n+1}}{n+1}\sin^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}\,dx.$$

75. We use integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x\,dx$. Then $du = (n-1)\sin^{n-2} x\cos x\ dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int \sin^n x\,dx &= \sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x\cos x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \cos^2 x\sin^{n-2} x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \left(1-\sin^2 x\right)\sin^{n-2} x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\left(\int \sin^{n-2} x\,dx - \int \sin^n x\,dx\right) \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.\end{aligned}$$

We add $(n-1)\int \sin^n x\,dx$ to each side, and then divide by n to obtain

$$\begin{aligned}(1+n-1)\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx \\ \int \sin^n x\,dx &= -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.\end{aligned}$$

77. (a) We use integration by parts with $u = x^2$ and $dv = e^{5x}\,dx$. Then $du = 2x\,dx$ and $v = \frac{1}{5}e^{5x}$. We obtain

$$\begin{aligned}\int x^2 e^{5x}\,dx &= x^2\left(\frac{1}{5}e^{5x}\right) - \int \left(\frac{1}{5}e^{5x}\right)(2x)\,dx \\ &= \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\int x e^{5x}\,dx.\end{aligned}$$

We use integration by parts again with $u = x$ and $dv = e^{5x}\,dx$. Then $du = dx$ and $v = \frac{1}{5}e^{5x}$. We obtain

$$\begin{aligned}\int x^2 e^{5x}\,dx &= \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\left[x\left(\frac{1}{5}e^{5x}\right) - \int \frac{1}{5}e^{5x}\,dx\right] \\ &= \frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{25}\int e^{5x}\,dx \\ &= \frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{25}\left(\frac{1}{5}e^{5x}\right) + C \\ &= \boxed{\tfrac{1}{5}x^2 e^{5x} - \tfrac{2}{25}x e^{5x} + \tfrac{2}{125}e^{5x} + C}.\end{aligned}$$

(b) We use integration by parts with $u = x^n$ and $dv = e^{kx}\ dx$. Then $du = nx^{n-1}\ dx$ and $v = \frac{1}{k}e^{kx}$. We obtain

$$\begin{aligned}\int x^n e^{kx}\,dx &= x^n\left(\frac{1}{k}e^{kx}\right) - \int \left(\frac{1}{k}e^{kx}\right)\left(nx^{n-1}\right)dx \\ &= \boxed{\tfrac{1}{k}x^n e^{kx} - \tfrac{n}{k}\int x^{n-1}e^{kx}\,dx}.\end{aligned}$$

79. (a) We use integration by parts with $u = \sin x$ and $dv = \cos x\,dx$. Then $du = \cos x\,dx$ and $v = \sin x$. We obtain

$$\begin{aligned}\int \sin x\,\cos x\,dx &= (\sin x)(\sin x) - \int (\sin x)(\cos x)\,dx \\ &= \sin^2 x - \int \sin x\,\cos x\,dx.\end{aligned}$$

We add $\int \sin x\,\cos x\,dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned}2\int \sin x\,\cos x\,dx &= \sin^2 x \\ \int \sin x\,\cos x\,dx &= \frac{1}{2}\sin^2 x + C_1.\end{aligned}$$

So we obtain $\boxed{f(x) = \frac{1}{2}\sin^2 x}$.

(b) We use integration by parts with $u = \cos x$ and $dv = \sin x\,dx$. Then $du = -\sin x\,dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int \sin x\,\cos x\,dx &= (\cos x)(-\cos x) - \int (-\cos x)(-\sin x)\,dx \\ &= -\cos^2 x - \int \sin x\,\cos x\,dx.\end{aligned}$$

We add $\int \sin x\,\cos x\,dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned}2\int \sin x\,\cos x\,dx &= -\cos^2 x \\ \int \sin x\,\cos x\,dx &= -\frac{1}{2}\cos^2 x + C_2.\end{aligned}$$

So we obtain $\boxed{g(x) = -\frac{1}{2}\cos^2 x}$.

(c) From the identity $\sin(2x) = 2\sin x\,\cos x$ we have $\sin x\,\cos x = \frac{1}{2}\sin(2x)$. So $\int \sin x\,\cos x\,dx = \frac{1}{2}\int \sin(2x)\,dx$. Let $u = 2x$, then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. We substitute and obtain

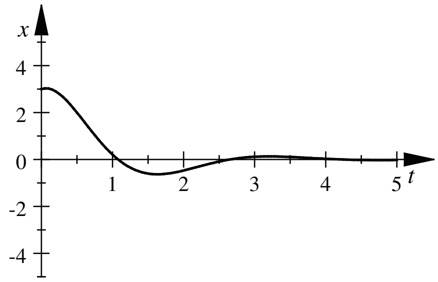
$$\begin{aligned}\int \sin x\,\cos x\,dx &= \frac{1}{2}\int \sin(2x)\,dx \\ &= \frac{1}{2}\int \sin u\left(\frac{1}{2}\,du\right) \\ &= \frac{1}{4}\int \sin u\,du \\ &= \frac{1}{4}(-\cos u) + C_3 \\ &= -\frac{1}{4}\cos(2x) + C_3.\end{aligned}$$

So we obtain $\boxed{h(x) = -\frac{1}{4}\cos(2x)}$.

(d) We have $f(x) + C_1 = g(x) + C_2$, so $\frac{1}{2}\sin^2 x + C_1 = -\frac{1}{2}\cos^2 x + C_2$. We obtain $C_2 = \frac{1}{2}\cos^2 x + \frac{1}{2}\sin^2 x + C_1 = \frac{1}{2} + C_1$, so $\boxed{C_2 = \frac{1}{2} + C_1}$.

(e) We have $f(x) + C_1 = h(x) + C_3$, so $\frac{1}{2}\sin^2 x + C_1 = -\frac{1}{4}\cos(2x) + C_3$. We obtain $C_3 = \frac{1}{4}\cos(2x) + \frac{1}{2}\sin^2 x + C_1 = \frac{1}{4}\left(1 - 2\sin^2 x\right) + \frac{1}{2}\sin^2 x + C_1 = \frac{1}{4} + C_1$. So $\boxed{C_3 = \frac{1}{4} + C_1}$.

81. (a)



(b) Set $3e^{-t}\cos(2t) + 2e^{-t}\sin(2t) = 0$ and solve for t. We obtain $\tan(2t) = -\frac{3}{2}$, so $2t = \tan^{-1}\left(-\frac{3}{2}\right) + k\pi$, where k is any integer. So the least positive number t such that $x(t) = 0$ is $t = \boxed{\frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(-\frac{3}{2}\right)}$.

(c) We integrate $\int_0^{\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{3}{2}\right)} 3e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\,dt$ using a Computer Algebra System, and we obtain $\int_0^{\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{3}{2}\right)} \left(3e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right)dt \approx \boxed{1.890}$.

83. We use integration by parts with $u = \sin(bx)$ and $dv = e^{ax}\,dx$. Then $du = b\cos(bx)\,dx$ and $v = \frac{1}{a}e^{ax}$. We obtain

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= (\sin(bx))\left(\frac{1}{a}e^{ax}\right) - \int\left(\frac{1}{a}e^{ax}\right)(b\cos(bx))\,dx \\ &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx\end{aligned}$$

We integrate by parts again with $u = \cos bx$ and $dv = e^{ax}\,dx$. Then $du = -b\sin bx\,dx$ and $v = \frac{1}{a}e^{ax}$. We obtain

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\left[(\cos(bx))\left(\frac{1}{a}e^{ax}\right) - \int\left(\frac{1}{a}e^{ax}\right)(-b\sin(bx))\,dx\right] \\ &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx) - \frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx\end{aligned}$$

We add $\frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx$ from each side and divide by $\frac{a^2+b^2}{a^2}$ to obtain

$$\begin{aligned}\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin(bx)\,dx &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx) \\ \int e^{ax}\sin(bx)\,dx &= \frac{a}{a^2+b^2}e^{ax}\sin(bx) - \frac{b}{a^2+b^2}e^{ax}\cos(bx) + C \\ &= \frac{e^{ax}\left[a\sin(bx) - b\cos(bx)\right]}{a^2+b^2} + C.\end{aligned}$$

85. (a) $\int_0^{\pi/2}\sin^6 x\,dx = \frac{(5)(3)(1)}{(6)(4)(2)}\left(\frac{\pi}{2}\right) = \boxed{\frac{5\pi}{32}}$.

(b) $\int_0^{\pi/2}\sin^5 x\,dx = \frac{(4)(2)}{(5)(3)(1)} = \boxed{\frac{8}{15}}$.

(c) $\int_0^{\pi/2}\cos^8 x\,dx = \int_0^{\pi/2}\sin^8 x\,dx = \frac{(7)(5)(3)(1)}{(8)(6)(4)(2)}\left(\frac{\pi}{2}\right) = \boxed{\frac{35\pi}{256}}$.

(d) $\int_0^{\pi/2}\cos^6 x\,dx = \int_0^{\pi/2}\sin^6 x\,dx = \frac{(5)(3)(1)}{(6)(4)(2)}\left(\frac{\pi}{2}\right) = \boxed{\frac{5\pi}{32}}$.

Challenge Problems

87. (a) We use integration by parts with $u = x^n$ and $dv = e^x\,dx$. Then $du = nx^{n-1}\,dx$ and $v = e^x$. We have

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

Repeating, we obtain for some constants c and k

$$\begin{aligned}\int x^n e^x\,dx &= x^n e^x - n\left(x^{n-1}e^x - (n-1)\int x^{n-2}e^x\,dx\right)\\ &= \left(x^n - nx^{n-1}\right)e^x + n(n-1)\int x^{n-2}e^x\,dx\\ &= \cdots\\ &= \left(x^n - nx^{n-1} + \cdots + kx\right)e^x + c\int e^x\,dx\\ &= \left(x^n - nx^{n-1} + \cdots + kx + c\right)e^x + C\\ &= p(x)\,e^x + C.\end{aligned}$$

(b) Since $\frac{d}{dx}(p(x)e^x) = x^n e^x$, we have $p(x)e^x + p'(x)e^x = x^n e^x$. Divide by e^x to obtain $p(x) + p'(x) = x^n$.

(c) We show $p(x) = \sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}x^{n-k}$ satisfies $p(x) + p'(x) = x^n$. We have

$$\begin{aligned}p(x)+p'(x) &= \sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}x^{n-k} + \frac{d}{dx}\left(\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}x^{n-k}\right)\\ &= x^n + \sum_{k=0}^{n-1}(-1)^k\frac{n!}{(n-k)!}x^{n-k} + \sum_{k=0}^{n-1}(-1)^k\frac{n!}{(n-k)!}(n-k)\,x^{n-k-1}\\ &= x^n + \sum_{k=0}^{n-1}(-1)^k\frac{n!}{(n-k)!}x^{n-k} + \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{(n-(k-1))!}(n-(k-1))\,x^{n-(k-1)-1}\\ &= x^n + \sum_{k=0}^{n-1}(-1)^k\frac{n!}{(n-k)!}x^{n-k} + \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{(n-k+1)!}(n-k+1)\,x^{n-k}\\ &= x^n + \sum_{k=0}^{n-1}(-1)^k\frac{n!}{(n-k)!}x^{n-k} + \sum_{k=1}^{n}(-1)^k(-1)^{-1}\frac{n!}{(n-k+1)(n-k)!}(n-k+1)\,x^{n-k}\\ &= x^n + \sum_{k=0}^{n-1}(-1)^k\frac{n!}{(n-k)!}x^{n-k} - \sum_{k=1}^{n}(-1)^k\frac{n!}{(n-k)!}x^{n-k}\\ &= x^n.\end{aligned}$$

89. We use integration by parts with $u = f(x)$ and $dv = e^x\,dx$. Then $du = f'(x)\,dx$ and $v = e^x$. We have

$$\int f(x)e^x\,dx = f(x)e^x - \int f'(x)e^x\,dx$$

We continue, using $u = f'(x)$, $dv = e^x$ and obtain

$$\begin{aligned}\int f(x)e^x\,dx &= f(x)e^x - \left(f'(x)e^x - \int f''(x)e^x\,dx\right)\\ &= \left(f(x) - f'(x)\right)e^x + \int f''(x)e^x\,dx\\ &= \cdots\\ &= \left(f(x) - f'(x) + \cdots \pm f^{(n-1)}(x)\right)e^x \pm \int f^{(n)}(x)e^x\,dx\end{aligned}$$

Since $f(x)$ is a polynomial of degree n, $f^{(n)}(x)$ is a constant, k, and we have

$$\begin{aligned}\int f(x)e^x\,dx &= \left(f(x)-f'(x)+\cdots\pm f^{(n-1)}(x)\right)e^x \pm \int ke^x\,dx \\ &= \left(f(x)-f'(x)+\cdots\pm f^{(n-1)}(x)\right)e^x \pm ke^x + C \\ &= \left(f(x)-f'(x)+\cdots\pm k\right)e^x + C.\end{aligned}$$

We nowhave $f(x) = g(x)e^x + C$, where $g(x) = f(x) - f'(x) + \cdots \pm k$ is a polynomial of degree n.

91. We use integration by parts with $u = f(x)$ and $dv = dx$. Then $du = f'(x)\,dx$ and take $v = x$. We obtain

$$\begin{aligned}\int_a^b f(x)\,dx &= [xf(x)]_a^b - \int_a^b xf'(x)\,dx \\ &= bf(b) - af(a) - \int_a^b xf'(x)\,dx.\end{aligned}$$

Let $x = f^{-1}(y)$, then $f(x) = y$, so $f'(x)\,dx = dy$. The limits of integration become $y = f(a)$ and $y = f(b)$. We substitute and obtain

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy,$$

so

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

AP® Practice Problems

1. The velocity of the object at time t is given by

$$v(t) = v(1) + \int_1^t a(x)\,dx = 2 + \int_1^t \ln(x+1)\,dx.$$

Use integration by parts to find $\int \ln(x+1)\,dx$.

Let $u = \ln(x+1)$ and $dv = dx$.

Then $du = \frac{1}{x+1}\,dx$ and $v = \int dx = x$.

$$\begin{aligned}\text{Now } \int \ln(x+1)\,dx &= x\ln(x+1) - \int x\cdot\frac{1}{x+1}\,dx \\ &= x\ln(x+1) - \int\left(1-\frac{1}{x+1}\right)dx \\ &= x\ln(x+1) - x + \ln(x+1) + C \\ &= (x+1)\ln(x+1) - x + C.\end{aligned}$$

$$\begin{aligned}\text{Therefore, } v(t) &= 2 + \int_1^t \ln(x+1)\,dx \\ &= 2 + [(x+1)\ln(x+1) - x]_1^t \\ &= 2 + [(t+1)\ln(t+1) - t] - (2\ln 2 - 1), \text{ and}\end{aligned}$$

$$v(3) = 2 + [4\ln 4 - 3] - (2\ln 2 - 1) = 2 + 4\ln(4) - \ln 4 - 3 + 1 = \boxed{3\ln 4}.$$

The answer is B.

3. Use integration by parts to find $\int \cos^{-1} x\,dx$.

   Let $u = \cos^{-1} x$ and $dv = dx$.

   Then $du = -\frac{1}{\sqrt{1-x^2}}\,dx$ and $v = \int dx = x$.

   Now $\displaystyle\int \cos^{-1} x\,dx = x\cos^{-1} x - \int x \cdot \left(-\frac{1}{\sqrt{1-x^2}}\,dx\right) = x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\,dx.$

   To evaluate $\int \frac{x}{\sqrt{1-x^2}}\,dx$, let $w = 1-x^2$. Then $dw = -2x\,dx$, $x\,dx = -\frac{dw}{2}$, and

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-x^2}}x\,dx\\ &= \int w^{-1/2}\left(-\frac{dw}{2}\right)\\ &= -\frac{1}{2}\int w^{-1/2}dw\\ &= -\frac{1}{2}\left(\frac{w^{1/2}}{\frac{1}{2}}\right) + C\\ &= -\sqrt{w} + C\\ &= -\sqrt{1-x^2} + C.\end{aligned}$$

$$\text{So, } \int \cos^{-1} x\,dx = x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\,dx = \boxed{x\cos^{-1} x - \sqrt{1-x^2} + C}.$$

   The answer is C.

5. Integrate $\int \left(3x^2+2\right)\sin x\,dx$ using integration by parts.

   Let $u = 3x^2 + 2$ and $dv = \sin x\,dx$.

   Then $du = 6x\,dx$ and $v = \int \sin x\,dx = -\cos x$.

$$\begin{aligned}\text{So } \int \left(3x^2+2\right)\sin x\,dx &= \left(3x^2+2\right)(-\cos x) - \int (-\cos x)(6x\,dx)\\ &= \boxed{-(3x^2+2)\cos x + 6\textstyle\int x\cos x\,dx}.\end{aligned}$$

   The answer is B.

7. The diameter of each semicircular slice is $\sqrt{\ln x}$. The radius is $\frac{1}{2}\sqrt{\ln x}$.

   The volume of each semicircular slice of thickness dx is $dV = \frac{1}{2}\cdot\pi\cdot\left(\frac{1}{2}\sqrt{\ln x}\right)^2 dx = \frac{\pi}{8}\ln x$.

   The volume of the solid is $V = \int_1^{e^2} dV = \frac{\pi}{8}\int_1^{e^2} \ln x\,dx$.

   Use integration by parts to find $\int \ln x\,dx$.

   Let $u = \ln x$ and $dv = dx$.

   Then $du = \frac{1}{x}\,dx$ and $v = \int dx = x$.

   Now $\displaystyle\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$

   So, $\displaystyle\frac{\pi}{8}\int_1^{e^2} \ln x\,dx = \frac{\pi}{8}[x\ln x - x]_1^{e^2} = \frac{\pi}{8}\left[\left(e^2\ln e^2 - e^2\right) - (0-1)\right] = \boxed{\tfrac{\pi}{8}(e^2+1)}.$

   The answer is A.

7.2 Integrals Containing Trigonometric Functions

Concepts and Vocabulary

1. True.

Skill Building

3. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}\int \cos^5 x\,dx &= \int \cos^4 x\,\cos x\,dx \\ &= \int \left(\cos^2 x\right)^2 \cos x\,dx \\ &= \int \left(1-\sin^2 x\right)^2 \cos x\,dx.\end{aligned}$$

Let $u = \sin x$, then $du = \cos x\,dx$. We substitute and obtain

$$\begin{aligned}\int \cos^5 x\,dx &= \int \left(1-u^2\right)^2 du \\ &= \int \left(u^4 - 2u^2 + 1\right) du \\ &= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C \\ &= \boxed{\tfrac{1}{5}\sin^5 x - \tfrac{2}{3}\sin^3 x + \sin x + C}.\end{aligned}$$

5. Use the identity $\sin^2 x = \frac{1-\cos(2x)}{2}$ and obtain

$$\begin{aligned}\int \sin^6 x\,dx &= \int \left(\sin^2 x\right)^3 dx \\ &= \int \left(\frac{1-\cos(2x)}{2}\right)^3 dx \\ &= \frac{1}{8}\int \left(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\right) dx \\ &= \frac{1}{8}\int dx - \frac{3}{8}\int \cos(2x)\,dx + \frac{3}{8}\int \cos^2(2x)\,dx - \frac{1}{8}\int \cos^3(2x)\,dx \\ &= \frac{1}{8}x - \frac{3}{16}\sin(2x) + \frac{3}{8}\int \cos^2(2x)\,dx - \frac{1}{8}\int \cos^3(2x)\,dx.\end{aligned}$$

We evaluate

$$\begin{aligned}\int \cos^2(2x)\,dx &= \int \frac{1+\cos(4x)}{2}\,dx \\ &= \frac{1}{2}\int (1+\cos(4x))\,dx \\ &= \frac{x}{2} + \frac{\sin(4x)}{8} + C.\end{aligned}$$

And also

$$\begin{aligned}\int \cos^3(2x)\,dx &= \int \cos^2(2x)\,\cos(2x)\,dx \\ &= \int \left(1-\sin^2(2x)\right)\cos(2x)\,dx.\end{aligned}$$

Let $u = \sin(2x)$, then $du = 2\cos(2x)\,dx$, so $\cos(2x)\,dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned} \int \cos^3(2x)\,dx &= \int (1-u^2)\,\frac{du}{2} \\ &= \frac{1}{2}\left(u - \frac{u^3}{3}\right) + C \\ &= \frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6} + C. \end{aligned}$$

We now obtain

$$\begin{aligned} \int \sin^6 x\,dx &= \frac{1}{8}x - \frac{3}{16}\sin(2x) + \frac{3}{8}\left(\frac{x}{2} + \frac{\sin(4x)}{8}\right) - \frac{1}{8}\left(\frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6}\right) + C \\ &= \boxed{\tfrac{5}{16}x - \tfrac{1}{4}\sin(2x) + \tfrac{3}{64}\sin(4x) + \tfrac{1}{48}\sin^3(2x) + C}. \end{aligned}$$

7. Use the identity $\sin^2(\pi x) = \frac{1-\cos(2\pi x)}{2}$ and obtain

$$\begin{aligned} \int \sin^2(\pi x)\,dx &= \int \frac{1-\cos(2\pi x)}{2}\,dx \\ &= \frac{1}{2}\int (1-\cos(2\pi x))\,dx \\ &= \frac{1}{2}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right) + C \\ &= \boxed{\tfrac{x}{2} - \tfrac{1}{4\pi}\sin(2\pi x) + C}. \end{aligned}$$

9. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned} \int_0^\pi \cos^5 x\,dx &= \int_0^\pi \cos^4 x\,\cos x\,dx \\ &= \int_0^\pi \left(\cos^2 x\right)^2 \cos x\,dx \\ &= \int_0^\pi \left(1-\sin^2 x\right)^2 \cos x\,dx. \end{aligned}$$

Let $u = \sin x$, then $du = \cos x\,dx$. The lower limit of integration is $u = \sin 0 = 0$ and the upper limit of integration is $u = \sin \pi = 0$. We substitute and obtain

$$\int_0^\pi \cos^5 x\,dx = \int_0^0 (1-u)^2\,du = \boxed{0}.$$

11. Factor out $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned} \int \sin^3 x\,\cos^2 x\,dx &= \int \sin^2 x\,\cos^2 x\,\sin x\,dx \\ &= \int \left(1-\cos^2 x\right)\cos^2 x\,\sin x\,dx. \end{aligned}$$

Let $u = \cos x$, then $du = -\sin x\,dx$, so $\sin x\,dx = -du$. We substitute and obtain

$$\begin{aligned}\int \sin^3 x \cos^2 x\,dx &= \int \left(1-u^2\right)\left(u^2\right)(-du)\\ &= \int \left(u^4 - u^2\right) du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \boxed{\tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C}.\end{aligned}$$

13. Integrate $\int_0^{\pi/2} \sin^3 x(\cos x)^{3/2}\,dx$ using trigonometric identities.

The exponent of $\sin x$ is 3, a positive, odd integer. Factor $\sin x$ from $\sin^3 x$ and write the rest of the integrand in terms of cosines.

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x(\cos x)^{3/2}\,dx &= \int_0^{\pi/2} \sin^2 x(\cos x)^{3/2} \sin x\,dx\\ &= \int_0^{\pi/2} \left(1-\cos^2 x\right)(\cos x)^{3/2} \sin x\,dx.\end{aligned}$$

Now use the substitution $u = \cos x$ and $du = -\sin x\,dx$. Then $\sin x\,dx = -du$. The lower limit of integration becomes $u = \cos 0 = 1$ and the upper limit of integration becomes $u = \cos \frac{\pi}{2} = 0$. Therefore,

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x(\cos x)^{3/2}\,dx &= \int_0^{\pi/2} \left(1-\cos^2 x\right)(\cos x)^{3/2} \sin x\,dx\\ &= \int_1^0 \left(1-u^2\right)u^{3/2}(-du) = \int_0^1 \left(1-u^2\right)u^{3/2}\,du.\end{aligned}$$

Use algebraic manipulation to rewrite $\left(1-u^2\right)u^{3/2}$ in a form whose antiderivative is recognizable: $\left(1-u^2\right)u^{3/2} = u^{3/2} - u^{7/2}$.

Then

$$\begin{aligned}\int_0^1 \left(1-u^2\right)u^{3/2}\,du &= \int_0^1 \left(u^{3/2} - u^{7/2}\right) du\\ &= \left[\frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2}\right]_0^1 = \left(\frac{2}{5} - \frac{2}{9}\right) - (0-0) = \boxed{\tfrac{8}{45}}.\end{aligned}$$

15. Integrate $\int \sin^3 x(\cos x)^{1/3}\,dx$ using trigonometric identities.

The exponent of $\sin x$ is 3, a positive, odd integer. Factor $\sin x$ from $\sin^3 x$ and write the rest of the integrand in terms of cosines.

$$\int \sin^3 x(\cos x)^{1/3}\,dx = \int \sin^2 x(\cos x)^{1/3} \sin x\,dx = \int \left(1-\cos^2 x\right)(\cos x)^{1/3} \sin x\,dx.$$

Now use the substitution $u = \cos x$ and $du = -\sin x\,dx$. So, $\sin x\,dx = -du$ and

$$\begin{aligned}\int \sin^3 x(\cos x)^{1/3}\,dx &= \int \left(1-\cos^2 x\right)(\cos x)^{1/3} \sin x\,dx\\ &= \int \left(1-u^2\right)u^{1/3}(-du) = -\int \left(1-u^2\right)u^{1/3}\,du.\end{aligned}$$

Use algebraic manipulation to rewrite $\left(1-u^2\right)u^{1/3}$ in a form whose antiderivative is recognizable: $\left(1-u^2\right)u^{1/3} = u^{1/3} - u^{7/3}$.

Then

$$\begin{aligned}\int \sin^3 x(\cos x)^{1/3}\,dx &= -\int \left(u^{1/3} - u^{7/3}\right) du = -\left(\frac{3}{4}u^{4/3} - \frac{3}{10}u^{10/3}\right) + C \\ &= -\left(\frac{3}{4}u^{4/3} - \frac{3}{10}u^{10/3}\right) + C \\ &= \boxed{\tfrac{3}{10}(\cos x)^{10/3} - \tfrac{3}{4}(\cos x)^{4/3} + C}.\end{aligned}$$

17. Factor out $\cos\left(\frac{x}{2}\right)$ and use the identity $\cos^2\left(\frac{x}{2}\right) = 1 - \sin^2\left(\frac{x}{2}\right)$.

$$\begin{aligned}\int \sin^2\left(\frac{x}{2}\right)\cos^3\left(\frac{x}{2}\right) dx &= \int \cos^2\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx \\ &= \int \left(1 - \sin^2\left(\frac{x}{2}\right)\right)\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx.\end{aligned}$$

Let $u = \sin\left(\frac{x}{2}\right)$, then $du = \frac{1}{2}\cos\left(\frac{x}{2}\right) dx$, so $\cos\left(\frac{x}{2}\right) dx = 2\,du$. We substitute and obtain

$$\begin{aligned}\int \sin^2\left(\frac{x}{2}\right)\cos^3\left(\frac{x}{2}\right) dx &= \int \left(1 - u^2\right)\left(u^2\right)(2\,du) \\ &= 2\int \left(u^2 - u^4\right) du \\ &= 2\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C \\ &= \boxed{\tfrac{2}{3}\sin^3\left(\tfrac{x}{2}\right) - \tfrac{2}{5}\sin^5\left(\tfrac{x}{2}\right) + C}.\end{aligned}$$

19. Integrate $\int \tan^3 x \sec^3 x\,dx$ using trigonometric identities.

    The exponent of $\tan x$ is 3, a positive, odd integer ≥ 3. Factor $\sec x \tan x$ from $\tan^3 x \sec^3 x$ and write the rest of the integrand in terms of secants. Use the identity $\tan^2 x = \sec^2 x - 1$.

$$\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x \cdot \sec x \tan x\,dx = \int \left(\sec^2 x - 1\right)\sec^2 x \cdot \sec x \tan x\,dx.$$

Now use the substitution $u = \sec x$ and $du = \sec x \tan x\,dx$.

$$\int \tan^3 x \sec^3 x\,dx = \int \left(\sec^2 x - 1\right)\sec^2 x \cdot \sec x \tan x\,dx = \int \left(u^2 - 1\right)u^2\,du.$$

Use algebraic manipulation to rewrite $\left(u^2 - 1\right)u^2$ in a form whose antiderivative is recognizable: $\left(u^2 - 1\right)u^2 = u^4 - u^2$.

Then

$$\int \tan^3 x \sec^3 x\,dx = \int \left(u^2 - 1\right)u^2\,du = \int \left(u^4 - u^3\right) du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \boxed{\tfrac{\sec^5 x}{5} - \tfrac{\sec^3 x}{3} + C}.$$

21. Integrate $\int \tan^{3/2} x \sec^4 x\,dx$ using trigonometric identities.

    The exponent of $\sec x$ is 4, a positive, even integer. Factor $\sec^2 x$ from $\sec^4 x$ and write the rest of the integrand in terms of tangents. Use the identity $\sec^2 x = 1 + \tan^2 x$.

$$\int \tan^{3/2} x \sec^4 x\,dx = \int \tan^{3/2} x \sec^2 x \sec^2 x\,dx = \int \tan^{3/2} x\left(1 + \tan^2 x\right)\sec^2 x\,dx.$$

Now use the substitution $u = \tan x$ and $du = \sec^2 x\,dx$.

$$\int \tan^{3/2} x \sec^4 x\,dx = \int \tan^{3/2} x \left(1+\tan^2 x\right)\sec^2 x\,dx = \int u^{3/2}\left(1+u^2\right)du.$$

Use algebraic manipulation to rewrite $u^{3/2}\left(1+u^2\right)$ in a form whose antiderivative is recognizable: $u^{3/2}\left(1+u^2\right) = u^{3/2} + u^{7/2}$.

Then

$$\begin{aligned}\int \tan^{3/2} x \sec^4 x\,dx &= \int \left(u^{3/2}+u^{7/2}\right)du = \frac{2}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C\\ &= \boxed{\tfrac{2}{5}\tan^{5/2} x + \tfrac{2}{9}\tan^{9/2} x + C}.\end{aligned}$$

23. Integrate $\int \tan^3 x\,(\sec x)^{3/2}\,dx$ using trigonometric identities.

The exponent of $\tan x$ is 3, a positive, odd integer. Factor $\tan x \sec x$ from $\tan^3 x\,(\sec x)^{3/2}$ and write the rest of the integrand in terms of secants. Use the identity $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}\int \tan^3 x\,(\sec x)^{3/2}\,dx &= \int \tan^2 x\,(\sec x)^{1/2}\tan x\sec x\,dx\\ &= \int \left(\sec^2 x - 1\right)(\sec x)^{1/2}\tan x \sec x\,dx.\end{aligned}$$

Now use the substitution $u = \sec x$ and $du = \tan x \sec x\,dx$.

$$\int \tan^3 x\,(\sec x)^{3/2}\,dx = \int \left(\sec^2 x - 1\right)(\sec x)^{1/2}\tan x\sec x\,dx = \int \left(u^2-1\right)u^{1/2}\,du.$$

Use algebraic manipulation to rewrite $\left(u^2-1\right)u^{1/2}$ in a form whose antiderivative is recognizable: $\left(u^2-1\right)u^{1/2} = u^{5/2} - u^{1/2}$.

Then

$$\begin{aligned}\int \tan^3 x\,(\sec x)^{3/2}\,dx &= \int \left(u^{5/2} - u^{1/2}\right)du\\ &= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C\\ &= \boxed{\tfrac{2}{7}(\sec x)^{7/2} - \tfrac{2}{3}(\sec x)^{3/2} + C}.\end{aligned}$$

25. Factor out $\csc x \cot x$, and use the identity $\cot^2 x = \csc^2 x - 1$.

$$\begin{aligned}\int \cot^3 x \csc x\,dx &= \int \cot^2 x\,\csc x \cot x\,dx\\ &= \int \left(\csc^2 x - 1\right)\csc x\cot x\,dx.\end{aligned}$$

Let $u = \csc x$, then $du = -\csc x \cot x\,dx$, so $\csc x\cot x\,dx = -du$. We substitute and obtain

$$\begin{aligned}\int \cot^3 x\csc x\,dx &= \int \left(\csc^2 x - 1\right)\csc x \cot x\,dx\\ &= \int \left(u^2-1\right)(-du)\\ &= \int \left(1-u^2\right)du\\ &= u - \frac{1}{3}u^3 + C\\ &= \boxed{\csc x - \tfrac{1}{3}\csc^3 x + C}.\end{aligned}$$

27. We use the identity $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ to obtain

$$\begin{aligned}\int \sin(3x)\cos x\,dx &= \frac{1}{2}\int (\sin(4x) + \sin(2x))\,dx \\ &= \frac{1}{2}\left(-\frac{1}{4}\cos(4x) - \frac{1}{2}\cos(2x)\right) + C \\ &= \boxed{-\tfrac{1}{8}\cos(4x) - \tfrac{1}{4}\cos(2x) + C}.\end{aligned}$$

29. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ to obtain

$$\begin{aligned}\int \cos x\,\cos(3x)\,dx &= \frac{1}{2}\int (\cos(4x) + \cos(-2x))\,dx \\ &= \frac{1}{2}\int (\cos(4x) + \cos(2x))\,dx \\ &= \frac{1}{2}\left(\frac{1}{4}\sin(4x) + \frac{1}{2}\sin(2x)\right) + C \\ &= \boxed{\tfrac{1}{8}\sin(4x) + \tfrac{1}{4}\sin(2x) + C}.\end{aligned}$$

31. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}\int \sin(2x)\,\sin(4x)\,dx &= \frac{1}{2}\int (\cos(-2x) - \cos(6x))\,dx \\ &= \frac{1}{2}\int (\cos(2x) - \cos(6x))\,dx \\ &= \frac{1}{2}\left(\frac{1}{2}\sin(2x) - \frac{1}{6}\sin(6x)\right) + C \\ &= \boxed{\tfrac{1}{4}\sin(2x) - \tfrac{1}{12}\sin(6x) + C}.\end{aligned}$$

33. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}\int_0^{\pi/2} \sin(2x)\,\sin x\,dx &= \frac{1}{2}\int_0^{\pi/2} (\cos(x) - \cos(3x))\,dx \\ &= \frac{1}{2}\left[\sin x - \frac{1}{3}\sin(3x)\right]_0^{\pi/2} \\ &= \frac{1}{2}\left[\left(\sin\frac{\pi}{2} - \frac{1}{3}\sin\left(\frac{3\pi}{2}\right)\right) - \left(\sin 0 - \frac{1}{3}\sin(3(0))\right)\right] \\ &= \frac{1}{2}\left(\frac{4}{3} - 0\right) \\ &= \boxed{\tfrac{2}{3}}.\end{aligned}$$

35. Integrate $\int_0^{\pi/2} \sin^2 x \cos^5 x\,dx$ using trigonometric identities.

The exponent of $\cos x$ is 5, a positive, odd integer. Factor $\cos x$ from $\sin^2 x \cos^5 x$ and write the rest of the integrand in terms of sines. Use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\int_0^{\pi/2} \sin^2 x \cos^5 x\,dx = \int_0^{\pi/2} \sin^2 x \cos^4 x \cos x\,dx = \int_0^{\pi/2} \sin^2 x\left(1 - \sin^2 x\right)^2 \cos x\,dx.$$

Now use the substitution $u = \sin x$ and $du = \cos x\,dx$. The lower limit of integration becomes $u = \sin 0 = 0$ and the upper limit of integration becomes $u = \sin\frac{\pi}{2} = 1$. Therefore,

$$\int_0^{\pi/2} \sin^2 x \cos^5 x\,dx = \int_0^{\pi/2} \sin^2 x\left(1-\sin^2 x\right)^2 \cos x\,dx = \int_0^1 u^2\left(1-u^2\right)^2 du.$$

Use algebraic manipulation to rewrite $u^2\left(1-u^2\right)^2$ in a form whose antiderivative is recognizable: $u^2\left(1-u^2\right)^2 = u^2 - 2u^4 + u^6$.

Then

$$\begin{aligned}\int_0^1 u^2\left(1-u^2\right)^2 du &= \int_0^1 \left(u^2 - 2u^4 + u^6\right) du\\ &= \left[\frac{1}{3}u^3 - 2\cdot\frac{1}{5}u^5 + \frac{1}{7}u^7\right]_0^1\\ &= \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) - 0 = \boxed{\tfrac{8}{105}}.\end{aligned}$$

37. Integrate $\int \frac{\sin^3 x}{\cos^2 x}\,dx$ using trigonometric identities.

The exponent of $\sin x$ is 3, a positive, odd integer. Factor $\sin x$ from $\sin^3 x$ and write the rest of the integrand in terms of cosines.

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \frac{\sin^2 x}{\cos^2 x}\sin x\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\sin x\,dx.$$

Now use the substitution $u = \cos x$ and $du = -\sin x\,dx$. Then $\sin x\,dx = -du$ and

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\sin x\,dx = \int \frac{1-u^2}{u^2}(-du) = -\int \frac{1-u^2}{u^2}\,du.$$

Use algebraic manipulation to rewrite $\frac{1-u^2}{u^2}$ in a form whose antiderivative is recognizable: $\frac{1-u^2}{u^2} = \frac{1}{u^2} - \frac{u^2}{u^2} = u^{-2} - 1$.

Then

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^2 x}\,dx &= -\int \left(u^{-2} - 1\right) du = -\left(-u^{-1} - u\right) + C = \frac{1}{u} + u + C\\ &= \boxed{\tfrac{1}{\cos x} + \cos x = \sec x + \cos x + C}.\end{aligned}$$

39. Integrate $\int_0^{\pi/3} \cos^3(3x)\,dx$ using trigonometric identities.

The exponent of $\cos(3x)$ is 3, a positive, odd integer. Factor $\cos(3x)$ from $\cos^3(3x)$ and write the rest of the integrand in terms of sines. Use the identity $\cos^2(3x) = 1 - \sin^2(3x)$.

$$\int_0^{\pi/3} \cos^3(3x)\,dx = \int_0^{\pi/3} \cos^2(3x)\cos(3x)\,dx = \int_0^{\pi/3} \left[1-\sin^2(3x)\right]\cos(3x)\,dx.$$

Now use the substitution $u = \sin(3x)$ and $du = 3\cos(3x)\,dx$. Then $\cos(3x)\,dx = \frac{du}{3}$. The lower limit of integration becomes $u = \sin 0 = 0$ and the upper limit of integration becomes $u = \sin\left(3\frac{\pi}{3}\right) = \sin(\pi) = 0$.

Therefore,

$$\int_0^{\pi/3} \cos^3(3x)\,dx = \int_0^{\pi/3} \left[1-\sin^2(3x)\right]\cos(3x)\,dx = \int_0^0 \left(1-u^2\right)\frac{du}{3} = \frac{1}{3}\int_0^0 \left(1-u^2\right) du.$$

Using a property of integrals, $\int_0^{\pi/3} \cos^3(3x)\,dx = \frac{1}{3}\int_0^0 \left(1-u^2\right) du = \boxed{0}$.

41. Factor out $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}\int_0^\pi \sin^3 x \cos^5 x \, dx &= \int_0^\pi \sin^2 x \cos^5 x \sin x \, dx \\ &= \int_0^\pi \left(1 - \cos^2 x\right) \cos^5 x \sin x \, dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x \; dx$, so $\sin x \, dx = -du$. The lower limit of integration is $u = \cos 0 = 1$, and the upper limit of integration is $u = \cos \pi = -1$. We substitute and obtain

$$\begin{aligned}\int_0^\pi \sin^3 x \cos^5 x \, dx &= \int_1^{-1} \left(1 - u^2\right) u^5 \, (-du) \\ &= \int_{-1}^1 \left(u^5 - u^7\right) du \\ &= \boxed{0},\end{aligned}$$

since we are integrating an odd function on a symmetric interval about 0.

43. Rewrite the integral, factor out $\sin x$, and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\begin{aligned}\int \tan^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} \, dx \\ &= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, dx \\ &= \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x \, dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x \, dx$, so $\sin x \, dx = -du$. We substitute and obtain

$$\begin{aligned}\int \tan^3 x \, dx &= \int \frac{1 - u^2}{u^3} (-du) \\ &= \int \left(\frac{1}{u} - u^{-3}\right) du \\ &= \ln|u| - \frac{u^{-2}}{-2} + C \\ &= \boxed{\ln|\cos x| + \tfrac{1}{2} \sec^2 x + C}.\end{aligned}$$

45. Factor out $\sec^2 x$ and use the identity $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned}\int \frac{\sec^6 x}{\tan^3 x} \, dx &= \int \frac{\sec^4 x}{\tan^3 x} \sec^2 x \, dx \\ &= \int \frac{\left(\sec^2 x\right)^2}{\tan^3 x} \sec^2 x \, dx \\ &= \int \frac{\left(1 + \tan^2 x\right)^2}{\tan^3 x} \sec^2 x \, dx.\end{aligned}$$

Let $u = \tan x$, then $du = \sec^2 x\, dx$. We substitute and obtain

$$\begin{aligned}\int \frac{\sec^6 x}{\tan^3 x}\, dx &= \int \frac{\left(1+u^2\right)^2}{u^3}\, du \\ &= \int \frac{1+2u^2+u^4}{u^3}\, du \\ &= \int \left(u^{-3} + \frac{2}{u} + u\right) du \\ &= \frac{u^{-2}}{-2} + 2\ln|u| + \frac{1}{2}u^2 + C \\ &= \boxed{-\tfrac{1}{2}\cot^2 x + 2\ln|\tan x| + \tfrac{1}{2}\tan^2 x + C}.\end{aligned}$$

47. Integrate $\int \csc^4 x \cot^3 x\, dx$ using trigonometric identities.

Notice that the exponent of $\csc x$ is 4 and the exponent of $\cot x$ is 3. So this integral can be found in two different ways, which will give two different (but equivalent) answers.

One solution:

The exponent of $\csc x$ is 4, a positive, even integer. Factor $\csc^2 x$ from $\csc^4 x \cot^3 x$ and write the rest of the integrand in terms of cotangents. Use the identity $\csc^2 x = 1 + \cot^2 x$.

$$\int \csc^4 x \cot^3 x\, dx = \int \csc^2 x \cot^3 x \csc^2 x\, dx = \int \left(1+\cot^2 x\right) \cot^3 x \csc^2 x\, dx.$$

Now use the substitution $u = \cot x$ and $du = -\csc^2 x\, dx$. Then $\csc^2 x\, dx = -du$ and

$$\begin{aligned}\int \csc^4 x \cot^3 x\, dx &= \int \left(1+\cot^2 x\right) \cot^3 x \left(\csc^2 x\, dx\right) \\ &= \int \left(1+u^2\right)u^3(-du) = -\int \left(1+u^2\right)u^3\, du.\end{aligned}$$

Use algebraic manipulation to rewrite $\left(1+u^2\right)u^3$ in a form whose antiderivative is recognizable: $\left(1+u^2\right)u^3 = u^3 + u^5$.

Then

$$\int \csc^4 x \cot^3 x\, dx = -\int \left(u^3+u^5\right) du = -\left(\frac{1}{4}u^4 + \frac{1}{6}u^6\right) + C = \boxed{-\tfrac{1}{4}\cot^4 x - \tfrac{1}{6}\cot^6 x + C}.$$

Another solution:

The exponent of $\cot x$ is 3, a positive, odd integer. Factor $\csc x \cot x$ from $\csc^4 x \cot^3 x$ and write the rest of the integrand in terms of cosecants. Use the identity $\cot^2 x = \csc^2 x - 1$.

$$\int \csc^4 x \cot^3 x\, dx = \int \csc^3 x \cot^2 x \csc x \cot x\, dx = \int \csc^3 x \left(\csc^2 x - 1\right) \csc x \cot x\, dx.$$

Now use the substitution $u = \csc x$ and $du = -\csc x \cot x\, dx$. Then $\csc x \cot x\, dx = -du$ and

$$\int \csc^4 x \cot^3 x\, dx = \int \csc^3 x \left(\csc^2 x - 1\right) \csc x \cot x\, dx = \int u^3\left(u^2-1\right)(-du) = \int u^3\left(1-u^2\right) du.$$

Use algebraic manipulation to rewrite $u^3\left(1-u^2\right)$ in a form whose antiderivative is recognizable: $u^3\left(1-u^2\right) = u^3 - u^5$.

Then

$$\int \csc^4 \cot^3 x\,dx = \int \left(u^3 - u^5\right) du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \boxed{\tfrac{1}{4}\csc^4 x - \tfrac{1}{6}\csc^6 x + C}.$$

The two answers are equivalent, because

$$\begin{aligned}
\frac{1}{4}\csc^4 x - \frac{1}{6}\csc^6 x + C_1 &= \frac{1}{4}\left(\csc^2 x\right)^2 - \frac{1}{6}\left(\csc^2 x\right)^3 + C_1 \\
&= \frac{1}{4}\left(1+\cot^2 x\right)^2 - \frac{1}{6}\left(1+\cot^2 x\right)^3 + C_1 \\
&= \frac{1}{4}\left(1+2\cot^2 x+\cot^4 x\right) - \frac{1}{6}\left(1+3\cot^2 x+3\cot^4 x+\cot^6 x\right) + C_1 \\
&= \frac{1}{4} + \frac{1}{2}\cot^2 x + \frac{1}{4}\cot^4 x - \frac{1}{6} - \frac{1}{2}\cot^2 x - \frac{1}{2}\cot^4 x - \frac{1}{6}\cot^6 x + C_1 \\
&= \left(\frac{1}{2}\cot^2 x - \frac{1}{2}\cot^2 x\right) + \left(\frac{1}{4}\cot^4 x - \frac{1}{2}\cot^4 x\right) - \frac{1}{6}\cot^6 x + \left(\frac{1}{4} - \frac{1}{6} + C_1\right) \\
&= -\frac{1}{4}\cot^4 x - \frac{1}{6}\cot^6 x + C_2.
\end{aligned}$$

49. Factor out $\csc(2x)\cot(2x)$ to obtain

$$\int \cot(2x)\csc^4(2x)\,dx = \int \csc^3(2x)\csc(2x)\cot(2x)\,dx.$$

Let $u = \csc(2x)$, then $du = -2\csc(2x)\cot(2x)\,dx$, so $\csc x \cot x\,dx = -\frac{du}{2}$ We substitute and obtain

$$\begin{aligned}
\int \cot(2x)\csc^4(2x)\,dx &= \int u^3\left(-\frac{du}{2}\right) \\
&= -\frac{1}{2}\int u^3\,du \\
&= -\frac{1}{2}\left(\frac{1}{4}u^4\right) + C \\
&= \boxed{-\tfrac{1}{8}\csc^4(2x) + C}.
\end{aligned}$$

51. We rewrite the integral using the identity $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}
\int_0^{\pi/4} \tan^4 x \sec^3 x\,dx &= \int_0^{\pi/4} \left(\tan^2 x\right)^2 \sec^3 x\,dx \\
&= \int_0^{\pi/4} \left(\sec^2 x - 1\right)^2 \sec^3 x\,dx \\
&= \int_0^{\pi/4} \left(\sec^4 x - 2\sec^2 x + 1\right) \sec^3 x\,dx \\
&= \int_0^{\pi/4} \sec^7 x\,dx - 2\int_0^{\pi/4} \sec^5 x\,dx + \int_0^{\pi/4} \sec^3 x\,dx.
\end{aligned}$$

From Example 7 we have $\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$, so

$$\begin{aligned}
\int_0^{\pi/4} \sec^3 x\,dx &= \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right]_0^{\pi/4} \\
&= \frac{1}{2}\left[\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \left(\sec 0 \tan 0 + \ln|\sec 0 + \tan 0|\right)\right] \\
&= \frac{1}{2}\ln\left(\sqrt{2}+1\right) + \frac{\sqrt{2}}{2}.
\end{aligned}$$

We use integration by parts for $\int_0^{\pi/4} \sec^5 x\,dx$ with $u = \sec^3 x$ and $dv = \sec^2 x\,dx$. Then $du = 3\sec^3 x \tan x\,dx$ and $v = \tan x$. We obtain

$$\begin{aligned}
\int_0^{\pi/4} \sec^5 x\,dx &= \left[\sec^3 x \tan x\right]_0^{\pi/4} - 3\int_0^{\pi/4} \left(\sec^3 x \tan x\right) \tan x\,dx \\
&= \sec^3 \frac{\pi}{4} \tan \frac{\pi}{4} - \left(\sec^3 0 \tan 0\right) - 3\int_0^{\pi/4} \sec^3 x \tan^2 x\,dx \\
&= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^3 x \left(\sec^2 x - 1\right) dx \\
&= 2\sqrt{2} - 3\left(\int_0^{\pi/4} \sec^5 x\,dx - \int_0^{\pi/4} \sec^3 x\,dx\right) \\
&= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^5 x\,dx + 3\left(\frac{1}{2}\ln\left(\sqrt{2}+1\right) + \frac{\sqrt{2}}{2}\right) \\
&= \frac{3}{2}\ln\left(\sqrt{2}+1\right) + \frac{7}{2}\sqrt{2} - 3\int_0^{\pi/4} \sec^5 x\,dx.
\end{aligned}$$

We add $3\int_0^{\pi/4} \sec^5 x\,dx$ to each side and divide by 4 to obtain

$$\begin{aligned}
4\int_0^{\pi/4} \sec^5 x\,dx &= \frac{3}{2}\ln\left(\sqrt{2}+1\right) + \frac{7}{2}\sqrt{2} \\
\int_0^{\pi/4} \sec^5 x\,dx &= \frac{3}{8}\ln\left(\sqrt{2}+1\right) + \frac{7}{8}\sqrt{2}.
\end{aligned}$$

We use integration by parts for $\int_0^{\pi/4} \sec^7 x\,dx$ with $u = \sec^5 x$ and $dv = \sec^2 x\,dx$. Then $du = 5\sec^5 x \tan x\,dx$ and $v = \tan x$. We obtain

$$\begin{aligned}
\int_0^{\pi/4} \sec^7 x\,dx &= \left[\sec^5 x \tan x\right]_0^{\pi/4} - 5\int_0^{\pi/4} \left(\sec^5 x \tan x\right) \tan x\,dx \\
&= \sec^5 \frac{\pi}{4} \tan \frac{\pi}{4} - \left(\sec^5 0 \tan 0\right) - 5\int_0^{\pi/4} \sec^5 x \tan^2 x\,dx \\
&= 4\sqrt{2} - 5\int_0^{\pi/4} \sec^5 x \left(\sec^2 x - 1\right) dx \\
&= 4\sqrt{2} - 5\left(\int_0^{\pi/4} \sec^7 x\,dx - \int_0^{\pi/4} \sec^5 x\,dx\right) \\
&= 4\sqrt{2} - 5\int_0^{\pi/4} \sec^7 x\,dx + 5\left(\frac{3}{8}\ln\left(\sqrt{2}+1\right) + \frac{7}{8}\sqrt{2}\right) \\
&= \frac{15}{8}\ln\left(\sqrt{2}+1\right) + \frac{67}{8}\sqrt{2} - 5\int_0^{\pi/4} \sec^7 x\,dx.
\end{aligned}$$

We add $5\int_0^{\pi/4} \sec^7 x\,dx$ to each side and divide by 6 to obtain

$$\begin{aligned}
6\int_0^{\pi/4} \sec^7 x\,dx &= \frac{15}{8}\ln\left(\sqrt{2}+1\right) + \frac{67}{8}\sqrt{2} \\
\int_0^{\pi/4} \sec^7 x\,dx &= \frac{5}{16}\ln\left(\sqrt{2}+1\right) + \frac{67}{48}\sqrt{2}.
\end{aligned}$$

We now obtain

$$\begin{aligned}
\int_0^{\pi/4} \tan^4 x \sec^3 x\,dx &= \int_0^{\pi/4} \sec^7 x\,dx - 2\int_0^{\pi/4} \sec^5 x\,dx + \int_0^{\pi/4} \sec^3 x\,dx \\
&= \left(\frac{5}{16}\ln\left(\sqrt{2}+1\right) + \frac{67}{48}\sqrt{2}\right) - 2\left(\frac{3}{8}\ln\left(\sqrt{2}+1\right) + \frac{7}{8}\sqrt{2}\right) \\
&\quad + \left(\frac{1}{2}\ln\left(\sqrt{2}+1\right) + \frac{\sqrt{2}}{2}\right) \\
&= \boxed{\tfrac{7}{48}\sqrt{2} + \tfrac{1}{16}\ln\left(\sqrt{2}+1\right)}.
\end{aligned}$$

53. Integrate $\int_0^{\pi/2} \sin\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)dx$ using the product-to-sum identity

$$2\sin A\cos B = \sin(A+B) + \sin(A-B).$$

Then

$$\begin{aligned}
\int_0^{\pi/2} \sin\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)dx &= \frac{1}{2}\int_0^{\pi/2}\left[\sin\left(\frac{x}{2}+\frac{3x}{2}\right) + \sin\left(\frac{x}{2}-\frac{3x}{2}\right)\right]dx \\
&= \frac{1}{2}\int_0^{\pi/2}[\sin(2x) + \sin(-x)]\,dx \\
&= \frac{1}{2}\int_0^{\pi/2}[\sin(2x) - \sin x]\,dx \text{ since } \sin(-x) = -\sin x \\
&= \frac{1}{2}\int_0^{\pi/2}\sin(2x)\,dx - \frac{1}{2}\int_0^{\pi/2}\sin x\,dx \\
&= \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/2} - \frac{1}{2}[-\cos(x)]_0^{\pi/2} \\
&= -\frac{1}{4}[\cos(\pi) - \cos(0)] + \frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] \\
&= -\frac{1}{4}(-1-1) + \frac{1}{2}(0-1) = \boxed{0}.
\end{aligned}$$

55. We use the identity $\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}
\int \sin\left(\frac{x}{2}\right)\sin\left(\frac{3x}{2}\right)dx &= \frac{1}{2}\int(\cos(-x) - \cos(2x))\,dx \\
&= \frac{1}{2}\int(\cos(x) - \cos(2x))\,dx \\
&= \frac{1}{2}\left(\sin x - \frac{1}{2}\sin(2x)\right) + C \\
&= \boxed{\tfrac{1}{2}\sin x - \tfrac{1}{4}\sin(2x) + C}.
\end{aligned}$$

57. The volume is given by the integral $\int_0^\pi \pi(\sin x)^2\, dx$. We write $\sin^2 x = \frac{1-\cos(2x)}{2}$ and obtain

$$\begin{aligned}\int_0^\pi \pi(\sin x)^2\, dx &= \int_0^\pi \pi\left(\frac{1-\cos(2x)}{2}\right) dx \\ &= \frac{\pi}{2}\int_0^\pi (1-\cos(2x))\, dx \\ &= \frac{\pi}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi \\ &= \frac{\pi}{2}\left[\pi - \frac{1}{2}\sin(2\pi) - \left(0 - \frac{1}{2}\sin(2(0))\right)\right] \\ &= \boxed{\tfrac{1}{2}\pi^2}.\end{aligned}$$

59. Using the method of disks, the volume is given by

$$V = \pi\int_0^{\pi/2} [f(x)]^2\, dx = \pi\int_0^{\pi/2}\left[\sin x(\cos x)^{3/2}\right]^2 dx = \pi\int_0^{\pi/2} \sin^2 x\cos^3 x\, dx.$$

The exponent of $\cos x$ is 3, a positive, odd integer. Factor $\cos x$ from $\cos^3 x$ and write the rest of the integrand in terms of sines. Use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\int_0^{\pi/2} \sin^2 x\cos^3 x\, dx = \int_0^{\pi/2} \sin^2 x\cos^2 x\cos x\, dx = \int_0^{\pi/2} \sin^2 x\left(1-\sin^2 x\right)\cos x\, dx.$$

Now use the substitution $u = \sin x$ and $du = \cos x\, dx$. The lower limit of integration becomes $u = \sin 0 = 0$ and the upper limit of integration becomes $u = \sin\frac{\pi}{2} = 1$. Therefore,

$$\int_0^{\pi/2} \sin^2 x\cos^3 x\, dx = \int_0^{\pi/2} \sin^2 x\left(1-\sin^2 x\right)\cos x\, dx = \int_0^1 u^2\left(1-u^2\right) du.$$

Use algebraic manipulation to rewrite $u^2\left(1-u^2\right)$ in a form whose antiderivative is recognizable: $u^2\left(1-u^2\right) = u^2 - u^4$.

Then

$$\int_0^1 u^2\left(1-u^2\right) du = \int_0^1 \left(u^2-u^4\right) du = \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 = \left(\frac{1}{3}-\frac{1}{5}\right) - (0-0) = \frac{2}{15}.$$

The volume of the solid is

$$V = \pi\int_0^{\pi/2} \sin^2 x\cos^3 x\, dx = \pi\int_0^1 u^2\left(1-u^2\right) du = \boxed{\tfrac{2\pi}{15}}.$$

61. (a) The average value is given by $\frac{1}{\pi-0}\int_0^\pi \sin x\, \cos^4 x\, dx$. Let $u = \cos x$, then $du = -\sin x\, dx$, so $\sin x\, dx = -du$. The lower limit of integration is $u = \cos 0 = 1$, and the upper limit of integration is $u = \cos\pi = -1$. We substitute and obtain

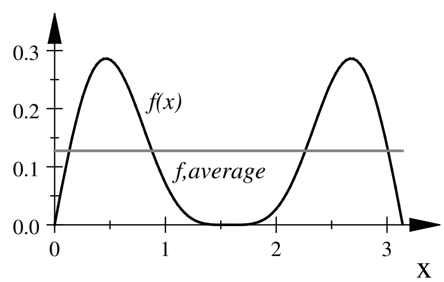
$$\begin{aligned}\frac{1}{\pi-0}\int_0^\pi \sin x\, \cos^4 x\, dx &= \frac{1}{\pi}\int_1^{-1} u^4\,(-du) \\ &= \frac{1}{\pi}\int_{-1}^1 u^4\, du \\ &= \frac{2}{\pi}\int_0^1 u^4\, du\end{aligned}$$

using that we have an even function on an interval symmetric about 0. So

$$\begin{aligned}\frac{1}{\pi-0}\int_0^{\pi}\sin x\,\cos^4 x\,dx &= \frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1\\ &= \frac{2}{\pi}\left[\frac{1}{5}1^5-\frac{1}{5}0^5\right]\\ &= \boxed{\frac{2}{5\pi}}.\end{aligned}$$

(b) The area under the graph of the function f is the same as the area of the rectangle with height $\frac{2}{5\pi}\approx 0.1273$ and width $\pi - 0 = \pi$.

(c)



63. The net displacement of the object from $t=0$ to $t=2\pi$ seconds is given by

$$\int_0^{2\pi} v(t)\,dt = \int_0^{2\pi}\sin^2 t\cos^2 t\,dt.$$

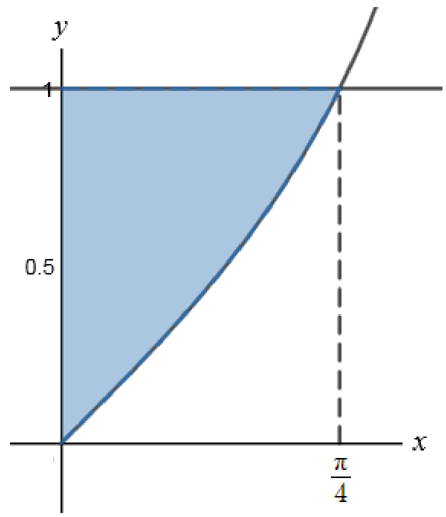
Use the identity $\sin^2\theta = \frac{1}{2}[1-\cos(2\theta)]$ and $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$.

Then

$$\begin{aligned}\int_0^{2\pi} v(t)\,dt &= \int_0^{2\pi}\frac{1}{2}[1-\cos(2t)]\cdot\frac{1}{2}[1+\cos(2t)]\,dt\\ &= \frac{1}{4}\int_0^{2\pi}\left[1-\cos^2(2t)\right]dt\\ &= \frac{1}{4}\int_0^{2\pi}\left[1-\frac{1}{2}[1+\cos(4t)]\right]dt\\ &= \frac{1}{8}\int_0^{2\pi}[1-\cos(4t)]\,dt\\ &= \frac{1}{8}\left[t-\frac{1}{4}\sin(4t)\right]_0^{2\pi}\\ &= \frac{1}{8}\left\{\left[(2\pi)-\frac{1}{4}\sin(8\pi)\right]-\left[(0)-\frac{1}{4}\sin(0)\right]\right\}\\ &= \boxed{\frac{\pi}{4}\approx 0.785}.\end{aligned}$$

After moving from $t=0$ to $t=2\pi$ seconds, the object is approximately 0.785 meters to the right of where it was at $t=0$.

65. (a) The region in the first quadrant bounded by the graph of $y = \tan x$ and the lines $x = 0$ and $y = 1$ is pictured below. The graph of $y = \tan x$ intersects with the line $y = 1$ when $\tan x = 1$; that is, when $x = \frac{\pi}{4}$. Since the region is above the x-axis, $\int_0^{\pi/4} (1 - \tan x)\, dx$ is the area of the desired region.



To evaluate $\int_0^{\pi/4} \tan x\, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx$, use the substitution $u = \cos x$ and $du = -\sin x\, dx$. Then $\sin x\, dx = -du$. The lower limit of integration becomes $u = \cos 0 = 1$ and the upper limit of integration becomes $u = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. So, $\int_0^{\pi/4} \tan x\, dx = \int_0^{\pi/4} \frac{1}{\cos x} \sin x\, dx = \int_1^{\sqrt{2}/2} \frac{1}{u}(-du) = -\int_1^{\sqrt{2}/2} \frac{1}{u}\, du$.
Using a property of integrals,

$$-\int_1^{\sqrt{2}/2} \frac{1}{u}\, du = \int_{\sqrt{2}/2}^{1} \frac{1}{u}\, du = [\ln|u|]_{\sqrt{2}/2}^{1} = \ln(1) - \ln\left(\frac{\sqrt{2}}{2}\right) = -\ln\left(\frac{\sqrt{2}}{2}\right) = \ln\left(\sqrt{2}\right).$$

Therefore, the area is

$$A = \int_0^{\pi/4} (1 - \tan x)\, dx = \int_0^{\pi/4} 1\, dx - \int_0^{\pi/4} \tan x\, dx = \boxed{\tfrac{\pi}{4} - \ln\left(\sqrt{2}\right)}.$$

(b) Using the method of washers, the volume is given by

$$V = \pi \int_0^{\pi/4} \left\{1^2 - [f(x)]^2\right\}\, dx = \pi\left(\int_0^{\pi/4} 1\, dx - \int_0^{\pi/4} \tan^2 x\, dx\right).$$

Use the identity $\tan^2 x = \sec^2 x - 1$ to evaluate $\int_0^{\pi/4} \tan^2 x\, dx$.

$$\begin{aligned}\int_0^{\pi/4} \tan^2 x\, dx &= \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx \\ &= [\tan x - x]_0^{\pi/4} \\ &= \left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)\right] \\ &= 1 - \frac{\pi}{4}\end{aligned}$$

So, $V = \pi\left(\int_0^{\pi/4} 1\, dx - \int_0^{\pi/4} \tan^2 x\, dx\right) = \pi\left[\frac{\pi}{4} - \left(1 - \frac{\pi}{4}\right)\right] = \boxed{\frac{\pi^2}{2} - \pi}$.

67. (a) Use the identity $\sin^2 x = \frac{1-\cos(2x)}{2}$ and obtain

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2 dx \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx \\
&= \frac{1}{4}\int dx - \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx \\
&= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \cos^2(2x)\,dx.
\end{aligned}$$

We evaluate

$$\begin{aligned}
\int \cos^2(2x)\,dx &= \int \frac{1+\cos(4x)}{2}\,dx \\
&= \frac{1}{2}\int (1+\cos(4x))\,dx \\
&= \frac{x}{2} + \frac{\sin(4x)}{8} + C.
\end{aligned}$$

We now obtain

$$\begin{aligned}
\int \cos^4 x\,dx &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4}\left(\frac{x}{2} + \frac{\sin(4x)}{8}\right) + C \\
&= \boxed{\tfrac{3}{8}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + C}.
\end{aligned}$$

(b) Use $\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$ with $n = 4$ to obtain

$$\begin{aligned}
\int \sin^4 x\,dx &= -\frac{\sin^3 x\cos x}{4} + \frac{4-1}{4}\int \sin^{4-2} x\,dx \\
&= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx.
\end{aligned}$$

And again with $n = 2$,

$$\begin{aligned}
\int \sin^4 x\,dx &= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\left(-\frac{\sin^{2-1} x\cos x}{2} + \frac{2-1}{2}\int \sin^{2-2} x\,dx\right) \\
&= -\frac{\sin^3 x\cos x}{4} - \frac{3\sin x\cos x}{8} + \frac{3}{8}\int dx \\
&= \boxed{-\tfrac{\sin^3 x\cos x}{4} - \tfrac{3\sin x\cos x}{8} + \tfrac{3}{8}x + C}.
\end{aligned}$$

(c) We have

$$\begin{aligned}
\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) &= \frac{3}{8}x - \frac{1}{4}(2\sin x\cos x) + \frac{1}{32}(2\sin(2x)\cos(2x)) \\
&= \frac{3}{8}x - \frac{1}{2}\sin x\cos x + \frac{1}{16}(2\sin x\cos x)\left(1 - 2\sin^2 x\right) \\
&= \frac{3}{8}x - \frac{1}{2}\sin x\cos x + \frac{1}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x \\
&= -\frac{\sin^3 x\cos x}{4} - -\frac{3\sin x\cos x}{8} + \frac{3}{8}x.
\end{aligned}$$

So the two antiderivatives are equal.

(d) Using a Computer Algebra System, we obtain

$$\int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C,$$

which agrees with the result obtained in part (a).

69. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}\int \sin(mx)\,\sin(nx)\,dx &= \frac{1}{2}\int (\cos((m-n)x) - \cos((m+n)x))\,dx \\ &= \frac{1}{2}\left(\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right) + C \\ &= \boxed{\tfrac{\sin((m-n)x)}{2(m-n)} - \tfrac{\sin((m+n)x)}{2(m+n)} + C}.\end{aligned}$$

71. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ to obtain

$$\begin{aligned}\int \cos(mx)\,\cos(nx)\,dx &= \frac{1}{2}\int (\cos((m+n)x) + \cos((m-n)x))\,dx \\ &= \frac{1}{2}\left(\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n}\right) + C \\ &= \boxed{\tfrac{\sin((m+n)x)}{2(m+n)} + \tfrac{\sin((m-n)x)}{2(m-n)} + C}.\end{aligned}$$

Challenge Problems

73. Let $\sqrt{x} = \sin y$, so $x = \sin^2 y$, and $dx = 2\sin y \cos y\,dy$. The lower limit of integration is $y = \sin^{-1} 0 = 0$, and the upper limit of integration is $y = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$. We substitute and obtain

$$\begin{aligned}\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\,dx &= \int_0^{\pi/4} \frac{\sin y}{\sqrt{1-\sin^2 y}}(2\sin y\cos y)\,dy \\ &= \int_0^{\pi/4} \frac{\sin y}{\sqrt{\cos^2 y}}(2\sin y\cos y)\,dy \\ &= \int_0^{\pi/4} \frac{\sin y}{\cos y}(2\sin y\cos y)\,dy \\ &= 2\int_0^{\pi/4} \sin^2 y\,dy.\end{aligned}$$

Use the identity $\sin^2 y = \frac{1-\cos(2y)}{2}$ and obtain

$$\begin{aligned}\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\,dx &= 2\int_0^{\pi/4} \frac{1-\cos(2y)}{2}\,dy \\ &= \int_0^{\pi/4} (1-\cos(2y))\,dy \\ &= \left[y - \frac{1}{2}\sin(2y)\right]_0^{\pi/4} \\ &= \left(\frac{\pi}{4} - \frac{1}{2}\sin\left(2\left(\frac{\pi}{4}\right)\right)\right) - \left(0 - \frac{1}{2}\sin(2(0))\right) \\ &= \boxed{\tfrac{1}{4}\pi - \tfrac{1}{2}}.\end{aligned}$$

75. (a) When writing $\left(\cos^2 x\right)^{3/2} = (\cos x)^3$, a mistake is made. When $\frac{\pi}{2} < x \le \pi$, $(\cos x)^3 < 0$, but $\left(\cos^2 x\right)^{3/2} > 0$. The correct equality would be $\left(\cos^2 x\right)^{3/2} = |\cos x|^3$.

(b) Use the identity $\cos^2 x = \frac{1+\cos(2x)}{2}$ and obtain

$$\begin{aligned}
\int_0^{\pi} \cos^4 x\,dx &= \int_0^{\pi} \left(\cos^2 x\right)^2 dx \\
&= \int_0^{\pi} \left(\frac{1+\cos(2x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos(2x) + \cos^2(2x)\right) dx \\
&= \frac{1}{4}\int_0^{\pi} dx + \frac{1}{2}\int_0^{\pi} \cos(2x)\,dx + \frac{1}{4}\int_0^{\pi} \cos^2(2x)\,dx \\
&= \left[\frac{1}{4}x\right]_0^{\pi} + \left[\frac{1}{4}\sin(2x)\right]_0^{\pi} + \frac{1}{4}\int_0^{\pi} \cos^2(2x)\,dx \\
&= \left[\frac{1}{4}\pi - \frac{1}{4}(0)\right] + \left[\frac{1}{4}\sin(2\pi) - \frac{1}{4}\sin(2(0))\right] + \frac{1}{4}\int_0^{\pi} \cos^2(2x)\,dx \\
&= \frac{\pi}{4} + \frac{1}{4}\int_0^{\pi} \cos^2(2x)\,dx.
\end{aligned}$$

We evaluate

$$\begin{aligned}
\int_0^{\pi} \cos^2(2x)\,dx &= \int_0^{\pi} \frac{1+\cos(4x)}{2}\,dx \\
&= \frac{1}{2}\int_0^{\pi} (1+\cos(4x))\,dx \\
&= \left[\frac{x}{2} + \frac{\sin(4x)}{8}\right]_0^{\pi} \\
&= \frac{\pi}{2} + \frac{\sin(4\pi)}{8} - \left(\frac{0}{2} + \frac{\sin(4(0))}{8}\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

We now obtain

$$\int_0^{\pi} \cos^4 x\,dx = \frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{3\pi}{8}}.$$

AP® Practice Problems

1. Integrate $\int \sin^3 x\,dx$ using trigonometric identities.

The exponent of $\sin x$ is 3, a positive, odd integer. Factor $\sin x$ from $\sin^3 x$ and write the rest of the integrand in terms of cosines.

$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int \left(1-\cos^2 x\right)\sin x\,dx.$$

Now use the substitution $u = \cos x$ and $du = -\sin x\,dx$. Then $\sin x\,dx = -du$ and

$$\int \sin^3 x\,dx = \int \left(1-\cos^2 x\right)\sin x\,dx = \int \left(1-u^2\right)(-du) = -\int \left(1-u^2\right)du$$
$$= -\left(u - \frac{1}{3}u^3\right) + C = -\left(\cos x - \frac{1}{3}\cos^3 x\right) + C = \boxed{-\cos x + \tfrac{\cos^3 x}{3} + C}.$$

The answer is C.

3. To evaluate $\int \sin x \cos(2x)\,dx$, use the product-to-sum identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.

$$\int \sin x\cos(2x)\,dx = \frac{1}{2}\int[\sin(x+2x)+\sin(x-2x)]\,dx = \frac{1}{2}\int[\sin(3x)+\sin(-x)]\,dx$$

Since $\sin(-x) = -\sin x$,

$$\int \sin x\cos(2x)\,dx = \frac{1}{2}\int[\sin(3x)-\sin x]\,dx = \boxed{\tfrac{1}{2}\left[-\tfrac{1}{3}\cos(3x)+\cos x\right]+C}.$$

The answer is D.

5. Integrate $\int_0^{\pi/4}\tan^3 x\sec x\,dx$ using trigonometric identities.

The exponent of $\tan x$ is 3, a positive, odd integer. Factor $\tan x\sec x$ from $\tan^3 x\sec x$ and write the rest of the integrand in terms of secants. Use the identity $\tan^2 x = \sec^2 x - 1$.

Then $$\int_0^{\pi/4}\tan^3 x\sec x\,dx = \int_0^{\pi/4}\tan^2 x\tan x\sec x\,dx = \int_0^{\pi/4}\left(\sec^2 x - 1\right)\tan x\sec x\,dx.$$

Now use the substitution $u = \sec x$ and $du = \tan x\sec x\,dx$.

The lower limit of integration becomes $u = \sec 0 = 1$ and the upper limit of integration becomes $u = \sec\frac{\pi}{4} = \sqrt{2}$.

So, $$\int_0^{\pi/4}\tan^3 x\sec x\,dx = \int_0^{\pi/4}\left(\sec^2 x-1\right)\tan x\sec x\,dx = \int_1^{\sqrt{2}}\left(u^2-1\right)du$$
$$= \left[\frac{u^3}{3}-u\right]_1^{\sqrt{2}} = \left(\frac{2\sqrt{2}}{3}-\sqrt{2}\right)-\left(\frac{1}{3}-1\right) = \boxed{\tfrac{1}{3}\left(2-\sqrt{2}\right)}.$$

The answer is C.

7.3 Integration Using Trigonometric Substitution

Concepts and Vocabulary

1. True.
3. (c), $x = 3\sec\theta$

Skill Building

5. Let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-(2\sin\theta)^2}\,(2\cos\theta)\,d\theta \\
&= 2\int \sqrt{4-4\sin^2\theta}\,\cos\theta\,d\theta \\
&= 2\int \sqrt{4\big(1-\sin^2\theta\big)}\,\cos\theta\,d\theta \\
&= 2\int \sqrt{4\cos^2\theta}\,\cos\theta\,d\theta \\
&= 2\int (2\cos\theta)\,\cos\theta\,d\theta \\
&= 4\int \cos^2\theta\,d\theta \\
&= 4\int \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= 2\int (1+\cos(2\theta))\,d\theta \\
&= 2\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C \\
&= 2\theta+\sin(2\theta)+C \\
&= 2\theta+2\sin\theta\cos\theta+C.
\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. We obtain

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= 2\sin^{-1}\left(\frac{x}{2}\right)+2\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right)+C \\
&= \boxed{2\sin^{-1}\left(\tfrac{x}{2}\right)+\tfrac{1}{2}x\sqrt{4-x^2}+C}.
\end{aligned}$$

7. Let $x = 4\sin\theta$, then $dx = 4\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= \int \frac{(4\sin\theta)^2}{\sqrt{16-(4\sin\theta)^2}}(4\cos\theta)\,d\theta \\
&= 64\int \frac{\sin^2\theta}{\sqrt{16-16\sin^2\theta}}\cos\theta\,d\theta \\
&= 64\int \frac{\sin^2\theta}{\sqrt{16\left(1-\sin^2\theta\right)}}\cos\theta\,d\theta \\
&= 64\int \frac{\sin^2\theta}{\sqrt{16\cos^2\theta}}\cos\theta\,d\theta \\
&= 16\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \\
&= 16\int \sin^2\theta\,d\theta \\
&= 16\int \frac{1-\cos(2\theta)}{2}\,d\theta \\
&= 8\int (1-\cos(2\theta))\,d\theta \\
&= 8\left(\theta-\frac{1}{2}\sin 2\theta\right)+C \\
&= 8\theta - 8\sin\theta\cos\theta + C.
\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{4}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/4)^2} = \frac{1}{4}\sqrt{16-x^2}$. We obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= 8\sin^{-1}\left(\frac{x}{4}\right) - 8\left(\frac{x}{4}\right)\left(\frac{1}{4}\sqrt{16-x^2}\right)+C \\
&= \boxed{8\sin^{-1}\left(\tfrac{x}{4}\right)-\tfrac{1}{2}x\sqrt{16-x^2}+C}.
\end{aligned}$$

9. Let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{(2\sin\theta)^2}(2\cos\theta)\,d\theta \\
&= \frac{1}{2}\int \frac{\sqrt{4-4\sin^2\theta}}{\sin^2\theta}\cos\theta\,d\theta \\
&= \frac{1}{2}\int \frac{\sqrt{4\left(1-\sin^2\theta\right)}}{\sin^2\theta}\cos\theta\,d\theta \\
&= \frac{1}{2}\int \frac{\sqrt{4\cos^2\theta}}{\sin^2\theta}\cos\theta\,d\theta \\
&= \int \frac{\cos\theta\,\cos\theta}{\sin^2\theta}\,d\theta \\
&= \int \cot^2\theta\,d\theta \\
&= \int \left(\csc^2\theta-1\right)d\theta \\
&= -\cot\theta-\theta+C.
\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. So $\cot\theta = \frac{\frac{1}{2}\sqrt{4-x^2}}{x/2} = \frac{\sqrt{4-x^2}}{x}$. We obtain

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = \boxed{-\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C}.$$

11. Let $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int x^2\sqrt{4-x^2}\,dx &= \int (2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}\,(2\cos\theta)\,d\theta \\
&= 8\int \left(\sin^2\theta\right)\sqrt{4-4\sin^2\theta}\,\cos\theta\,d\theta \\
&= 8\int \left(\sin^2\theta\right)\sqrt{4\left(1-\sin^2\theta\right)}\,\cos\theta\,d\theta \\
&= 8\int \left(\sin^2\theta\right)\sqrt{4\cos^2\theta}\,\cos\theta\,d\theta \\
&= 8\int \left(\sin^2\theta\right)(2\cos\theta)\,\cos\theta\,d\theta \\
&= 16\int \sin^2\theta\,\cos^2\theta\,d\theta \\
&= 16\int \left(\frac{\sin 2\theta}{2}\right)^2 d\theta \\
&= 4\int \sin^2(2\theta)\,d\theta \\
&= 4\int \frac{1-\cos(4\theta)}{2}\,d\theta \\
&= 2\int (1-\cos(4\theta))\,d\theta \\
&= 2\left(\theta - \frac{1}{4}\sin(4\theta)\right) + C \\
&= 2\theta - \frac{1}{2}\sin(4\theta) + C \\
&= 2\theta - \sin 2\theta\cos 2\theta + C \\
&= 2\theta - (2\sin\theta\cos\theta)\left(1-2\sin^2\theta\right) + C.
\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. We obtain

$$\begin{aligned}
\int x^2\sqrt{4-x^2}\,dx &= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right)\left(1-2\left(\frac{x}{2}\right)^2\right) + C \\
&= \boxed{\frac{x\left(x^2-2\right)}{4}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + C}.
\end{aligned}$$

13. Let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{1}{(4-x^2)^{3/2}}\,dx &= \int \frac{1}{\left(4-4\sin^2\theta\right)^{3/2}}\,(2\cos\theta)\,d\theta \\ &= 2\int \frac{1}{4^{3/2}(\cos^2\theta)^{3/2}}\,\cos\theta\,d\theta \\ &= \frac{1}{4}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta \\ &= \frac{1}{4}\int \sec^2\theta\,d\theta \\ &= \frac{1}{4}\tan\theta + C.\end{aligned}$$

We have $\sin\theta = \frac{x}{2}$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. So $\tan\theta = \frac{x/2}{\frac{1}{2}\sqrt{4-x^2}} = \frac{x}{\sqrt{4-x^2}}$. We obtain

$$\begin{aligned}\int \frac{1}{(4-x^2)^{3/2}}\,dx &= \frac{1}{4}\tan\theta + C \\ &= \boxed{\frac{x}{4\sqrt{4-x^2}} + C}.\end{aligned}$$

15. Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{4+x^2}\,dx &= \int \sqrt{4+(2\tan\theta)^2}\left(2\sec^2\theta\right)d\theta \\ &= 2\int \sqrt{4+4\tan^2\theta}\,\sec^2\theta\,d\theta \\ &= 2\int (2\sec\theta)\sec^2\theta\,d\theta \\ &= 4\int \sec^3\theta\,d\theta \\ &= 4\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right] + C \\ &= 2\sec\theta\tan\theta + 2\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

We have $\tan\theta = \frac{x}{2}$, and $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+(x/2)^2} = \frac{1}{2}\sqrt{4+x^2}$. We obtain

$$\begin{aligned}\int \sqrt{4+x^2}\,dx &= 2\left(\frac{1}{2}\sqrt{4+x^2}\right)\left(\frac{x}{2}\right) + 2\ln\left|\frac{1}{2}\sqrt{4+x^2} + \left(\frac{x}{2}\right)\right| + C \\ &= \boxed{\frac{1}{2}x\sqrt{4+x^2} + 2\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + C}\end{aligned}$$

17. Let $x = 4\tan\theta$, then $dx = 4\sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{1}{\sqrt{(4\tan\theta)^2+16}}\left(4\sec^2\theta\right) d\theta \\
&= 4\int \frac{1}{\sqrt{16\tan^2\theta+16}}\left(\sec^2\theta\right) d\theta \\
&= 4\int \frac{1}{\sqrt{16\left(\tan^2\theta+1\right)}}\left(\sec^2\theta\right) d\theta \\
&= 4\int \frac{1}{\sqrt{16\sec^2\theta}}\left(\sec^2\theta\right) d\theta \\
&= \int \frac{1}{\sec\theta}\left(\sec^2\theta\right) d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

We have $\tan\theta = \frac{x}{4}$, and $\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{(x/4)^2+1} = \frac{1}{4}\sqrt{x^2+16}$. We obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+16}} &= \ln\left|\frac{1}{4}\sqrt{x^2+16}+\left(\frac{x}{4}\right)\right| + C \\
&= \boxed{\ln\left|\tfrac{\sqrt{x^2+16}+x}{4}\right| + C}.
\end{aligned}$$

19. Let $x = \frac{1}{3}\tan\theta$, then $dx = \frac{1}{3}\sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \sqrt{1+9x^2}\, dx &= \int \sqrt{1+9\left(\frac{1}{3}\tan\theta\right)^2}\left(\frac{1}{3}\sec^2\theta\right) d\theta \\
&= \frac{1}{3}\int \sqrt{1+\tan^2\theta}\sec^2\theta\, d\theta \\
&= \frac{1}{3}\int (\sec\theta)\sec^2\theta\, d\theta \\
&= \frac{1}{3}\int \sec^3\theta\, d\theta \\
&= \frac{1}{3}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right] + C \\
&= \frac{1}{6}\sec\theta\tan\theta + \frac{1}{6}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

We have $\tan\theta = 3x$, and $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+(3x)^2} = \sqrt{1+9x^2}$. We obtain

$$\begin{aligned}
\int \sqrt{1+9x^2}\, dx &= \frac{1}{6}\left(\sqrt{1+9x^2}\right)(3x) + \frac{1}{6}\ln\left|\sqrt{1+9x^2}+(3x)\right| + C \\
&= \boxed{\tfrac{1}{2}x\sqrt{1+9x^2} + \tfrac{1}{6}\ln\left(\sqrt{1+9x^2}+3x\right) + C}.
\end{aligned}$$

21. Let $x = \frac{2}{3}\tan\theta$, then $dx = \frac{2}{3}\sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4+9x^2}}\,dx &= \int \frac{\left(\frac{2}{3}\tan\theta\right)^2}{\sqrt{4+9\left(\frac{2}{3}\tan\theta\right)^2}}\left(\frac{2}{3}\sec^2\theta\right)d\theta \\
&= \frac{8}{27}\int \frac{\tan^2\theta}{\sqrt{4+4\tan^2\theta}}\sec^2\theta\,d\theta \\
&= \frac{8}{27}\int \frac{\tan^2\theta}{\sqrt{4\left(1+\tan^2\theta\right)}}\sec^2\theta\,d\theta \\
&= \frac{8}{27}\int \frac{\tan^2\theta}{\sqrt{4\sec^2\theta}}\sec^2\theta\,d\theta \\
&= \frac{4}{27}\int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta \\
&= \frac{4}{27}\int \tan^2\theta\sec\theta\,d\theta \\
&= \frac{4}{27}\int \left(\sec^2\theta-1\right)\sec\theta\,d\theta \\
&= \frac{4}{27}\int \left(\sec^3\theta-\sec\theta\right)d\theta \\
&= \frac{4}{27}\left[\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|-\ln|\sec\theta+\tan\theta|\right]+C \\
&= \frac{2}{27}\sec\theta\tan\theta-\frac{2}{27}\ln|\sec\theta+\tan\theta|+C.
\end{aligned}$$

We have $\tan\theta = \frac{3x}{2}$, and $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{3x}{2}\right)^2} = \frac{1}{2}\sqrt{4+9x^2}$. We obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4+9x^2}}\,dx &= \frac{2}{27}\left(\frac{1}{2}\sqrt{4+9x^2}\right)\left(\frac{3x}{2}\right)-\frac{2}{27}\ln\left|\frac{1}{2}\sqrt{4+9x^2}+\frac{3x}{2}\right|+C \\
&= \boxed{\tfrac{1}{18}x\sqrt{4+9x^2}-\tfrac{2}{27}\ln\left(\tfrac{\sqrt{4+9x^2}+3x}{2}\right)+C}.
\end{aligned}$$

23. Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}}\,d\theta \\
&= \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{4\tan^2\theta+4}}\,d\theta \\
&= \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{4\left(\tan^2\theta+1\right)}}\,d\theta \\
&= \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.
\end{aligned}$$

Let $u = \sin\theta$, then $du = \cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \frac{1}{4}\int \frac{1}{u^2}\, du \\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C \\ &= -\frac{1}{4}\csc\theta + C.\end{aligned}$$

We have $\tan\theta = \frac{x}{2}$, so $\cot\theta = \frac{2}{x}$, and $\csc\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\left(\frac{2}{x}\right)^2} = \frac{1}{x}\sqrt{x^2+4}$. We obtain

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= -\frac{1}{4}\frac{1}{x}\sqrt{x^2+4} + C \\ &= \boxed{-\tfrac{\sqrt{x^2+4}}{4x} + C}.\end{aligned}$$

25. Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{(x^2+4)^{3/2}} &= \int \frac{2\sec^2\theta}{\left((2\tan\theta)^2+4\right)^{3/2}}\, d\theta \\ &= 2\int \frac{\sec^2\theta}{\left(4\tan^2\theta+4\right)^{3/2}}\, d\theta \\ &= 2\int \frac{\sec^2\theta}{\left(4\left(\tan^2\theta+1\right)\right)^{3/2}}\, d\theta \\ &= \frac{1}{4}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta \\ &= \frac{1}{4}\int \cos\theta\, d\theta \\ &= \frac{1}{4}\sin\theta + C.\end{aligned}$$

We have $\tan\theta = \frac{x}{2}$, so $\cot\theta = \frac{2}{x}$, and $\csc\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\left(\frac{2}{x}\right)^2} = \frac{1}{x}\sqrt{x^2+4}$. So $\sin\theta = \frac{x}{\sqrt{x^2+4}}$ and we obtain

$$\begin{aligned}\int \frac{dx}{(x^2+4)^{3/2}} &= \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C \\ &= \boxed{\tfrac{x}{4\sqrt{x^2+4}} + C}.\end{aligned}$$

27. Let $x = 5\sec\theta$, then $dx = 5\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-25}}\,dx &= \int \frac{(5\sec\theta)^2}{\sqrt{(5\sec\theta)^2-25}}(5\sec\theta\tan\theta)\,d\theta \\
&= 125\int \frac{\sec^2\theta}{\sqrt{25\sec^2\theta-25}}\sec\theta\tan\theta\;d\theta \\
&= 25\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\;d\theta \\
&= 25\int \frac{\sec^2\theta}{\tan\theta}\sec\theta\tan\theta\;d\theta \\
&= 25\int \sec^3\theta\,d\theta \\
&= 25\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) + C \\
&= \frac{25}{2}\sec\theta\tan\theta + \frac{25}{2}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

We have $\sec\theta = \frac{x}{5}$, so $\tan\theta = \sqrt{\sec^2\theta-1} = \sqrt{\left(\frac{x}{5}\right)^2-1} = \frac{1}{5}\sqrt{x^2-25}$. We obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-25}}\,dx &= \frac{25}{2}\left(\frac{x}{5}\right)\left(\frac{1}{5}\sqrt{x^2-25}\right) + \frac{25}{2}\ln\left|\frac{x}{5}+\frac{1}{5}\sqrt{x^2-25}\right| + C \\
&= \boxed{\tfrac{1}{2}x\sqrt{x^2-25} + \tfrac{25}{2}\ln\left|\tfrac{x+\sqrt{x^2-25}}{5}\right| + C}.
\end{aligned}$$

29. Let $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\sqrt{(\sec\theta)^2-1}}{\sec\theta}(\sec\theta\tan\theta)\,d\theta \\
&= \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\;d\theta \\
&= \int \tan^2\theta\,d\theta \\
&= \int \left(\sec^2\theta-1\right)d\theta \\
&= \tan\theta - \theta + C.
\end{aligned}$$

We have $\theta = \sec^{-1}x$, so $\tan\theta = \sqrt{\sec^2\theta-1} = \sqrt{x^2-1}$. We obtain

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \boxed{\sqrt{x^2-1} - \sec^{-1}x + C}.$$

31. Let $x = 6\sec\theta$, then $dx = 6\sec\theta\tan\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-36}} &= \int \frac{6\sec\theta\tan\theta}{(6\sec\theta)^2\sqrt{(6\sec\theta)^2-36}}\, d\theta \\
&= \frac{1}{6}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{36\sec^2\theta-36}}\, d\theta \\
&= \frac{1}{36}\int \frac{\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\, d\theta \\
&= \frac{1}{36}\int \frac{\tan\theta}{\sec\theta\tan\theta}\, d\theta \\
&= \frac{1}{36}\int \cos\theta\, d\theta \\
&= \frac{1}{36}\sin\theta + C.
\end{aligned}$$

We have $\sec\theta = \frac{x}{6}$, so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{6}\right)^2 - 1} = \frac{1}{6}\sqrt{x^2-36}$. So $\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{\frac{1}{6}\sqrt{x^2-36}}{\frac{x}{6}} = \frac{\sqrt{x^2-36}}{x}$. We obtain

$$\int \frac{dx}{x^2\sqrt{x^2-36}} = \boxed{\frac{1}{36}\frac{\sqrt{x^2-36}}{x} + C}.$$

33. Let $x = \frac{3}{2}\sec\theta$, then $dx = \frac{3}{2}\sec\theta\tan\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x^2-9}} &= \int \frac{\frac{3}{2}\sec\theta\tan\theta}{\sqrt{4\left(\frac{3}{2}\sec\theta\right)^2-9}}\, d\theta \\
&= \frac{3}{2}\int \frac{\sec\theta\tan\theta}{\sqrt{9\sec^2\theta-9}}\, d\theta \\
&= = \frac{3}{2}\int \frac{\sec\theta\tan\theta}{\sqrt{9(\sec^2\theta-1)}}\, d\theta \\
&= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta \\
&= \frac{1}{2}\int \sec\theta\, d\theta \\
&= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.
\end{aligned}$$

We have $\sec\theta = \frac{2x}{3}$, so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{2x}{3}\right)^2 - 1} = \frac{1}{3}\sqrt{4x^2-9}$. We obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x^2-9}} &= \frac{1}{2}\ln\left|\frac{2x}{3} + \frac{1}{3}\sqrt{4x^2-9}\right| + C \\
&= \boxed{\frac{1}{2}\ln\left|\frac{2x+\sqrt{4x^2-9}}{3}\right| + C}.
\end{aligned}$$

35. Let $x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{(x^2-9)^{3/2}} &= \int \frac{3\sec\theta\tan\theta}{\left((3\sec\theta)^2-9\right)^{3/2}}\,d\theta \\ &= 3\int \frac{\sec\theta\tan\theta}{(9\sec^2\theta-9)^{3/2}}\,d\theta \\ &= 3\int \frac{\sec\theta\tan\theta}{(9(\sec^2\theta-1))^{3/2}}\,d\theta \\ &= \frac{1}{9}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta \\ &= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\ &= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta.\end{aligned}$$

Let $u = \sin\theta$, then $du = \cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{(x^2-9)^{3/2}} &= \frac{1}{9}\int \frac{du}{u^2} \\ &= \frac{1}{9}\left(-\frac{1}{u}\right)+C \\ &= -\frac{1}{9}\csc\theta + C.\end{aligned}$$

We have $\sec\theta = \frac{x}{3}$, so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3}\sqrt{x^2-9}$. Then $\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{x/3}{\frac{1}{3}\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}$. We obtain

$$\int \frac{dx}{(x^2-9)^{3/2}} = \boxed{-\frac{1}{9}\frac{x}{\sqrt{x^2-9}}+C}.$$

37. Let $x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{x^2\,dx}{(x^2-9)^{3/2}} &= \int \frac{(3\sec\theta)^2(3\sec\theta\tan\theta)}{\left((3\sec\theta)^2-9\right)^{3/2}}\,d\theta \\ &= 27\int \frac{\sec^3\theta\tan\theta}{(9\sec^2\theta-9)^{3/2}}\,d\theta \\ &= 27\int \frac{\sec^3\theta\tan\theta}{(9(\sec^2\theta-1))^{3/2}}\,d\theta \\ &= \int \frac{\sec^3\theta\tan\theta}{\tan^3\theta}\,d\theta \\ &= \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta \\ &= \int \frac{\sec\theta\left(\tan^2\theta+1\right)}{\tan^2\theta}\,d\theta \\ &= \int \left(\sec\theta + \frac{\sec\theta}{\tan^2\theta}\right)d\theta \\ &= \ln|\sec\theta+\tan\theta| + \int \frac{\cos\theta}{\sin^2\theta}\,d\theta.\end{aligned}$$

Let $u = \sin\theta$, then $du = \cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2\,dx}{\left(x^2-9\right)^{3/2}} &= \ln|\sec\theta+\tan\theta| + \int \frac{du}{u^2} \\
&= \ln|\sec\theta+\tan\theta| + \left(-\frac{1}{u}\right) + C \\
&= \ln|\sec\theta+\tan\theta| - \csc\theta + C.
\end{aligned}$$

We have $\sec\theta = \frac{x}{3}$, so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3}\sqrt{x^2-9}$. Then $\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{x/3}{\frac{1}{3}\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}$. We obtain

$$\begin{aligned}
\int \frac{x^2\,dx}{\left(x^2-9\right)^{3/2}} &= \ln\left|\frac{x}{3} + \frac{1}{3}\sqrt{x^2-9}\right| - \frac{x}{\sqrt{x^2-9}} + C \\
&= \boxed{\ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| - \frac{x}{\sqrt{x^2-9}} + C}.
\end{aligned}$$

39. Let $x = 4\tan\theta$, then $dx = 4\sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{16+x^2}\,dx &= \int \frac{\left(4\tan\theta\right)^2}{16+\left(4\tan\theta\right)^2}\left(4\sec^2\theta\right)d\theta \\
&= 64\int \frac{\tan^2\theta}{16+16\tan^2\theta}\sec^2\theta\, d\theta \\
&= 4\int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\, d\theta \\
&= 4\int \tan^2\theta\, d\theta \\
&= 4\int \left(\sec^2\theta - 1\right)d\theta \\
&= 4(\tan\theta - \theta) + C \\
&= 4\left(\frac{x}{4}\right) - 4\tan^{-1}\left(\frac{x}{4}\right) + C \\
&= \boxed{x - 4\tan^{-1}\left(\tfrac{x}{4}\right) + C}.
\end{aligned}$$

41. Let $x = \frac{2}{5}\sin\theta$, then $dx = \frac{2}{5}\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \sqrt{4-25x^2}\,dx &= \int \sqrt{4-25\left(\frac{2}{5}\sin\theta\right)^2}\left(\frac{2}{5}\cos\theta\right)d\theta \\
&= \frac{2}{5}\int \sqrt{4-4\sin^2\theta}\cos\theta\,d\theta \\
&= = \frac{2}{5}\int \sqrt{4\bigl(1-\sin^2\theta\bigr)}\cos\theta\,d\theta \\
&= \frac{4}{5}\int \cos^2\theta\,d\theta \\
&= \frac{4}{5}\int \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{2}{5}\int (1+\cos(2\theta))\,d\theta \\
&= \frac{2}{5}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C \\
&= \frac{2}{5}\theta+\frac{1}{5}\sin(2\theta)+C \\
&= \frac{2}{5}\theta+\frac{2}{5}\sin\theta\cos\theta+C.
\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{5x}{2}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{5x}{2}\right)^2} = \frac{1}{2}\sqrt{4-25x^2}$. We obtain

$$\begin{aligned}
\int \sqrt{4-25x^2}\,dx &= \frac{2}{5}\sin^{-1}\left(\frac{5x}{2}\right)+\frac{2}{5}\left(\frac{5x}{2}\right)\left(\frac{1}{2}\sqrt{4-25x^2}\right)+C \\
&= \boxed{\tfrac{2}{5}\sin^{-1}\left(\tfrac{5x}{2}\right)+\tfrac{1}{2}x\sqrt{4-25x^2}+C}.
\end{aligned}$$

43. Let $x = \frac{2}{5}\sin\theta$, then $dx = \frac{2}{5}\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{1}{(4-25x^2)^{3/2}}\,dx &= \int \frac{1}{\left(4-25\left(\frac{2}{5}\sin\theta\right)^2\right)^{3/2}}\left(\frac{2}{5}\cos\theta\right)d\theta \\
&= \frac{2}{5}\int \frac{1}{4^{3/2}(\cos^2\theta)^{3/2}}\cos\theta\,d\theta \\
&= \frac{1}{20}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta \\
&= \frac{1}{20}\int \sec^2\theta\,d\theta \\
&= \frac{1}{20}\tan\theta+C.
\end{aligned}$$

We have $\sin\theta = \frac{5x}{2}$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{5x}{2}\right)^2} = \frac{1}{2}\sqrt{4-25x^2}$. So $\tan\theta = \frac{5x/2}{\frac{1}{2}\sqrt{4-25x^2}} = \frac{5x}{\sqrt{4-25x^2}}$. We obtain

$$\begin{aligned}
\int \frac{1}{(4-25x^2)^{3/2}}\,dx &= \frac{1}{20}\left(\frac{5x}{\sqrt{4-25x^2}}\right)+C \\
&= \boxed{\tfrac{x}{4\sqrt{4-25x^2}}+C}.
\end{aligned}$$

45. Let $x = \frac{2}{5}\tan\theta$, then $dx = \frac{2}{5}\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \sqrt{4+25x^2}\,dx &= \int \sqrt{4+25\left(\frac{2}{5}\tan\theta\right)^2}\left(\frac{2}{5}\sec^2\theta\right)d\theta \\
&= \frac{2}{5}\int \sqrt{4+4\tan^2\theta}\sec^2\theta\,d\theta \\
&= \frac{2}{5}\int \sqrt{4\left(1+\tan^2\theta\right)}\sec^2\theta\,d\theta \\
&= \frac{2}{5}\int (2\sec\theta)\sec^2\theta\,d\theta \\
&= \frac{4}{5}\int \sec^3\theta\,d\theta \\
&= \frac{4}{5}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]+C \\
&= \frac{2}{5}\sec\theta\tan\theta + \frac{2}{5}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

We have $\tan\theta = \frac{5x}{2}$, and $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{5x}{2}\right)^2} = \frac{1}{2}\sqrt{4+25x^2}$. We obtain

$$\begin{aligned}
\int \sqrt{4+25x^2}\,dx &= \frac{2}{5}\left(\frac{1}{2}\sqrt{4+25x^2}\right)\left(\frac{5x}{2}\right) + \frac{2}{5}\ln\left|\frac{1}{2}\sqrt{4+25x^2}+\frac{5x}{2}\right| + C \\
&= \boxed{\tfrac{1}{2}x\sqrt{4+25x^2} + \tfrac{2}{5}\ln\left(\tfrac{\sqrt{4+25x^2}+5x}{2}\right) + C}.
\end{aligned}$$

47. Let $x = 4\sec\theta$, then $dx = 4\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-16}} &= \int \frac{4\sec\theta\tan\theta}{(4\sec\theta)^3\sqrt{(4\sec\theta)^2-16}}\,d\theta \\
&= \frac{1}{16}\int \frac{\sec\theta\tan\theta}{\sec^3\theta\sqrt{16\sec^2\theta-16}}\,d\theta \\
&= \frac{1}{64}\int \frac{\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}\,d\theta \\
&= \frac{1}{64}\int \frac{\tan\theta}{\sec^2\theta\tan\theta}\,d\theta \\
&= \frac{1}{64}\int \cos^2\theta\,d\theta \\
&= \frac{1}{64}\int \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{1}{128}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{1}{128}\theta + \frac{1}{256}(2\sin\theta\cos\theta) + C \\
&= \frac{1}{128}\theta + \frac{1}{128}\sin\theta\cos\theta + C.
\end{aligned}$$

We have $\theta = \sec^{-1}\left(\frac{x}{4}\right)$, so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{4}\right)^2 - 1} = \frac{1}{4}\sqrt{x^2-16}$. So $\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{\frac{1}{4}\sqrt{x^2-16}}{\frac{x}{4}} = \frac{\sqrt{x^2-16}}{x}$, and $\cos\theta = \frac{4}{x}$. We obtain

$$\begin{aligned}\int \frac{dx}{x^3\sqrt{x^2-16}} &= \frac{1}{128}\sec^{-1}\left(\frac{x}{4}\right) + \frac{1}{128}\left(\frac{\sqrt{x^2-16}}{x}\right)\left(\frac{4}{x}\right) + C \\ &= \boxed{\frac{1}{128}\sec^{-1}\left(\frac{x}{4}\right) + \frac{\sqrt{x^2-16}}{32x^2} + C}.\end{aligned}$$

49. Let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$. The lower limit of integration is $\theta = \sin^{-1} 0 = 0$, and the upper limit of integration is $\theta = \sin^{-1} 1 = \frac{\pi}{2}$. We substitute and obtain

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\, dx &= \int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,(\cos\theta)\, d\theta \\ &= \int_0^{\pi/2} \cos^2\theta\, d\theta \\ &= \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta \\ &= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} \\ &= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\left(2\left(\frac{\pi}{2}\right)\right)\right) - \left(0 + \frac{1}{2}\sin(2(0))\right)\right] \\ &= \boxed{\frac{1}{4}\pi}.\end{aligned}$$

51. Let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$. The lower limit of integration is $\theta = \tan^{-1} 0 = 0$, and the upper limit of integration is $\theta = \tan^{-1} 1 = \frac{\pi}{4}$. We substitute and obtain

$$\begin{aligned}\int_0^1 \sqrt{1+x^2}\, dx &= \int_0^{\pi/4} \sqrt{1+(\tan\theta)^2}\left(\sec^2\theta\right) d\theta \\ &= \int_0^{\pi/4} (\sec\theta)\sec^2\theta\, d\theta \\ &= \int_0^{\pi/4} \sec^3\theta\, d\theta \\ &= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \\ &= \frac{1}{2}\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \frac{1}{2}\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \left(\frac{1}{2}\sec 0\tan 0 + \frac{1}{2}\ln|\sec 0 + \tan 0|\right) \\ &= \boxed{\frac{\sqrt{2}+\ln\left(\sqrt{2}+1\right)}{2}}.\end{aligned}$$

53. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3\sec\theta$, then $dx = 3\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-9}}\,dx &= \int \frac{(3\sec\theta)^2}{\sqrt{(3\sec\theta)^2-9}}(3\sec\theta\tan\theta\,)\,d\theta \\
&= 27\int \frac{\sec^2\theta}{\sqrt{9\sec^2\theta-9}}\sec\theta\tan\theta\;d\theta \\
&= 9\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\;d\theta \\
&= 9\int \frac{\sec^2\theta}{\tan\theta}\sec\theta\tan\theta\;d\theta \\
&= 9\int \sec^3\theta\,d\theta \\
&= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

Since $\sec\theta = \frac{x}{3}$, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3}\sqrt{x^2-9}$. So

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-9}}\,dx &= \frac{9}{2}\left(\frac{1}{3}\sqrt{x^2-9}\right)\frac{x}{3} + \frac{9}{2}\ln\left|\frac{1}{3}\sqrt{x^2-9}+\frac{x}{3}\right| + C \\
&= \frac{1}{2}x\sqrt{x^2-9} + \frac{9}{2}\ln\left|\frac{1}{3}\sqrt{x^2-9}+\frac{x}{3}\right| + C.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_4^5 \frac{x^2}{\sqrt{x^2-9}}\,dx &= \left[\frac{1}{2}x\sqrt{x^2-9} + \frac{9}{2}\ln\left|\frac{1}{3}\sqrt{x^2-9}+\frac{x}{3}\right|\right]_4^5 \\
&= \left(\frac{1}{2}5\sqrt{5^2-9} + \frac{9}{2}\ln\left|\frac{1}{3}\sqrt{5^2-9}+\frac{5}{3}\right|\right) - \left(\frac{1}{2}4\sqrt{4^2-9} + \frac{9}{2}\ln\left|\frac{1}{3}\sqrt{4^2-9}+\frac{4}{3}\right|\right) \\
&= \frac{9}{2}\ln 3 + 10 - \left(\frac{9}{2}\ln\left(\frac{1}{3}\sqrt{7}+\frac{4}{3}\right) + 2\sqrt{7}\right) \\
&= \frac{9}{2}\ln 3 + 10 - \frac{9}{2}\ln\left(\sqrt{7}+4\right) + \frac{9}{2}\ln 3 - 2\sqrt{7} \\
&= \boxed{10 - 2\sqrt{7} + 9\ln 3 - \tfrac{9}{2}\ln\left(4+\sqrt{7}\right)}.
\end{aligned}$$

55. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 4\sin\theta$, then $dx = 4\cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{\left(16-x^2\right)^{3/2}}\,dx &= \int \frac{(4\sin\theta)^2}{\left(16-(4\sin\theta)^2\right)^{3/2}}(4\cos\theta)\,d\theta \\
&= 64\int \frac{\sin^2\theta}{\left(16\cos^2\theta\right)^{3/2}}\cos\theta\,d\theta \\
&= \int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}\,d\theta \\
&= \int \tan^2\theta\,d\theta \\
&= \int \left(\sec^2\theta-1\right)d\theta \\
&= \tan\theta-\theta+C \\
&= \frac{\sin\theta}{\cos\theta}-\theta+C.
\end{aligned}$$

Since $\sin\theta = \frac{x}{4}$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{4}\right)^2} = \frac{1}{4}\sqrt{16-x^2}$. So we have

$$\begin{aligned}
\int \frac{x^2}{\left(16-x^2\right)^{3/2}}\,dx &= \frac{\frac{x}{4}}{\frac{1}{4}\sqrt{16-x^2}}-\sin^{-1}\frac{x}{4}+C \\
&= \frac{x}{\sqrt{16-x^2}}-\sin^{-1}\frac{x}{4}+C.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^2 \frac{x^2}{\left(16-x^2\right)^{3/2}}\,dx &= \left[\frac{x}{\sqrt{16-x^2}}-\sin^{-1}\frac{x}{4}\right]_0^2 \\
&= \left(\frac{2}{\sqrt{16-2^2}}-\sin^{-1}\frac{2}{4}\right)-\left(\frac{0}{\sqrt{4-0^2}}-\sin^{-1}\frac{0}{4}\right) \\
&= \frac{\sqrt{3}}{3}-\frac{\pi}{6}-0 \\
&= \boxed{\frac{\sqrt{3}}{3}-\frac{\pi}{6}}.
\end{aligned}$$

57. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3\tan\theta$, then $dx = 3\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{9+x^2}\,dx &= \int \frac{(3\tan\theta)^2}{9+(3\tan\theta)^2}\left(3\sec^2\theta\right)d\theta \\
&= 27\int \frac{\tan^2\theta}{9\tan^2\theta+9}\sec^2\theta\,d\theta \\
&= 3\int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\,d\theta \\
&= 3\int \tan^2\theta\,d\theta \\
&= 3\int \left(\sec^2\theta-1\right)d\theta \\
&= 3\left(\tan\theta-\theta\right)+C.
\end{aligned}$$

Since $\tan\theta = \frac{x}{3}$, and $\theta = \tan^{-1}\frac{x}{3}$. So

$$\begin{aligned}\int \frac{x^2}{9+x^2}\,dx &= 3\left(\frac{x}{3} - \tan^{-1}\frac{x}{3}\right) + C \\ &= x - 3\tan^{-1}\frac{x}{3} + C.\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_0^3 \frac{x^2}{9+x^2}\,dx &= \left[x - 3\tan^{-1}\frac{x}{3}\right]_0^3 \\ &= \left(3 - 3\tan^{-1}\frac{3}{3}\right) - \left(0 - 3\tan^{-1}\frac{0}{3}\right) \\ &= \left(3 - \frac{3}{4}\pi\right) - (0) \\ &= \boxed{3 - \frac{3}{4}\pi}.\end{aligned}$$

Applications and Extensions

59. The volume is given by $\int_0^1 \pi\left(\frac{1}{x^2+4}\right)^2 dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \pi\left(\frac{1}{x^2+4}\right)^2 dx &= \pi\int\left(\frac{1}{(2\tan\theta)^2+4}\right)^2 \left(2\sec^2\theta\right)d\theta \\ &= 2\pi\int\left(\frac{1}{4\sec^2\theta}\right)^2 \sec^2\theta\,d\theta \\ &= \frac{\pi}{8}\int\frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\ &= \frac{\pi}{8}\int\cos^2\theta\,d\theta \\ &= \frac{\pi}{8}\int\frac{1+\cos(2\theta)}{2}\,d\theta \\ &= \frac{\pi}{16}\int(1+\cos(2\theta))\,d\theta \\ &= \frac{\pi}{16}\left(\theta + \frac{1}{2}\sin(2\theta)\right) \\ &= \frac{\pi}{16}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{\pi}{16}\left(\theta + \frac{\tan\theta}{\sec^2\theta}\right) + C.\end{aligned}$$

Since $\tan\theta = \frac{x}{2}$, $\sec\theta = \sqrt{\tan^2\theta + 1} = \sqrt{\left(\frac{x}{2}\right)^2 + 1} = \frac{1}{2}\sqrt{x^2+4}$, and $\theta = \tan^{-1}\frac{x}{2}$. We have

$$\begin{aligned}\int \pi\left(\frac{1}{x^2+4}\right)^2 dx &= \frac{\pi}{16}\left(\tan^{-1}\frac{x}{2} + \frac{\frac{x}{2}}{\left(\frac{1}{2}\sqrt{x^2+4}\right)^2}\right) + C \\ &= \frac{\pi}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C.\end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_0^1 \pi\left(\frac{1}{x^2+4}\right)^2 dx &= \left[\frac{\pi}{16}\left(\tan^{-1}\frac{x}{2}+\frac{2x}{x^2+4}\right)\right]_0^1 \\ &= \left(\frac{\pi}{16}\left(\tan^{-1}\frac{1}{2}+\frac{2(1)}{(1)^2+4}\right)\right)-\left(\frac{\pi}{16}\left(\tan^{-1}\frac{0}{2}+\frac{2(0)}{(0)^2+4}\right)\right) \\ &= \frac{\pi}{16}\left(\tan^{-1}\frac{1}{2}+\frac{2}{5}\right)-0 \\ &= \boxed{\tfrac{\pi}{40}+\tfrac{\pi}{16}\tan^{-1}\tfrac{1}{2}}.\end{aligned}$$

61. The average value of $f(x)=\frac{1}{\sqrt{9-4x^2}}$ over the interval $\left[0,\frac{1}{2}\right]$ is

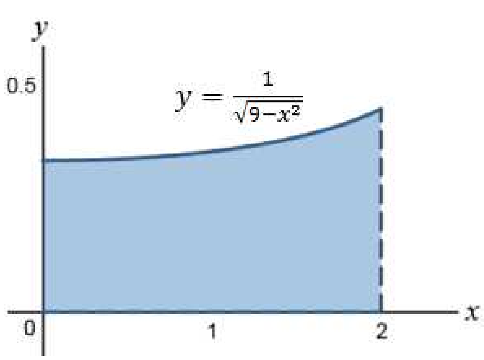
$$\bar{f}=\frac{1}{\frac{1}{2}-0}\int_0^{1/2}\frac{1}{\sqrt{9-4x^2}}\,dx=2\int_0^{1/2}\frac{1}{\sqrt{9-4x^2}}\,dx.$$

Use the substitution $x=\frac{3}{2}\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dx=\frac{3}{2}\cos\theta\,d\theta$ and $\sqrt{9-4x^2}=\sqrt{9-9\sin^2\theta}=3\sqrt{1-\sin^2\theta}=3\sqrt{\cos^2\theta}=3\cos\theta$ since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Since $\theta=\sin^{-1}\frac{2x}{3}$, the lower limit of integration becomes $\theta=\sin^{-1}0=0$ and the upper limit of integration becomes $u=\sin^{-1}\frac{1}{3}$.
The integral becomes

$$\begin{aligned}2\int_0^{1/2}\frac{1}{\sqrt{9-4x^2}}\,dx &= 2\int_0^{\sin^{-1}\frac{1}{3}}\frac{1}{3\cos\theta}\left(\frac{3}{2}\cos\theta\,d\theta\right) \\ &= \int_0^{\sin^{-1}\frac{1}{3}}d\theta \\ &= [\theta]_0^{\sin^{-1}\frac{1}{3}} \\ &= \boxed{\sin^{-1}\tfrac{1}{3}\approx 0.340}.\end{aligned}$$

63. Since $y=\frac{1}{\sqrt{9-x^2}}$ is nonnegative on the interval $[0,2]$, $A=\int_0^2\frac{1}{\sqrt{9-x^2}}\,dx$ is the area under the graph of $y=\frac{1}{\sqrt{9-x^2}}$ from $x=0$ to $x=2$.



Use the substitution $x=3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dx=3\cos\theta\,d\theta$ and $\sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=3\sqrt{1-\sin^2\theta}=3\sqrt{\cos^2\theta}=3\cos\theta$ since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Since $\theta = \sin^{-1}\frac{x}{3}$, the lower limit of integration becomes $\theta = \sin^{-1} 0 = 0$ and the upper limit of integration becomes $u = \sin^{-1}\frac{2}{3}$.

The integral becomes

$$\begin{aligned} A = \int_0^2 \frac{1}{\sqrt{9-x^2}}\,dx &= \int_0^{\sin^{-1}\frac{2}{3}} \frac{1}{3\cos\theta}(3\cos\theta\,d\theta) \\ &= \int_0^{\sin^{-1}\frac{2}{3}} d\theta \\ &= [\theta]_0^{\sin^{-1}\frac{2}{3}} \\ &= \boxed{\sin^{-1}\tfrac{2}{3}}. \end{aligned}$$

65. The area under the graph is given by $\int_3^5 \frac{x^2}{\sqrt{x^2-1}}\,dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2-1}}\,dx &= \int \frac{(\sec\theta)^2}{\sqrt{(\sec\theta)^2-1}}(\sec\theta\tan\theta\,)\,d\theta \\ &= \int \frac{\sec^2\theta}{\tan\theta}\sec\theta\tan\theta\;d\theta \\ &= \int \sec^3\theta\,d\theta \\ &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C. \end{aligned}$$

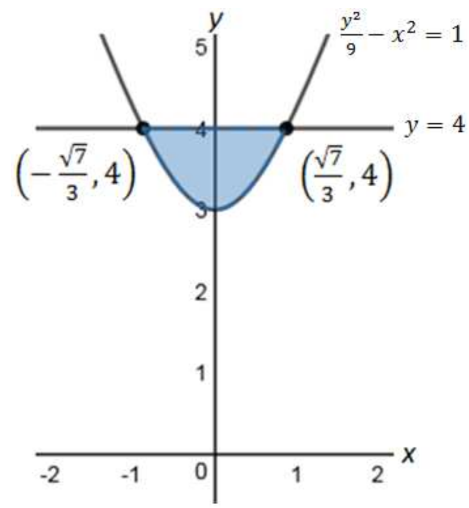
Since $\sec\theta = x$, $\tan\theta = \sqrt{x^2-1}$. So

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C.$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_3^5 \frac{x^2}{\sqrt{x^2-1}}\,dx &= \left(\frac{1}{2}5\sqrt{5^2-1} + \frac{1}{2}\ln\left|5+\sqrt{5^2-1}\right|\right) - \left(\frac{1}{2}3\sqrt{3^2-1} + \frac{1}{2}\ln\left|3+\sqrt{3^2-1}\right|\right) \\ &= \frac{1}{2}\ln\left(2\sqrt{6}+5\right) + 5\sqrt{6} - \left(\frac{1}{2}\ln\left(2\sqrt{2}+3\right) + 3\sqrt{2}\right) \\ &= \boxed{\tfrac{1}{2}\ln\left(2\sqrt{6}+5\right) - \tfrac{1}{2}\ln\left(2\sqrt{2}+3\right) - 3\sqrt{2} + 5\sqrt{6}}. \end{aligned}$$

67. (a) The region enclosed by the hyperbola $\frac{y^2}{9} - x^2 = 1$ and the line $y = 4$ is pictured below.



Solving the equation of the hyperbola for x, $x = \pm\sqrt{\frac{y^2}{9} - 1} = \pm\frac{1}{3}\sqrt{y^2 - 9}$.
Using symmetry, the area enclosed by the hyperbola and the line $y = 4$ is given by

$$A = 2\int_3^4 \frac{1}{3}\sqrt{y^2 - 9}\, dy = \frac{2}{3}\int_3^4 \sqrt{y^2 - 9}\, dy.$$

Use the substitution $y = 3\sec\theta$ $(0 \le \theta < \frac{\pi}{2},\ \pi \le \theta < \frac{3\pi}{2})$ to evaluate $\int \sqrt{y^2 - 9}\, dy$.
Then $dy = 3\tan\theta\sec\theta\, d\theta$ and

$$\sqrt{y^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta$$

since $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.
Then

$$\begin{aligned}\int \sqrt{y^2 - 9}\, dy &= \int (3\tan\theta)(3\tan\theta\sec\theta\, d\theta)\\ &= 9\int \tan^2\theta\sec\theta\, d\theta\\ &= 9\int \left(\sec^2\theta - 1\right)\sec\theta\, d\theta\\ &= 9\int \left(\sec^3\theta - \sec\theta\right)d\theta.\end{aligned}$$

Evaluate $\int \sec^3\theta\, d\theta$ using integration by parts.
Let $u = \sec\theta$ and $dv = \sec^2\theta\, d\theta$.
Then $du = \tan\theta\sec\theta\, dx$ and $v = \int \sec^2\theta\, d\theta = \tan\theta$.
Now

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int (\tan\theta)(\tan\theta\sec\theta\, dx) = \tan\theta\sec\theta - \int \tan^2\theta\sec\theta\, d\theta.$$

Use the identity $\tan^2\theta = \sec^2\theta - 1$.

$$\int \sec^3\theta\, d\theta = \tan\theta\sec\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\, d\theta = \tan\theta\sec\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta.$$

Add $\int \sec^3\theta\, d\theta$ to both sides.

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \int \sec\theta\, d\theta = \tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|$$

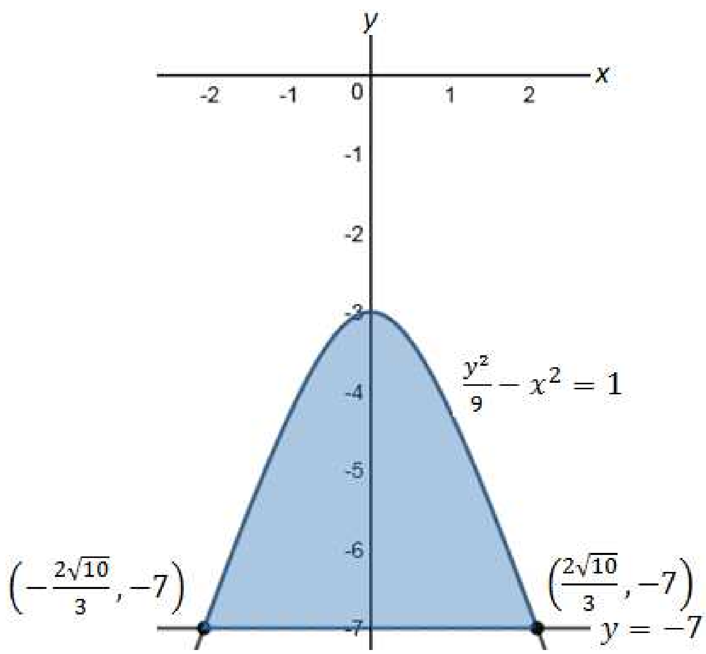
and $\int \sec^3\theta\, d\theta = \frac{1}{2}[\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|] + C$.
Therefore,

$$\begin{aligned}
\int \sqrt{y^2-9}\, dy &= 9\int \left(\sec^3\theta - \sec\theta\right) d\theta \\
&= 9\left\{\frac{1}{2}[\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|] - \ln|\sec\theta+\tan\theta|\right\} + C \\
&= \frac{9}{2}[\tan\theta\sec\theta - \ln|\sec\theta+\tan\theta|] + C \\
&= \frac{9}{2}\left[\frac{\sqrt{y^2-9}}{3}\cdot\frac{y}{3} - \ln\left|\frac{y}{3} + \frac{\sqrt{y^2-9}}{3}\right|\right] + C \\
&= \frac{1}{2}\left[y\sqrt{y^2-9} - 9\ln\left|y+\sqrt{y^2-9}\right| - 9\ln 3\right] + C \\
&= \frac{1}{2}\left[y\sqrt{y^2-9} - 9\ln\left|y+\sqrt{y^2-9}\right|\right] + C \text{ after combining the constants.}
\end{aligned}$$

So,

$$\begin{aligned}
A &= \frac{2}{3}\int_3^4 \sqrt{y^2-9}\, dy \\
&= \frac{2}{3}\cdot\frac{1}{2}\left[y\sqrt{y^2-9} - 9\ln\left|y+\sqrt{y^2-9}\right|\right]_3^4 \\
&= \frac{1}{3}\left\{\left[4\sqrt{7} - 9\ln\left(4+\sqrt{7}\right)\right] - [0 - 9\ln(3)]\right\} \\
&= \frac{1}{3}\left[4\sqrt{7} + 9\ln\left(\frac{3}{4+\sqrt{7}}\right)\right] \\
&= \boxed{\tfrac{4\sqrt{7}}{3} - 3\ln\tfrac{4+\sqrt{7}}{3}}.
\end{aligned}$$

(b) The region enclosed by the hyperbola $\frac{y^2}{9} - x^2 = 1$ and the line $y = -7$ is pictured below.



Solving the equation of the hyperbola for x, $x = \pm\sqrt{\frac{y^2}{9} - 1} = \pm\frac{1}{3}\sqrt{y^2 - 9}$.
Using symmetry, the area enclosed by the hyperbola and the line $y = -7$ is given by

$$A = 2\int_{-7}^{-3} \frac{1}{3}\sqrt{y^2 - 9}\, dy = \frac{2}{3}\int_{-7}^{-3} \sqrt{y^2 - 9}\, dy.$$

Using the results from part (a),
So,

$$\begin{aligned}
A &= \frac{2}{3}\int_{-7}^{-3} \sqrt{y^2 - 9}\, dy \\
&= \frac{2}{3}\cdot\frac{1}{2}\left[y\sqrt{y^2 - 9} - 9\ln\left|y + \sqrt{y^2 - 9}\right|\right]_{-7}^{-3} \\
&= \frac{1}{3}\left\{[0 - 9\ln|-3|] - \left[-7\sqrt{40} - 9\ln\left|-7 + \sqrt{40}\right|\right]\right\} \\
&= \frac{1}{3}\left[7\sqrt{40} + 9\ln\frac{7 - \sqrt{40}}{3}\right] \\
&= \frac{1}{3}\left[7\cdot 2\sqrt{10} + 9\ln\left(\frac{7 - \sqrt{40}}{3}\cdot\frac{7 + \sqrt{40}}{7 + \sqrt{40}}\right)\right] \\
&= \frac{1}{3}\left[14\sqrt{10} + 9\ln\frac{9}{3(7 + \sqrt{40})}\right] \\
&= \frac{14\sqrt{10}}{3} + 3\ln\frac{3}{7 + \sqrt{40}} \\
&= \boxed{\tfrac{14\sqrt{10}}{3} - 3\ln\tfrac{7 + 2\sqrt{10}}{3}}.
\end{aligned}$$

69. The length of the graph is given by $L = \int_0^5 \sqrt{1 + \left[\frac{d}{dx}(5x - x^2)\right]^2}\, dx = \int_0^5 \sqrt{1 + (5 - 2x)^2}\, dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $u = 5 - 2x$, so $du = -2\, dx$ and $dx = -\frac{1}{2}\, du$. We substitute and obtain

$$\begin{aligned}
\int \sqrt{1 + (5 - 2x)^2}\, dx &= \int \sqrt{1 + u^2}\left(-\frac{1}{2}\, du\right) \\
&= -\frac{1}{2}\int \sqrt{1 + u^2}\, du.
\end{aligned}$$

Let $u = \tan\theta$, then $du = \sec^2\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}
\int \sqrt{1 + (5 - 2x)^2}\, dx &= -\frac{1}{2}\int \sqrt{1 + \tan^2\theta}\sec^2\theta\, d\theta \\
&= -\frac{1}{2}\int \sec^3\theta\, d\theta \\
&= -\frac{1}{4}\sec\theta\tan\theta - \frac{1}{4}\ln|\sec\theta + \tan\theta| + C.
\end{aligned}$$

Since $\tan\theta = u$, $\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + u^2}$, and also substituting $u = 5 - 2x$, we obtain

$$\begin{aligned}
\int \sqrt{1 + (5 - 2x)^2}\, dx &= -\frac{1}{4}\sqrt{1 + u^2}\,(u) - \frac{1}{4}\ln\left|\sqrt{1 + u^2} + u\right| + C \\
&= -\frac{1}{4}(5 - 2x)\sqrt{1 + (5 - 2x)^2} - \frac{1}{4}\ln\left|\sqrt{1 + (5 - 2x)^2} + (5 - 2x)\right| + C.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^5 \sqrt{1+(5-2x)^2}\,dx &= \left[-\frac{1}{4}(5-2x)\sqrt{1+(5-2x)^2}-\frac{1}{4}\ln\left|\sqrt{1+(5-2x)^2}+(5-2x)\right|\right]_0^5 \\
&= \left(-\frac{(5-2(5))\sqrt{1+(5-2(5))^2}}{4}-\frac{\ln\left|\sqrt{1+(5-2(5))^2}+5-2(5)\right|}{4}\right) \\
&\quad -\left(-\frac{(5-2(0))\sqrt{1+(5-2(0))^2}}{4}-\frac{\ln\left|\sqrt{1+(5-2(0))^2}+5-2(0)\right|}{4}\right) \\
&= \frac{5}{4}\sqrt{26}-\frac{1}{4}\ln\left(\sqrt{26}-5\right)-\left(-\frac{1}{4}\ln\left(\sqrt{26}+5\right)-\frac{5}{4}\sqrt{26}\right) \\
&= \boxed{\tfrac{1}{4}\ln\left(\sqrt{26}+5\right)-\tfrac{1}{4}\ln\left(\sqrt{26}-5\right)+\tfrac{5}{2}\sqrt{26}}.
\end{aligned}$$

71. (a) We equate $x^2 = 4 - y^2$ with $x^2 = 1-(y-2)^2$ to get the points of intersection. We obtain

$$\begin{aligned}
4-y^2 &= 1-(y-2)^2 \\
4-y^2 &= -y^2+4y-3 \\
4 &= 4y-3 \\
\frac{7}{4} &= y.
\end{aligned}$$

So $x^2 = 4-\left(\frac{7}{4}\right)^2 = \frac{15}{16}$, and $x = \pm\frac{\sqrt{15}}{4}$. The area is given by

$$A = \int_{-\sqrt{15}/4}^{\sqrt{15}/4}\left[\sqrt{4-x^2}-\left(2-\sqrt{1-x^2}\right)\right]\,dx = 2\int_0^{\sqrt{15}/4}\left[\sqrt{4-x^2}-\left(2-\sqrt{1-x^2}\right)\right]\,dx.$$

We expand to obtain

$$\begin{aligned}
A &= 2\int_0^{\sqrt{15}/4}\sqrt{4-x^2}\,dx+2\int_0^{\sqrt{15}/4}\sqrt{1-x^2}\,dx-2\int_0^{\sqrt{15}/4}2\,dx \\
&= 2\int_0^{\sqrt{15}/4}\sqrt{4-x^2}\,dx+2\int_0^{\sqrt{15}/4}\sqrt{1-x^2}\,dx-\sqrt{15}.
\end{aligned}$$

For the first integral, we evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
2\int\sqrt{4-x^2}\,dx &= 2\int\sqrt{4-(2\sin\theta)^2}(2\cos\theta)\,d\theta \\
&= 4\int\sqrt{4-4\sin^2\theta}\cos\theta\,d\theta \\
&= 8\int\cos^2\theta\,d\theta \\
&= 8\int\frac{1+\cos(2\theta)}{2}\,d\theta \\
&= 4\int(1+\cos(2\theta))\,d\theta \\
&= 4\left(\theta+\frac{1}{2}\sin 2\theta\right)+C \\
&= 4\theta+4\cos\theta\sin\theta+C.
\end{aligned}$$

Since $\sin\theta = \frac{x}{2}$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{2}\right)^2} = \frac{1}{2}\sqrt{4-x^2}$, and $\theta = \sin^{-1}\frac{x}{2}$. We obtain

$$\begin{aligned} 2\int \sqrt{4-x^2}\,dx &= 4\sin^{-1}\frac{x}{2} + 4\left(\frac{1}{2}\sqrt{4-x^2}\right)\left(\frac{x}{2}\right) + C \\ &= 4\sin^{-1}\frac{x}{2} + x\sqrt{4-x^2} + C. \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} 2\int_0^{\sqrt{15}/4} \sqrt{4-x^2}\,dx &= \left[4\sin^{-1}\frac{x}{2} + x\sqrt{4-x^2}\right]_0^{\sqrt{15}/4} \\ &= \left(4\sin^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{2}\right) + \frac{\sqrt{15}}{4}\sqrt{4-\left(\frac{\sqrt{15}}{4}\right)^2}\right) - \left(4\sin^{-1}\frac{0}{2} + 0\sqrt{4-0^2}\right) \\ &= \left(4\sin^{-1}\left(\frac{\sqrt{15}}{8}\right) + \frac{7\sqrt{15}}{16}\right) - 0 \\ &= 4\sin^{-1}\left(\frac{\sqrt{15}}{8}\right) + \frac{7\sqrt{15}}{16}. \end{aligned}$$

For the second integral, we evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = \sin\theta$, then $dx = \cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned} 2\int \sqrt{1-x^2}\,dx &= 2\int \sqrt{1-\sin^2\theta}(\cos\theta)\,d\theta \\ &= 2\int \cos^2\theta\,d\theta \\ &= 2\int \frac{1+\cos(2\theta)}{2}\,d\theta \\ &= \int (1+\cos(2\theta))\,d\theta \\ &= \theta + \frac{1}{2}\sin 2\theta + C \\ &= \theta + \cos\theta\sin\theta + C. \end{aligned}$$

Since $\sin\theta = x$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$, and $\theta = \sin^{-1} x$. We obtain

$$\begin{aligned} 2\int \sqrt{1-x^2}\,dx &= \sin^{-1} x + \sqrt{1-x^2}\,(x) + C \\ &= \sin^{-1} x + x\sqrt{1-x^2} + C. \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} 2\int_0^{\sqrt{15}/4} \sqrt{1-x^2}\,dx &= \left[\sin^{-1} x + x\sqrt{1-x^2}\right]_0^{\sqrt{15}/4} \\ &= \left(\sin^{-1}\frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{4}\sqrt{1-\left(\frac{\sqrt{15}}{4}\right)^2}\right) - \left(\sin^{-1} 0 + 0\sqrt{1-0^2}\right) \\ &= \left(\sin^{-1}\frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{16}\right) - 0 \\ &= \sin^{-1}\frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{16}. \end{aligned}$$

So we obtain $A = 2\int_0^{\sqrt{15}/4} \sqrt{4-x^2}\,dx + 2\int_0^{\sqrt{15}/4} \sqrt{1-x^2}\,dx - \sqrt{15} = \left(4\sin^{-1}\frac{\sqrt{15}}{8} + \frac{7\sqrt{15}}{16}\right) + \left(\sin^{-1}\frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{16}\right) - \sqrt{15} = \boxed{\sin^{-1}\frac{\sqrt{15}}{4} + 4\sin^{-1}\frac{\sqrt{15}}{8} - \frac{1}{2}\sqrt{15}}.$

(b) Here we subtract the area of the smaller lune from the area of the upper circle. We obtain

$$\begin{aligned} A &= \pi(1)^2 - \left(\sin^{-1}\frac{\sqrt{15}}{4} + 4\sin^{-1}\frac{\sqrt{15}}{8} - \frac{1}{2}\sqrt{15}\right) \\ &= \boxed{\pi - \sin^{-1}\frac{\sqrt{15}}{4} - 4\sin^{-1}\frac{\sqrt{15}}{8} + \frac{1}{2}\sqrt{15}}. \end{aligned}$$

73. Let $u = x - 2$, then $du = dx$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{1-(x-2)^2}} &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1} u + C \\ &= \boxed{\sin^{-1}(x-2) + C}. \end{aligned}$$

75. To evaluate $\int \frac{dx}{\sqrt{(2x-1)^2-4}}$, use the substitution $u = 2x - 1$. Then $du = 2\,dx$, $dx = \frac{1}{2}\,du$, and

$$\int \frac{dx}{\sqrt{(2x-1)^2-4}} = \int \frac{1}{\sqrt{u^2-4}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int \frac{1}{\sqrt{u^2-4}}\,du.$$

Use the substitution $u = 2\sec\theta$ $(0 \le \theta < \frac{\pi}{2},\ \pi \le \theta < \frac{3\pi}{2})$ to evaluate $\int \frac{1}{\sqrt{u^2-4}}dx$.

Then $du = 2\tan\theta\sec\theta\,d\theta$ and

$$\sqrt{u^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta} = 2\tan\theta$$

since $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.

Therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{(2x-1)^2-4}} &= \frac{1}{2}\int \frac{1}{\sqrt{u^2-4}}\,du \\
&= \frac{1}{2}\int \frac{1}{2\tan\theta}\cdot 2\tan\theta\sec\theta\,d\theta \\
&= \frac{1}{2}\int \sec\theta\,d\theta \\
&= \frac{1}{2}\ln|\sec\theta+\tan\theta|+C \\
&= \frac{1}{2}\ln\left|\frac{u}{2}+\frac{\sqrt{u^2-4}}{2}\right|+C \\
&= \frac{1}{2}\ln\left|\frac{u+\sqrt{u^2-4}}{2}\right|+C \\
&= \frac{1}{2}\ln\left|\frac{(2x-1)+\sqrt{(2x-1)^2-4}}{2}\right|+C \text{ or, equivalently,} \\
& \boxed{\tfrac{1}{2}\ln\left|\tfrac{(2x-1)+\sqrt{4x^2-4x-3}}{2}\right|+C}.
\end{aligned}$$

77. Let $u=e^x$, then $du=e^x\,dx$. We substitute and obtain

$$\int e^x\sqrt{25-e^{2x}}\,dx=\int\sqrt{25-u^2}\,du.$$

Let $u=5\sin\theta$, then $du=5\cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int e^x\sqrt{25-e^{2x}}\,dx &= \int\sqrt{25-(5\sin\theta)^2}(5\cos\theta)\,d\theta \\
&= 5\int\sqrt{25-25\sin^2\theta}\cos\theta\,d\theta \\
&= 25\int\cos^2\theta\,d\theta \\
&= 25\int\frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{25}{2}\int(1+\cos(2\theta))\,d\theta \\
&= \frac{25}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C \\
&= \frac{25}{2}\theta+\frac{25}{2}\sin\theta\cos\theta+C.
\end{aligned}$$

We have $\theta=\sin^{-1}\left(\frac{u}{5}\right)$, so $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\left(\frac{u}{5}\right)^2}=\frac{1}{5}\sqrt{25-u^2}$. We obtain

$$\begin{aligned}
\int e^x\sqrt{25-e^{2x}}\,dx &= \frac{25}{2}\sin^{-1}\left(\frac{u}{5}\right)+\frac{25}{2}\left(\frac{u}{5}\right)\left(\frac{1}{5}\sqrt{25-u^2}\right)+C \\
&= \frac{25}{2}\sin^{-1}\left(\frac{u}{5}\right)+\frac{1}{2}u\sqrt{25-u^2}+C.
\end{aligned}$$

And since $u=e^x$, we have

$$\int e^x\sqrt{25-e^{2x}}\,dx=\boxed{\tfrac{25}{2}\sin^{-1}\left(\tfrac{e^x}{5}\right)+\tfrac{1}{2}e^x\sqrt{25-e^{2x}}+C.}$$

79. Let $u = \sin^{-1} x$ and $dv = x\,dx$. Then $du = \frac{1}{\sqrt{1-x^2}}\,dx$ and $v = \frac{1}{2}x^2$. We use integration by parts and obtain

$$\begin{aligned}\int x\sin^{-1} x\,dx &= \frac{1}{2}x^2\sin^{-1} x - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{\sqrt{1-x^2}}\right)dx \\ &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx.\end{aligned}$$

Let $x = \sin\theta$, then $dx = \cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int x\sin^{-1} x\,dx &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}(\cos\theta)\,d\theta \\ &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta \\ &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}\,d\theta \\ &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{4}\int (1-\cos(2\theta))\,d\theta \\ &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \\ &= \frac{1}{2}x^2\sin^{-1} x - \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C.\end{aligned}$$

We have $\theta = \sin^{-1} x$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. We obtain

$$\int x\sin^{-1} x\,dx = \boxed{\tfrac{1}{2}x^2\sin^{-1} x - \tfrac{1}{4}\sin^{-1} x + \tfrac{1}{4}x\sqrt{1-x^2} + C}.$$

81. (a) Let $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{x^2+a^2}\,dx &= \int \sqrt{(a\tan\theta)^2 + a^2}\,\left(a\sec^2\theta\right)d\theta \\ &= a^2\int \sec^3\theta\,d\theta \\ &= a^2\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) + C \\ &= \frac{a^2}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C.\end{aligned}$$

We have $\tan\theta = \frac{x}{a}$, and $\sec\theta = \sqrt{\tan^2\theta + 1} = \sqrt{\left(\frac{x}{a}\right)^2 + 1} = \frac{1}{a}\sqrt{x^2+a^2}$. We obtain

$$\begin{aligned}\int \sqrt{x^2+a^2}\,dx &= \frac{a^2}{2}\left(\frac{1}{a}\sqrt{x^2+a^2}\left(\frac{x}{a}\right) + \ln\left|\frac{1}{a}\sqrt{x^2+a^2} + \frac{x}{a}\right|\right) + C \\ &= \boxed{\tfrac{1}{2}a^2\ln\left|\tfrac{x+\sqrt{a^2+x^2}}{a}\right| + \tfrac{1}{2}x\sqrt{a^2+x^2} + C}.\end{aligned}$$

(b) Let $x = a\sinh\theta$, then $dx = a\cosh\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int\sqrt{x^2+a^2}\,dx &= \int\sqrt{(a\sinh\theta)^2+a^2}(a\cosh\theta)\,d\theta \\
&= a^2\int\sqrt{\sinh^2\theta+1}\cosh\theta\,d\theta \\
&= a^2\int\cosh^2\theta\,d\theta \\
&= a^2\int\left(\frac{e^\theta+e^{-\theta}}{2}\right)^2 d\theta \\
&= \frac{a^2}{4}\int\left(e^{2\theta}+2+e^{-2\theta}\right)d\theta \\
&= \frac{a^2}{4}\left(\frac{1}{2}e^{2\theta}+2\theta-\frac{1}{2}e^{-2\theta}\right)+C \\
&= \frac{1}{2}a^2\theta+\frac{a^2}{8}\left(e^{2\theta}-e^{-2\theta}\right)+C \\
&= \frac{1}{2}a^2\theta+\frac{a^2}{2}\left(\frac{e^\theta-e^{-\theta}}{2}\right)\left(\frac{e^\theta+e^{-\theta}}{2}\right)+C \\
&= \frac{1}{2}a^2\theta+\frac{a^2}{2}\sinh\theta\cosh\theta+C.
\end{aligned}$$

Since $\sinh\theta = \frac{x}{a}$, $\cosh\theta = \sqrt{\sinh^2\theta+1} = \sqrt{(x/a)^2+1} = \frac{1}{a}\sqrt{x^2+a^2}$. We obtain

$$\begin{aligned}
\int\sqrt{x^2+a^2}\,dx &= \frac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right)+\frac{a^2}{2}\left(\frac{x}{a}\right)\left(\frac{1}{a}\sqrt{x^2+a^2}\right)+C \\
&= \boxed{\tfrac{1}{2}a^2\sinh^{-1}\left(\tfrac{x}{a}\right)+\tfrac{1}{2}x\sqrt{a^2+x^2}+C}.
\end{aligned}$$

83. Let $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}
\int\frac{dx}{\sqrt{a^2-x^2}} &= \int\frac{a\cos\theta}{\sqrt{a^2-(a\sin\theta)^2}}\,d\theta \\
&= \int\frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\,d\theta \\
&= \frac{1}{a}\int\frac{a\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta \\
&= \int\frac{\cos\theta}{\cos\theta}\,d\theta \\
&= \theta+C \\
&= \boxed{\sin^{-1}\left(\tfrac{x}{a}\right)+C}.
\end{aligned}$$

85. Let $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{(a\sec\theta)^2-a^2}}\,d\theta \\ &= \frac{1}{a}\int \frac{\tan\theta}{\sqrt{\sec^2\theta-1}}\,d\theta \\ &= \frac{1}{a}\int \frac{\tan\theta}{\tan\theta}\,d\theta \\ &= \frac{1}{a}\theta + C \\ &= \boxed{\tfrac{1}{a}\sec^{-1}\left(\tfrac{x}{a}\right)+C}.\end{aligned}$$

87. Let $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \frac{1}{\sqrt{(a\tan\theta)^2+a^2}}\left(a\sec^2\theta\right)d\theta \\ &= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta| + C_1.\end{aligned}$$

We have $\tan\theta = \frac{x}{a}$, and $\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\left(\frac{x}{a}\right)^2+1} = \frac{1}{a}\sqrt{x^2+a^2}$. We obtain

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \ln\left|\frac{1}{a}\sqrt{x^2+a^2}+\frac{x}{a}\right| + C_1 \\ &= \ln\left|x+\sqrt{x^2+a^2}\right| - \ln|a| + C_1 \\ &= \boxed{\ln\left(x+\sqrt{x^2+a^2}\right)+C},\end{aligned}$$

where $C = -\ln|a| + C_1$.

Challenge Problems

89. Let $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{x^2-a^2}\,dx &= \int \sqrt{(a\sec\theta)^2-a^2}(a\sec\theta\tan\theta)\,d\theta \\ &= a^2\int (\tan\theta)(\sec\theta\tan\theta)\,d\theta \\ &= a^2\int \tan^2\theta\sec\theta\,d\theta \\ &= a^2\int \left(\sec^2\theta-1\right)\sec\theta\,d\theta \\ &= a^2\int \left(\sec^3\theta-\sec\theta\right)d\theta \\ &= a^2\left[\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right] + C \\ &= a^2\left[\frac{1}{2}\sec\theta\tan\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|\right] + C.\end{aligned}$$

We have $\sec\theta = \frac{x}{a}$, and $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1} = \frac{1}{a}\sqrt{x^2-a^2}$. We obtain

$$\begin{aligned}\int \sqrt{x^2-a^2}\,dx &= a^2\left[\frac{1}{2}\left(\frac{x}{a}\right)\left(\frac{1}{a}\sqrt{x^2-a^2}\right) - \frac{1}{2}\ln\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2-a^2}\right|\right] + C \\ &= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{1}{2}a^2\ln|a| + C \\ &= \boxed{\tfrac{1}{2}x\sqrt{x^2-a^2} - \tfrac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right| + C}.\end{aligned}$$

91. Let $u = \tan x$, then $du = \sec^2 x\,dx$. We substitute and obtain

$$\begin{aligned}\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6\tan x + 8}}dx &= \int \frac{du}{\sqrt{u^2-6u+8}} \\ &= \int \frac{du}{\sqrt{u^2-6u+9-1}} \\ &= \int \frac{du}{\sqrt{(u-3)^2-1}}.\end{aligned}$$

Let $y = u - 3$, then $dy = du$ We substitute and obtain

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6\tan x + 8}}dx = \int \frac{dy}{\sqrt{y^2-1}}.$$

Let $y = \sec\theta$, then $dy = \sec\theta\tan\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6\tan x + 8}}dx &= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}}\,d\theta \\ &= \int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

We have $\sec\theta = y$, and $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{y^2-1}$. We obtain

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6\tan x + 8}}dx = \ln\left|y + \sqrt{y^2-1}\right| + C.$$

Since $y = u - 3 = \tan x - 3$, we now have

$$\begin{aligned}\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6\tan x + 8}}dx &= \ln\left|\tan x - 3 + \sqrt{(\tan x - 3)^2 - 1}\right| + C \\ &= \boxed{\ln\left|\tan x - 3 + \sqrt{\tan^2 x - 6\tan x + 8}\right| + C}.\end{aligned}$$

AP® Practice Problems

1. To evaluate $\int_0^3 \sqrt{9-x^2}\,dx$, use the substitution $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

   Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta$ since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

The lower limit of integration becomes $\theta=\sin^{-1}0=0$, and the upper limit of integration becomes $\theta=\sin^{-1}\frac{3}{3}=\frac{\pi}{2}$.

The integral becomes

$$\begin{aligned}\int_0^3\sqrt{9-x^2}\,dx &= \int_0^{\pi/2}(3\cos\theta)(3\cos\theta\,d\theta)=9\int_0^{\pi/2}\cos^2\theta\,d\theta\\ &=9\int_0^{\pi/2}\frac{1}{2}[1+\cos(2\theta)]d\theta=\frac{9}{2}\left[\theta+\frac{1}{2}\sin(2\theta)\right]_0^{\pi/2}=\frac{9}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]=\boxed{\tfrac{9}{4}\pi}.\end{aligned}$$

The answer is C.

3. To evaluate $\int_0^1\frac{1}{(x^2+1)^{3/2}}\,dx$, use the substitution $x=\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

   Then $dx=\sec^2\theta\,d\theta$ and $\sqrt{x^2+1}=\sqrt{\tan^2\theta+1}=\sqrt{\sec^2\theta}=\sec\theta$ since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

   The lower limit of integration becomes $\theta=\tan^{-1}0=0$, and the upper limit of integration becomes $\theta=\tan^{-1}1=\frac{\pi}{4}$.

   So, $\int_0^1\frac{1}{(x^2+1)^{3/2}}\,dx=\int_0^{\pi/4}\frac{1}{\sec^3\theta}\sec^2\theta\,d\theta=\int_0^{\pi/4}\frac{1}{\sec\theta}=\int_0^{\pi/4}\cos\theta=[\sin\theta]_0^{\pi/4}=\boxed{\frac{\sqrt{2}}{2}}$.

   The answer is B.

5. Use the substitution $x=\frac{1}{3}\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, to evaluate $\int\frac{dx}{\sqrt{1+9x^2}}$.

   Then $dx=\frac{1}{3}\sec^2\theta\,d\theta$ and $\sqrt{1+9x^2}=\sqrt{1+9\left(\frac{1}{3}\tan\theta\right)^2}=\sqrt{1+\tan^2\theta}=\sqrt{\sec^2\theta}=\sec\theta$ since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

   Then $\int\frac{dx}{\sqrt{1+9x^2}}\,dx=\int\frac{\frac{1}{3}\sec^2\theta\,d\theta}{\sec\theta}=\frac{1}{3}\int\sec\theta\,d\theta=\frac{1}{3}\ln|\sec\theta+\tan\theta|+C=\boxed{\frac{1}{3}\ln\left|\sqrt{1+9x^2}+3x\right|+C}$.

   The answer is D.

7.4 Integrands Containing ax^2+bx+c

Skill Building

1. We complete the square: $x^2+4x+5=(x+2)^2+1$. Then we let $u=x+2$, so $du=dx$, and we obtain

$$\begin{aligned}\int\frac{dx}{x^2+4x+5} &= \int\frac{dx}{(x+2)^2+1}\\ &= \int\frac{du}{u^2+1}\\ &= \tan^{-1}u+C\\ &= \boxed{\tan^{-1}(x+2)+C}.\end{aligned}$$

3. We complete the square: $x^2+4x+8=(x+2)^2+4$. Then we let $u=x+2$, so $du=dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{x^2+4x+8} &= \int \frac{dx}{(x+2)^2+4}\\ &= \int \frac{du}{u^2+2^2}\\ &= \frac{1}{2}\tan^{-1}\frac{u}{2}+C\\ &= \boxed{\tfrac{1}{2}\tan^{-1}\left(\tfrac{x+2}{2}\right)+C}.\end{aligned}$$

5. We complete the square: $3+2x+2x^2=2\left(x+\frac{1}{2}\right)^2+\frac{5}{2}$. Then we let $u=x+\frac{1}{2}$ to obtain

$$\begin{aligned}\int \frac{2\,dx}{3+2x+2x^2} &= \int \frac{2\,dx}{2\left(x+\frac{1}{2}\right)^2+\frac{5}{2}}\\ &= 2\int \frac{du}{2u^2+\frac{5}{2}}\\ &= \int \frac{du}{u^2+\left(\frac{\sqrt{5}}{2}\right)^2}\\ &= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}/2}\right)+C\\ &= \frac{2\sqrt{5}}{5}\tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{5}}\right)+C\\ &= \boxed{\tfrac{2\sqrt{5}}{5}\tan^{-1}\left(\tfrac{2x+1}{\sqrt{5}}\right)+C}.\end{aligned}$$

7. We force the the derivative of the denominator to appear in the numerator. We then complete the square with $2x^2+2x+3=2\left(x+\frac{1}{2}\right)^2+\frac{5}{2}$, and let $u=x+\frac{1}{2}$, so $du=dx$. We obtain

$$\begin{aligned}\int \frac{x\,dx}{2x^2+2x+3} &= \int \frac{\frac{1}{4}(4x+2)-\frac{1}{2}}{2x^2+2x+3}\,dx\\ &= \frac{1}{4}\int \frac{4x+2}{2x^2+2x+3}\,dx-\frac{1}{2}\int \frac{dx}{2x^2+2x+3}\\ &= \frac{1}{4}\ln\left|2x^2+2x+3\right|-\frac{1}{2}\int \frac{dx}{2x^2+2x+3}\\ &= \frac{1}{4}\ln\left(2x^2+2x+3\right)-\frac{1}{2}\int \frac{dx}{2\left(x+\frac{1}{2}\right)^2+\frac{5}{2}}\\ &= \frac{1}{4}\ln\left(2x^2+2x+3\right)-\frac{1}{4}\int \frac{du}{u^2+\left(\frac{\sqrt{5}}{2}\right)^2}\\ &= \frac{1}{4}\ln\left(2x^2+2x+3\right)-\frac{1}{4}\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}/2}\right)+C\\ &= \frac{1}{4}\ln\left(2x^2+2x+3\right)-\frac{\sqrt{5}}{10}\tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{5}}\right)+C\\ &= \boxed{\tfrac{1}{4}\ln\left(2x^2+2x+3\right)-\tfrac{\sqrt{5}}{10}\tan^{-1}\left(\tfrac{2x+1}{\sqrt{5}}\right)+C}.\end{aligned}$$

9. We complete the square: $8 + 2x - x^2 = 9 - (x-1)^2$. Then we let $u = x - 1$ to obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{8+2x-x^2}} &= \int \frac{dx}{\sqrt{9-(x-1)^2}} \\ &= \int \frac{du}{\sqrt{3^2-u^2}} \\ &= \sin^{-1}\left(\frac{u}{3}\right) + C \\ &= \boxed{\sin^{-1}\left(\tfrac{x-1}{3}\right) + C}.\end{aligned}$$

11. We complete the square: $4x - x^2 = 4 - (x-2)^2$. Then we let $u = x - 2$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{4x-x^2}} &= \int \frac{dx}{\sqrt{4-(x-2)^2}} \\ &= \int \frac{du}{\sqrt{2^2-u^2}} \\ &= \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= \boxed{\sin^{-1}\left(\tfrac{x-2}{2}\right) + C}.\end{aligned}$$

13. We complete the square: $x^2 + 2x + 2 = (x+1)^2 + 1$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}} &= \int \frac{dx}{(x+1)\sqrt{(x+1)^2+1}} \\ &= \int \frac{du}{u\sqrt{u^2+1^2}} \\ &= \ln\left|\frac{\sqrt{u^2+1}-1}{u}\right| + C \\ &= \ln\left|\frac{\sqrt{(x+1)^2+1}-1}{x+1}\right| + C \\ &= \boxed{\ln\left|\tfrac{\sqrt{x^2+2x+2}-1}{x+1}\right| + C}.\end{aligned}$$

15. We complete the square: $24 - 2x - x^2 = 25 - (x+1)^2$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{24-2x-x^2}} &= \int \frac{dx}{\sqrt{25-(x+1)^2}} \\ &= \int \frac{du}{\sqrt{5^2-u^2}} \\ &= \sin^{-1}\left(\frac{u}{5}\right) + C \\ &= \boxed{\sin^{-1}\left(\tfrac{x+1}{5}\right) + C}.\end{aligned}$$

17. We force the the derivative of the denominator to appear in the numerator, and obtain two integrals.

$$\begin{aligned}\int \frac{x-5}{\sqrt{x^2-2x+5}}\,dx &= \int \frac{\frac{1}{2}(2x-2)-4}{\sqrt{x^2-2x+5}}\,dx \\ &= \frac{1}{2}\int \frac{(2x-2)}{\sqrt{x^2-2x+5}}\,dx - 4\int \frac{dx}{\sqrt{(x-1)^2+4}}.\end{aligned}$$

Now for the first integral let $w = x^2-2x+5$, so $dw = (2x-2)\ dx$. In the second integral, complete the square with $x^2-2x+5 = (x-1)^2+4$. ; and let $u = x-1$, so $du = dx$. We obtain

$$\begin{aligned} &= \frac{1}{2}\int w^{-1/2}\,dw - 4\int \frac{du}{\sqrt{u^2+2^2}} \\ &= \frac{1}{2}(2\sqrt{w}) - 4\ln\left(u+\sqrt{u^2+4}\right) + C \\ &= \sqrt{x^2-2x+5} - 4\ln\left(x-1+\sqrt{(x-1)^2+4}\right) + C \\ &= \boxed{\sqrt{x^2-2x+5} - 4\ln\left(x-1+\sqrt{x^2-2x+5}\right) + C}.\end{aligned}$$

19. We complete the square: $x^2-2x+5 = (x-1)^2+4$, and let $u = x-1$ to obtain

$$\begin{aligned}\int_1^3 \frac{dx}{\sqrt{x^2-2x+5}} &= \int_1^3 \frac{dx}{\sqrt{(x-1)^2+4}} \\ &= \int_0^2 \frac{du}{\sqrt{u^2+2^2}} \\ &= \left[\ln\left(u+\sqrt{u^2+4}\right)\right]_{u=0}^{u=2} \\ &= \ln\left(2+\sqrt{2^2+4}\right) - \ln\left(0+\sqrt{0^2+4}\right) \\ &= \ln\left(2\sqrt{2}+2\right) - \ln 2 \\ &= \boxed{\ln\left(\sqrt{2}+1\right)}.\end{aligned}$$

21. We complete the square: $e^{2x}+e^x+1 = \left(e^x+\frac{1}{2}\right)^2+\frac{3}{4}$. Then we let $u = e^x+\frac{1}{2}$ to obtain

$$\begin{aligned}\int \frac{e^x\,dx}{\sqrt{e^{2x}+e^x+1}} &= \int \frac{e^x\,dx}{\sqrt{\left(e^x+\frac{1}{2}\right)^2+\frac{3}{4}}} \\ &= \int \frac{du}{\sqrt{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}} \\ &= \ln\left(u+\sqrt{u^2+\frac{3}{4}}\right) + C \\ &= \ln\left(e^x+\frac{1}{2}+\sqrt{\left(e^x+\frac{1}{2}\right)^2+\frac{3}{4}}\right) + C \\ &= \boxed{\ln\left(e^x+\tfrac{1}{2}+\sqrt{e^{2x}+e^x+1}\right) + C}.\end{aligned}$$

23. We force the the derivative of the quadratic to appear in the numerator.

$$\begin{aligned}\int \frac{2x\,-3}{\sqrt{4x-x^2-3}}\,dx &= \int \frac{-(4-2x)+1}{\sqrt{4x-x^2-3}}\,dx \\ &= -\int \frac{4-2x}{\sqrt{4x-x^2-3}}\,dx+\int \frac{1}{\sqrt{1-(x-2)^2}}\,dx.\end{aligned}$$

For the first integral let $w=4x-x^2-3$ so $dw=(4-2x)\,dx$. In the second integral, complete the square with $4x-x^2-3=1-(x-2)^2$, and let $u=x-2$, so $du=dx$. We obtain

$$\begin{aligned}\int \frac{2x\,-3}{\sqrt{4x-x^2-3}}\,dx &= -\int w^{-1/2}\,dw+\int \frac{du}{\sqrt{1-u^2}} \\ &= -\left(2\sqrt{w}\right)+\sin^{-1}u+C \\ &= \boxed{-2\sqrt{4x-x^2-3}+\sin^{-1}(x-2)+C}.\end{aligned}$$

25. We complete the square: $x^2-2x+10=(x-1)^2+9$. Then we let $u=x-1$, so $du=dx$, and obtain

$$\begin{aligned}\int \frac{dx}{\left(x^2-2x+10\right)^{3/2}} &= \int \frac{dx}{\left((x-1)^2+9\right)^{3/2}} \\ &= \int \frac{du}{\left(u^2+9\right)^{3/2}}.\end{aligned}$$

Let $u=3\tan\theta$, then $du=3\sec^2\theta\,d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{\left(x^2-2x+10\right)^{3/2}} &= \int \frac{3\sec^2\theta}{\left((3\tan\theta)^2+9\right)^{3/2}}\,d\theta \\ &= 3\int \frac{\sec^2\theta}{\left(9\tan^2\theta+9\right)^{3/2}}\,d\theta \\ &= \frac{1}{9}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta \\ &= \frac{1}{9}\int \cos\theta\,d\theta \\ &= \frac{1}{9}\sin\theta+C.\end{aligned}$$

We have $\tan\theta=\frac{u}{3}$, so $\cot\theta=\frac{3}{u}$, and $\csc\theta=\sqrt{1+\cot^2\theta}=\sqrt{1+\left(\frac{3}{u}\right)^2}=\frac{1}{u}\sqrt{u^2+9}$. So $\sin\theta=\frac{u}{\sqrt{u^2+9}}$ and we obtain

$$\begin{aligned}\int \frac{dx}{\left(x^2-2x+10\right)^{3/2}} &= \frac{1}{9}\frac{u}{\sqrt{u^2+9}}+C \\ &= \frac{x-1}{9\sqrt{(x-1)^2+9}}+C \\ &= \boxed{\tfrac{x-1}{9\sqrt{x^2-2x+10}}+C}.\end{aligned}$$

27. We complete the square: $x^2+2x-3=(x+1)^2-4$. Then we let $u=x+1$, so $du=dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+2x-3}} &= \int \frac{dx}{\sqrt{(x+1)^2-4}} \\ &= \int \frac{du}{\sqrt{u^2-4}} \\ &= \ln\left|u+\sqrt{u^2-4}\right|+C \\ &= \ln\left|x+1+\sqrt{(x+1)^2-4}\right|+C \\ &= \boxed{\ln\left|x+1+\sqrt{x^2+2x-3}\right|+C}.\end{aligned}$$

29. We complete the square: $5+4x-x^2=9-(x-2)^2$. Then we let $u=x-2$, so $du=dx$, and we obtain

$$\begin{aligned}\int \frac{\sqrt{5+4x-x^2}}{x-2}\,dx &= \int \frac{\sqrt{9-(x-2)^2}}{x-2}\,dx \\ &= \int \frac{\sqrt{9-u^2}}{u}\,du.\end{aligned}$$

Let $u=3\sin\theta$, so $du=3\cos\theta$, and we have

$$\begin{aligned}\int \frac{\sqrt{5+4x-x^2}}{x-2}\,dx &= \int \frac{\sqrt{9-(3\sin\theta)^2}}{3\sin\theta}(3\cos\theta)\,d\theta \\ &= 3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta \\ &= 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta \\ &= 3\int(\csc\theta-\sin\theta)\,d\theta \\ &= 3(-\ln|\csc\theta+\cot\theta|+\cos\theta)+C \\ &= 3\left(-\ln\left|\frac{3}{u}+\frac{\sqrt{9-u^2}}{u}\right|+\frac{1}{3}\sqrt{9-u^2}\right)+C \\ &= -3\ln\left|\frac{3}{x-2}+\frac{\sqrt{9-(x-2)^2}}{x-2}\right|+\sqrt{9-(x-2)^2}+C \\ &= \boxed{3\ln|x-2|-3\ln\left|3+\sqrt{5+4x-x^2}\right|+\sqrt{5+4x-x^2}+C}.\end{aligned}$$

31. We force the the derivative of the quadratic to appear in the numerator.

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2+2x-3}} &= \int \frac{\frac{1}{2}(2x+2)-1}{\sqrt{x^2+2x-3}}\,dx \\ &= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-3}}\,dx-\int \frac{dx}{\sqrt{(x+1)^2-4}}.\end{aligned}$$

In the first integral, let $w = x^2 + 2x - 3$, so $dw = (2x+2)\,dx$. In the second integral, complete the square with $x^2 + 2x - 3 = (x+1)^2 - 4$, and let $u = x + 1$, so $du = dx$. We obtain

$$\begin{aligned}
\int \frac{x\,dx}{\sqrt{x^2+2x-3}} &= \frac{1}{2}\int w^{-1/2}\,dw - \int \frac{du}{\sqrt{u^2-4}} \\
&= \frac{1}{2}\left(2\sqrt{w}\right) - \ln\left|u + \sqrt{u^2-4}\right| + C \\
&= \sqrt{x^2+2x-3} - \ln\left|x + 1 + \sqrt{(x+1)^2 - 4}\right| + C \\
&= \boxed{\sqrt{x^2+2x-3} - \ln\left|x+1+\sqrt{x^2+2x-3}\right| + C}.
\end{aligned}$$

Applications and Extensions

33. Let $u = x + h$, then

$$\int \frac{dx}{\sqrt{(x+h)^2 + k}} = \int \frac{du}{\sqrt{u^2+k}}.$$

We then apply the result of problem 87, section 7.3, to obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{(x+h)^2+k}} &= \ln\left[u + \sqrt{u^2+k}\right] + C \\
&= \boxed{\ln\left[\sqrt{(x+h)^2+k} + x + h\right] + C}.
\end{aligned}$$

Challenge Problems

35. We rewrite

$$\sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}} = \sqrt{\frac{(a+x)^2}{a^2-x^2}} = \frac{\sqrt{(a+x)^2}}{\sqrt{a^2-x^2}} = \frac{a+x}{\sqrt{a^2-x^2}}.$$

Then we obtain

$$\begin{aligned}
\int \sqrt{\frac{a+x}{a-x}}\,dx &= \int \frac{a+x}{\sqrt{a^2-x^2}}\,dx \\
&= \int \frac{a}{\sqrt{a^2-x^2}}\,dx + \int \frac{x}{\sqrt{a^2-x^2}}\,dx.
\end{aligned}$$

Let $u = a^2 - x^2$, so $du = -2x\,dx$. Then we have

$$\begin{aligned}
\int \sqrt{\frac{a+x}{a-x}}\,dx &= a\sin^{-1}\frac{x}{a} + \int \frac{-\frac{1}{2}du}{\sqrt{u}} \\
&= a\sin^{-1}\frac{x}{a} - \sqrt{u} + C \\
&= \boxed{a\sin^{-1}\tfrac{x}{a} - \sqrt{a^2-x^2} + C}.
\end{aligned}$$

AP® Practice Problems

1. The integrand of $\int \frac{1}{x^2+6x+13}\,dx$ contains the quadratic expression $x^2+6x+13$.

   So, we complete the square in the denominator.

$$\int \frac{1}{x^2+6x+13}\,dx = \int \frac{1}{(x^2+6x+9)+(13-9)}\,dx = \int \frac{1}{(x+3)^2+4}\,dx.$$

   Now use the substitution $u = x+3$. Then $du = dx$ and $\int \frac{1}{x^2+6x+13}\,dx = \int \frac{1}{(x+3)^2+4}\,dx = \int \frac{1}{u^2+4}\,du = \frac{1}{2}\tan^{-1}\frac{u}{2} + C = \boxed{\frac{1}{2}\tan^{-1}\frac{x+3}{2} + C}$.

   The answer is B.

7.5 Integration of Rational Functions Using Partial Fractions; The Logistic Model

Concepts and Vocabulary

1. (a) less than

3. True

Skill Building

5. Apply polynomial division to $\frac{x^2+1}{x+1}$.

$$\begin{array}{r|l} & \quad\quad\;\; x-1 \\ \hline (x+1) & x^2+0x+1 \\ & -(x^2+\;\;x) \\ \hline & \quad\;\; -x+1 \\ & \quad -(-x-1) \\ \hline & \quad\quad\quad\;\; 2 \end{array}$$

   The quotient is $x-1$ and the remainder is 2.
   Therefore,

$$\frac{x^2+1}{x+1} = \boxed{x-1+\frac{2}{(x+1)}}.$$

7. Apply polynomial division to $\frac{x^3+3x-4}{x-2}$.

$$\begin{array}{r|l} & \quad\quad x^2+2x+7 \\ \hline (x-2) & x^3+0x^2+3x-4 \\ & -(x^3-2x^2) \\ \hline & \quad\quad 2x^2+3x \\ & \quad -(2x^2-4x) \\ \hline & \quad\quad\quad\quad 7x-\;\;4 \\ & \quad\quad\quad -(7x-14) \\ \hline & \quad\quad\quad\quad\quad\;\; 10 \end{array}$$

   The quotient is x^2+2x+7 and the remainder is 10.
   Therefore,

$$\frac{x^3+3x-4}{x-2} = \boxed{x^2+2x+7+\frac{10}{x-2}}.$$

9. Apply polynomial division to $\frac{2x^3+3x^2-17x-27}{x^2-9}$.

$$\begin{array}{r|l} & \qquad\qquad 2x + 3 \\ \hline (x^2-9) & 2x^3 + 3x^2 - 17x - 27 \\ & -(2x^3 \qquad\quad - 18x) \\ \hline & \qquad 3x^2 + x - 27 \\ & \quad -(3x^2 \qquad - 27) \\ \hline & \qquad\qquad x \end{array}$$

The quotient is $2x+3$ and the remainder is x.
Therefore,

$$\frac{2x^3+3x^2-17x-27}{x^2-9} = \boxed{2x+3+\frac{x}{x^2-9}}.$$

11. Apply polynomial division to $\frac{x^4-1}{x(x+4)} = \frac{x^4-1}{x^2+4x}$.

$$\begin{array}{r|l} & \qquad\qquad x^2 - 4x + 16 \\ \hline (x^2+4x) & x^4 + 0x^3 + 0x^2 + 0x - 1 \\ & -(x^4 + 4x^3) \\ \hline & \quad -4x^3 + 0x^2 \\ & -(-4x^3 - 16x^2) \\ \hline & \qquad 16x^2 + 0x \\ & \quad -(16x^2 + 64x) \\ \hline & \qquad\qquad -64x - 1 \end{array}$$

The quotient is $x^2-4x+16$ and the remainder is $-64x-1$.
Therefore,

$$\frac{x^4-1}{x(x+4)} = \boxed{x^2-4x+16-\frac{64x+1}{x^2+4x}}.$$

13. Apply polynomial division to $\frac{2x^4+x^2-2}{x^2+4}$.

$$\begin{array}{r|l} & \qquad\qquad 2x^2 \qquad - 7 \\ \hline (x^2+4) & 2x^4 + 0x^3 + x^2 + 0x - 2 \\ & -(2x^4 \qquad + 8x^2) \\ \hline & \qquad -7x^2 \qquad - 2 \\ & \quad -(-7x^2 \qquad - 28) \\ \hline & \qquad\qquad\qquad 26 \end{array}$$

The quotient is $2x^2-7$ and the remainder is 26.
Therefore,

$$\frac{2x^4+x^2-2}{x^2+4} = \boxed{2x^2-7+\frac{26}{x^2+4}}.$$

15. We use long division to obtain

$$\begin{aligned} \int \frac{x^2+1}{x+1}\,dx &= \int \left(x-1+\frac{2}{x+1}\right)dx \\ &= \boxed{\tfrac{1}{2}x^2 - x + 2\ln|x+1| + C}. \end{aligned}$$

17. We use long division to obtain

$$\begin{aligned}\int \frac{x^3+3x-4}{x-2}\,dx &= \int \left(x^2+2x+7+\frac{10}{x-2}\right)dx \\ &= \boxed{\tfrac{1}{3}x^3+x^2+7x+10\ln|x-2|+C}.\end{aligned}$$

19. To evaluate $\int \frac{2x^3+3x^2-17x-27}{x^2-9}\,dx$, apply polynomial division to $\frac{2x^3+3x^2-17x-27}{x^2-9}$.

$$\begin{array}{r|l} & \quad\quad\quad\quad 2x + \ \ 3 \\ \hline (x^2-9) & \ 2x^3+3x^2-17x-27 \\ & -(2x^3 \qquad\quad -18x) \\ \hline & \qquad\quad 3x^2+ \quad x-27 \\ & \qquad -(3x^2 \qquad\quad -27) \\ \hline & \qquad\qquad\qquad x \end{array}$$

The quotient is $2x+3$ and the remainder is x.
So, $\frac{2x^3+3x^2-17x-27}{x^2-9} = 2x+3+\frac{x}{x^2-9}$.
Therefore,

$$\int \frac{3x^3-2x^2-3x+2}{x^2-1}\,dx = \int\left(2x+3+\frac{x}{x^2-9}\right)dx = \boxed{x^2+3x+\tfrac{1}{2}\ln\left|x^2-9\right|+C}.$$

21. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(x-2)(x+1)} &= \frac{A}{x-2}+\frac{B}{x+1} \\ 1 &= A(x+1)+B(x-2)\end{aligned}$$

When $x=-1$ we obtain $B=-\frac{1}{3}$, and when $x=2$ we have $A=\frac{1}{3}$. So we obtain

$$\begin{aligned}\int \frac{1}{(x-2)(x+1)}\,dx &= \int\left(\frac{1/3}{x-2}-\frac{1/3}{x+1}\right)dx \\ &= \boxed{\tfrac{1}{3}\ln|x-2|-\tfrac{1}{3}\ln|x+1|+C}.\end{aligned}$$

23. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{(x-1)(x-2)} &= \frac{A}{x-1}+\frac{B}{x-2} \\ x &= A(x-2)+B(x-1)\end{aligned}$$

When $x=2$ we obtain $B=2$, and when $x=1$ we have $A=-1$. So we obtain

$$\begin{aligned}\int \frac{x\,dx}{(x-1)(x-2)} &= \int\left(-\frac{1}{x-1}+\frac{2}{x-2}\right)dx \\ &= \boxed{-\ln|x-1|+2\ln|x-2|+C}.\end{aligned}$$

25. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{(3x-2)(2x+1)} &= \frac{A}{3x-2}+\frac{B}{2x+1} \\ x &= A(2x+1)+B(3x-2)\end{aligned}$$

When $x=-\frac{1}{2}$ we obtain $B=\frac{1}{7}$, and when $x=\frac{2}{3}$ we have $A=\frac{2}{7}$. So we obtain

$$\begin{aligned}\int \frac{x\,dx}{(3x-2)(2x+1)} &= \int\left(\frac{2/7}{3x-2}+\frac{1/7}{2x+1}\right)dx\\ &= \boxed{\tfrac{2}{21}\ln|3x-2|+\tfrac{1}{14}\ln|2x+1|+C}.\end{aligned}$$

27. We use partial fractions to obtain

$$\begin{aligned}\frac{x-3}{(x+2)(x+1)^2} &= \frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\\ x-3 &= A(x+1)^2+B(x+1)(x+2)+C(x+2)\end{aligned}$$

When $x=-1$ we obtain $C=-4$, and when $x=-2$ we have $A=-5$. When $x=0$ we obtain $B=5$. So we obtain

$$\begin{aligned}\int \frac{x-3}{(x+2)(x+1)^2}\,dx &= \int\left(\frac{-5}{x+2}+\frac{5}{x+1}+\frac{-4}{(x+1)^2}\right)dx\\ &= \int \frac{-5}{x+2}\,dx+\int\frac{5}{x+1}\,dx+\int\frac{-4}{(x+1)^2}\,dx\\ &= \boxed{-5\ln|x+2|+5\ln|x+1|+\tfrac{4}{x+1}+C}.\end{aligned}$$

29. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2}{(x-1)^2(x+1)} &= \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\\ x^2 &= A(x-1)^2+B(x-1)(x+1)+C(x+1)\end{aligned}$$

When $x=1$ we obtain $C=1/2$, and when $x=-1$ we have $A=1/4$. When $x=0$ we obtain $B=3/4$. So we obtain

$$\begin{aligned}\int \frac{x^2}{(x-1)^2(x+1)}\,dx &= \int\left(\frac{1/4}{x+1}+\frac{3/4}{x-1}+\frac{1/2}{(x-1)^2}\right)dx\\ &= \int \frac{1/4}{x+1}\,dx+\int\frac{3/4}{x-1}\,dx+\int\frac{1/2}{(x-1)^2}\,dx\\ &= \boxed{\tfrac{1}{4}\ln|x+1|+\tfrac{3}{4}\ln|x-1|-\tfrac{1}{2(x-1)}+C}.\end{aligned}$$

31. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x(x^2+1)} &= \frac{A}{x}+\frac{Bx+C}{x^2+1}\\ 1 &= A\left(x^2+1\right)+(Bx+C)x\end{aligned}$$

When $x=0$ we obtain $A=1$. So

$$\begin{aligned}1 &= (1)\left(x^2+1\right)+(Bx+C)x\\ 1 &= (B+1)x^2+Cx+1\end{aligned}$$

Equating coefficients, we obtain $B=-1$ and $C=0$. We now have

$$\begin{aligned}\int\frac{dx}{x(x^2+1)} &= \int\left(\frac{1}{x}+\frac{-x}{x^2+1}\right)dx\\ &= \int\frac{1}{x}\,dx+\int\frac{-x}{x^2+1}\,dx\\ &= \boxed{\ln|x|-\tfrac{1}{2}\ln\left(x^2+1\right)+C}.\end{aligned}$$

33. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2+2x+3}{(x+1)(x^2+2x+4)} &= \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+4}\\ x^2+2x+3 &= A\big(x^2+2x+4\big)+(Bx+C)(x+1)\end{aligned}$$

When $x=-1$ we obtain $A=2/3$. So

$$\begin{aligned}x^2+2x+3 &= A\big(x^2+2x+4\big)+(Bx+C)(x+1)\\ x^2+2x+3 &= \left(B+\frac{2}{3}\right)x^2+\left(B+C+\frac{4}{3}\right)x+\left(C+\frac{8}{3}\right)\end{aligned}$$

Equating coefficients, we obtain $B=1/3$ and $C=1/3$. We now have

$$\begin{aligned}\int\frac{x^2+2x+3}{(x+1)(x^2+2x+4)}\,dx &= \int\left(\frac{2/3}{x+1}+\frac{(1/3)x+1/3}{x^2+2x+4}\right)dx\\ &= \int\frac{2/3}{x+1}\,dx+\frac{1}{6}\int\frac{2x+2}{x^2+2x+4}\,dx\\ &= \boxed{\tfrac{2}{3}\ln|x+1|+\tfrac{1}{6}\ln\left(x^2+2x+4\right)+C}.\end{aligned}$$

35. We expand to obtain

$$\begin{aligned}\int\frac{2x+1}{\left(x^2+16\right)^2}\,dx &= \int\frac{2x}{\left(x^2+16\right)^2}\,dx+\int\frac{1}{\left(x^2+16\right)^2}\,dx\\ &= -\frac{1}{x^2+16}+\int\frac{1}{\left(x^2+16\right)^2}\,dx.\end{aligned}$$

Let $x=4\tan\theta$ to obtain

$$\begin{aligned}\int\frac{2x+1}{\left(x^2+16\right)^2}\,dx &= -\frac{1}{x^2+16}+\int\frac{4\sec^2\theta}{256\sec^4\theta}\,d\theta\\ &= -\frac{1}{x^2+16}+\frac{1}{64}\int\cos^2\theta\,d\theta\\ &= -\frac{1}{x^2+16}+\frac{1}{128}\int(1+\cos 2\theta)\,d\theta\\ &= -\frac{1}{x^2+16}+\frac{1}{128}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C\\ &= -\frac{1}{x^2+16}+\frac{1}{128}\theta+\frac{1}{128}\cos\theta\sin\theta+C\\ &= -\frac{1}{x^2+16}+\frac{1}{128}\tan^{-1}\left(\frac{x}{4}\right)+\frac{1}{128}\frac{4}{\sqrt{x^2+16}}\frac{x}{\sqrt{x^2+16}}+C\\ &= \boxed{-\tfrac{1}{x^2+16}+\tfrac{1}{128}\tan^{-1}\left(\tfrac{x}{4}\right)+\tfrac{x}{32(x^2+16)}+C}.\end{aligned}$$

37. We use partial fractions to obtain

$$\begin{aligned}\frac{x^3}{\left(x^2+16\right)^3} &= \frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}+\frac{Ex+F}{\left(x^2+16\right)^3}\\ x^3 &= (Ax+B)\big(x^2+16\big)^2+(Cx+D)\big(x^2+16\big)+Ex+F\\ x^3 &= Ax^5+Bx^4+(32A+C)x^3+(32B+D)x^2\\ &\qquad +(256A+16C+E)x+(256B+F+16D).\end{aligned}$$

Equating coefficients, we obtain $A = 0$, $B = 0$, $C = 1$, $D = 0$, $E = -16$, and $F = 0$. So

$$\begin{aligned}\int \frac{x^3\,dx}{(x^2+16)^3} &= \int \left(\frac{x}{(x^2+16)^2} + \frac{(-16)x}{(x^2+16)^3} \right) dx \\ &= \int \frac{x}{(x^2+16)^2}\,dx - 16\int \frac{x}{(x^2+16)^3}\,dx \\ &= \boxed{-\frac{1}{2(x^2+16)} + \frac{4}{(x^2+16)^2} + C}.\end{aligned}$$

39. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{x^2+2x-3} &= \frac{x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} \\ x &= A(x-1) + B(x+3)\end{aligned}$$

When $x = 1$ we obtain $B = 1/4$, and when $x = -3$ we have $A = 3/4$. So we obtain

$$\begin{aligned}\int \frac{x\,dx}{x^2+2x-3} &= \int \left(\frac{3/4}{x+3} + \frac{1/4}{x-1} \right) dx \\ &= \boxed{\tfrac{3}{4}\ln|x+3| + \tfrac{1}{4}\ln|x-1| + C}.\end{aligned}$$

41. We use partial fractions to obtain

$$\begin{aligned}\frac{10x^2+2x}{(x-1)^2(x^2+2)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2} \\ 10x^2+2x &= A(x-1)\left(x^2+2\right) + B\left(x^2+2\right) + (Cx+D)(x-1)^2\end{aligned}$$

When $x = 1$ we obtain $B = 4$. We expand and obtain

$$\begin{aligned}10x^2+2x &= A(x-1)\left(x^2+2\right) + 4\left(x^2+2\right) + (Cx+D)(x-1)^2 \\ 10x^2+2x &= (A+C)x^3 + (-A-2C+D+4)x^2 + (2A+C-2D)x + (-2A+D+8)\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}0 &= A+C \\ 10 &= -A-2C+D+4 \\ 2 &= 2A+C-2D \\ 0 &= -2A+D+8\end{aligned}$$

We obtain $A = -C$, so $10 = -(-C) - 2C + D + 4 = D - C + 4$ and $D = 6 + C$. Then $0 = -2A + D + 8 = -2(-C) + (6 + C) + 8 = 3C + 14$, and $C = -14/3$. We now obtain $A = 14/3$, and $D = 4/3$. So

$$\begin{aligned}\int \frac{10x^2+2x}{(x-1)^2(x^2+2)}\,dx &= \int \left(\frac{14/3}{x-1} + \frac{4}{(x-1)^2} + \frac{(-14/3)x+4/3}{x^2+2} \right) dx \\ &= \int \frac{14/3}{x-1}\,dx + \int \frac{4}{(x-1)^2}\,dx - \frac{14}{3}\int \frac{x}{x^2+2}\,dx + \frac{4}{3}\int \frac{dx}{x^2+2} \\ &= \boxed{\tfrac{14}{3}\ln|x-1| - \tfrac{4}{x-1} - \tfrac{7}{3}\ln\left(x^2+2\right) + \tfrac{2\sqrt{2}}{3}\tan^{-1}\left(\tfrac{\sqrt{2}x}{2}\right) + C}.\end{aligned}$$

43. We use partial fractions to obtain

$$\begin{aligned}\frac{7x+3}{x^3-2x^2-3x} &= \frac{7x+3}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3} \\ 7x+3 &= A(x+1)(x-3) + Bx(x-3) + Cx(x+1)\end{aligned}$$

When $x = 3$ we obtain $C = 2$, when $x = -1$, we get $B = -1$, and when $x = 0$, we have $A = -1$. So

$$\begin{aligned}\int \frac{7x+3}{x^3-2x^2-3x}\,dx &= \int \left(\frac{-1}{x} + \frac{-1}{x+1} + \frac{2}{x-3}\right) dx \\ &= \int \frac{-1}{x}\,dx + \int \frac{-1}{x+1}\,dx + \int \frac{2}{x-3}\,dx \\ &= \boxed{-\ln|x| - \ln|x+1| + 2\ln|x-3| + C}.\end{aligned}$$

45. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2}{(x-2)(x-1)^2} &= \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ x^2 &= A(x-1)^2 + B(x-2)(x-1) + C(x-2).\end{aligned}$$

When $x = 1$ we obtain $C = -1$, and when $x = 2$, we have $A = 4$. So

$$\begin{aligned}x^2 &= 4(x-1)^2 + B(x-2)(x-1) + (-1)(x-2) \\ x^2 &= (B+4)x^2 + (-3B-9)x + (2B+6).\end{aligned}$$

Equating coefficients, we obtain $B = -3$. We now obtain

$$\begin{aligned}\int \frac{x^2}{(x-2)(x-1)^2}\,dx &= \int \left(\frac{4}{x-2} + \frac{-3}{x-1} + \frac{-1}{(x-1)^2}\right) dx \\ &= \int \frac{4}{x-2}\,dx + \int \frac{-3}{x-1}\,dx + \int \frac{-1}{(x-1)^2}\,dx \\ &= \boxed{4\ln|x-2| - 3\ln|x-1| + \tfrac{1}{x-1} + C}.\end{aligned}$$

47. We use partial fractions to obtain

$$\begin{aligned}\frac{2x+1}{x^3-1} &= \frac{2x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \\ 2x+1 &= A\left(x^2+x+1\right) + (Bx+C)(x-1).\end{aligned}$$

When $x = 1$ we obtain $A = 1$. So

$$\begin{aligned}2x+1 &= (1)\left(x^2+x+1\right) + (Bx+C)(x-1) \\ 2x+1 &= (B+1)x^2 + (C-B+1)x + (1-C).\end{aligned}$$

Equating coefficients, we obtain $B = -1$, and $C = 0$. We now obtain

$$\begin{aligned}\int \frac{2x+1}{x^3-1}\,dx &= \int \left(\frac{1}{x-1} + \frac{-x}{x^2+x+1}\right) dx \\ &= \int \frac{1}{x-1}\,dx + \int \frac{-\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1}\,dx \\ &= \int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx \\ &= \boxed{\ln|x-1| - \tfrac{1}{2}\ln\left(x^2+x+1\right) + \tfrac{\sqrt{3}}{3}\tan^{-1}\left(\tfrac{\sqrt{3}(2x+1)}{3}\right) + C}.\end{aligned}$$

49. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x^2-9} &= \frac{1}{(x-3)(x+3)} = \frac{A}{x-3}+\frac{B}{x+3}\\ 1 &= A(x+3)+B(x-3).\end{aligned}$$

When $x=-3$ we obtain $B=-\frac{1}{6}$, and when $x=3$ we have $A=\frac{1}{6}$. So we obtain

$$\begin{aligned}\int_0^1 \frac{dx}{x^2-9} &= \int_0^1\left(\frac{1/6}{x-3}+\frac{-1/6}{x+3}\right)dx\\ &= \left[\frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|\right]_0^1\\ &= \frac{1}{6}\ln|1-3|-\frac{1}{6}\ln|1+3|-\left(\frac{1}{6}\ln|0-3|-\frac{1}{6}\ln|0+3|\right)\\ &= \boxed{-\tfrac{1}{6}\ln 2}.\end{aligned}$$

51. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{16-x^2} &= \frac{-1}{(x-4)(x+4)} = \frac{A}{x-4}+\frac{B}{x+4}\\ -1 &= A(x+4)+B(x-4).\end{aligned}$$

When $x=-4$ we obtain $B=\frac{1}{8}$, and when $x=4$ we have $A=\frac{-1}{8}$. So we obtain

$$\begin{aligned}\int_{-2}^3 \frac{dx}{16-x^2} &= \int_{-2}^3\left(\frac{-1/8}{x-4}+\frac{1/8}{x+4}\right)dx\\ &= \left[\frac{-1}{8}\ln|x-4|+\frac{1}{8}\ln|x+4|\right]_{-2}^3\\ &= \frac{-1}{8}\ln|3-4|+\frac{1}{8}\ln|3+4|-\left(\frac{-1}{8}\ln|-2-4|+\frac{1}{8}\ln|-2+4|\right)\\ &= \frac{1}{8}\ln 7-\frac{1}{8}\ln 2+\frac{1}{8}\ln 6 = \boxed{\tfrac{1}{8}\ln 21}.\end{aligned}$$

53. Let $x=\sin\theta$, and substitute to obtain

$$\int \frac{\cos\theta}{\sin^2\theta+\sin\theta-6}\,d\theta = \int\frac{dx}{x^2+x-6} = \int\frac{dx}{(x-2)(x+3)}.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(x-2)(x+3)} &= \frac{A}{x-2}+\frac{B}{x+3}\\ 1 &= A(x+3)+B(x-2)\end{aligned}$$

When $x=-3$ we obtain $B=-\frac{1}{5}$, and when $x=2$ we have $A=\frac{1}{5}$. So we obtain

$$\begin{aligned}\int\frac{\cos\theta}{\sin^2\theta+\sin\theta-6}\,d\theta &= \int\left(\frac{1/5}{x-2}+\frac{-1/5}{x+3}\right)dx\\ &= \frac{1}{5}\ln|x-2|-\frac{1}{5}\ln|x+3|+C\\ &= \boxed{\tfrac{1}{5}\ln(2-\sin\theta)-\tfrac{1}{5}\ln(\sin\theta+3)+C}.\end{aligned}$$

55. Let $x = \cos\theta$, and substitute to obtain

$$\int \frac{\sin\theta}{\cos^3\theta + \cos\theta}\, d\theta = \int \frac{-dx}{x^3 + x} = \int \frac{-dx}{x(x^2+1)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{-1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx + C}{x^2+1} \\ -1 &= A\left(x^2+1\right) + (Bx + C)x \end{aligned}$$

When $x = 0$ we obtain $A = -1$. So

$$\begin{aligned} -1 &= (-1)\left(x^2+1\right) + (Bx + C)x \\ -1 &= (B-1)x^2 + Cx - 1 \end{aligned}$$

Equating coefficients, we obtain $B = 1$ and $C = 0$. We now have

$$\begin{aligned} \int \frac{\sin\theta}{\cos^3\theta + \cos\theta}\, d\theta &= \int \left(\frac{-1}{x} + \frac{x}{x^2+1}\right) dx \\ &= \int \frac{-1}{x}\, dx + \int \frac{x}{x^2+1}\, dx \\ &= -\ln|x| + \frac{1}{2}\ln\left(x^2+1\right) + C \\ &= \boxed{-\ln|\cos\theta| + \tfrac{1}{2}\ln\left(\cos^2\theta + 1\right) + C}. \end{aligned}$$

57. Let $x = e^t$, and substitute to obtain

$$\int \frac{e^t}{e^{2t} + e^t - 2}\, dt = \int \frac{dx}{x^2 + x - 2} = \int \frac{dx}{(x-1)(x+2)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{1}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2} \\ 1 &= A(x+2) + B(x-1) \end{aligned}$$

When $x = -2$ we obtain $B = -\frac{1}{3}$, and when $x = 1$ we have $A = \frac{1}{3}$. So we obtain

$$\begin{aligned} \int \frac{e^t}{e^{2t} + e^t - 2}\, dt &= \int \left(\frac{1/3}{x-1} + \frac{-1/3}{x+2}\right) dx \\ &= \int \frac{1/3}{x-1}\, dx + \int \frac{-1/3}{x+2}\, dx \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C \\ &= \boxed{\tfrac{1}{3}\ln\left|e^t - 1\right| - \tfrac{1}{3}\ln\left(e^t + 2\right) + C}. \end{aligned}$$

59. Let $y = e^x$, and substitute to obtain

$$\int \frac{e^x}{e^{2x} - 1}\, dx = \int \frac{dy}{y^2 - 1} = \int \frac{dy}{(y-1)(y+1)}.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(y-1)(y+1)} &= \frac{A}{y-1}+\frac{B}{y+1}\\ 1 &= A(y+1)+B(y-1)\end{aligned}$$

When $y=-1$ we obtain $B=-\frac{1}{2}$, and when $y=1$ we have $A=\frac{1}{2}$. So we obtain

$$\begin{aligned}\int \frac{e^x}{e^{2x}-1}\,dx &= \int \left(\frac{1/2}{y-1}+\frac{-1/2}{y+1}\right)dy\\ &= \int \frac{1/2}{y-1}\,dy+\int \frac{-1/2}{y+1}\,dy\\ &= \frac{1}{2}\ln|y-1|-\frac{1}{2}\ln|y+1|+C\\ &= \boxed{\tfrac{1}{2}\ln|e^x-1|-\tfrac{1}{2}\ln(e^x+1)+C}.\end{aligned}$$

61. We rewrite the integral as

$$\int \frac{dt}{e^{2t}+1}=\int \frac{e^t}{e^t(e^{2t}+1)}\,dt.$$

Let $x=e^t$, and substitute to obtain

$$\int \frac{dt}{e^{2t}+1}=\int \frac{dx}{x(x^2+1)}.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x(x^2+1)} &= \frac{A}{x}+\frac{Bx+C}{x^2+1}\\ 1 &= A\left(x^2+1\right)+(Bx+C)x\end{aligned}$$

When $x=0$ we obtain $A=1$. So

$$\begin{aligned}1 &= (1)\left(x^2+1\right)+(Bx+C)x\\ 1 &= (B+1)x^2+Cx+1\end{aligned}$$

Equating coefficients, we obtain $B=-1$ and $C=0$. We now have

$$\begin{aligned}\int \frac{dt}{e^{2t}+1} &= \int \left(\frac{1}{x}+\frac{-x}{x^2+1}\right)dx\\ &= \int \frac{1}{x}\,dx+\int \frac{-x}{x^2+1}\,dx\\ &= \ln|x|-\frac{1}{2}\ln\left(x^2+1\right)+C\\ &= \ln\left(e^t\right)-\frac{1}{2}\ln\left(e^{2t}+1\right)+C\\ &= \boxed{t-\tfrac{1}{2}\ln\left(e^{2t}+1\right)+C}.\end{aligned}$$

63. Let $u = \sin x - 1$, and substitute to obtain

$$\begin{aligned}\int \frac{\sin x \cos x}{(\sin x - 1)^2}\,dx &= \int \frac{u+1}{u^2}\,du \\ &= \int \left(\frac{1}{u} + u^{-2}\right) du \\ &= \ln|u| - \frac{1}{u} + C \\ &= \boxed{\ln|\sin x - 1| - \tfrac{1}{\sin x - 1} + C}.\end{aligned}$$

65. Let $u = \sin x$, and substitute to obtain

$$\int \frac{\cos x}{(\sin^2 x + 9)^2}\,dx = \int \frac{du}{(u^2+9)^2}.$$

Let $u = 3\tan\theta$, and substitute to obtain

$$\begin{aligned}\int \frac{\cos x}{(\sin^2 x + 9)^2}\,dx &= \int \frac{3\sec^2\theta\,d\theta}{\left((3\tan\theta)^2 + 9\right)^2} \\ &= \frac{1}{27}\int \cos^2\theta\,d\theta \\ &= \frac{1}{27}\int \frac{1+\cos(2\theta)}{2}\,d\theta \\ &= \frac{1}{54}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\ &= \frac{1}{54}\tan^{-1}\frac{u}{3} + \frac{1}{54}\sin\theta\cos\theta + C \\ &= \frac{1}{54}\tan^{-1}\frac{u}{3} + \frac{1}{54}\frac{\tan\theta}{\sec^2\theta} + C.\end{aligned}$$

Since $\tan\theta = \frac{u}{3}$, $\sec\theta = \sqrt{\tan^2\theta + 1} = \sqrt{\left(\frac{u}{3}\right)^2 + 1} = \frac{1}{3}\sqrt{u^2+9}$, and we have

$$\begin{aligned}\int \frac{\cos x}{(\sin^2 x + 9)^2}\,dx &= \frac{1}{54}\tan^{-1}\frac{u}{3} + \frac{1}{54}\frac{\frac{u}{3}}{\left(\frac{1}{3}\sqrt{u^2+9}\right)^2} + C \\ &= \frac{1}{54}\tan^{-1}\frac{u}{3} + \frac{1}{18}\frac{u}{u^2+9} + C \\ &= \boxed{\tfrac{1}{54}\tan^{-1}\tfrac{\sin x}{3} + \tfrac{1}{18}\tfrac{\sin x}{\sin^2 x + 9} + C}.\end{aligned}$$

67. The area under the graph is given by

$$A = \int_3^5 \frac{4}{x^2-4}\,dx = \int_3^5 \frac{4}{(x+2)(x-2)}\,dx.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{4}{(x+2)(x-2)} &= \frac{A}{x+2} + \frac{B}{x-2} \\ 4 &= A(x-2) + B(x+2)\end{aligned}$$

When $x = 2$ we obtain $B = 1$, and when $x = -2$ we have $A = -1$. So we obtain

$$\begin{aligned} A &= \int_3^5 \left(\frac{-1}{x+2} + \frac{1}{x-2} \right) dx \\ &= \left[-\ln(x+2) + \ln(x-2) \right]_3^5 \\ &= -\ln(5+2) + \ln(5-2) - (-\ln(3+2) + \ln(3-2)) \\ &= \ln 3 + \ln 5 - \ln 7 = \boxed{\ln \tfrac{15}{7}}. \end{aligned}$$

69. The area under the graph is given by

$$A = \int_0^2 \frac{8}{x^3+1}\, dx = \int_0^2 \frac{8}{(x+1)(x^2-x+1)}\, dx.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{8}{(x+1)(x^2-x+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\ 8 &= A\big(x^2-x+1\big) + (Bx+C)(x+1). \end{aligned}$$

When $x = -1$ we obtain $A = 8/3$. We expand and obtain

$$\begin{aligned} 8 &= (8/3)\big(x^2-x+1\big) + (Bx+C)(x+1) \\ 8 &= \left(B + \frac{8}{3}\right)x^2 + \left(B + C - \frac{8}{3}\right)x + \left(C + \frac{8}{3}\right). \end{aligned}$$

Equating coefficients, we have $B = -8/3$ and $C = 16/3$. We now have

$$\begin{aligned} A &= \int_0^2 \left(\frac{8/3}{x+1} + \frac{(-8/3)x + 16/3}{x^2-x+1} \right) dx \\ &= \left[\frac{8}{3} \ln(x+1) \right]_0^2 + \int_0^2 \frac{(-4/3)(2x-1)x + 4}{x^2-x+1}\, dx \\ &= \frac{8}{3}\ln(2+1) - \frac{8}{3}\ln(0+1) - \frac{4}{3}\int_0^2 \frac{2x-1}{x^2-x+1}\, dx + 4\int_0^2 \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{8}{3}\ln 3 - \frac{4}{3}\left[\ln\left(x^2-x+1\right)\right]_0^2 + 4\left[\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(2x-1)}{3}\right)\right]_0^2 \\ &= \frac{8}{3}\ln 3 - \frac{4}{3}\left(\ln\left(2^2-2+1\right) - \ln\left(0^2-0+1\right)\right) \\ &\quad +4\left[\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(2(2)-1)}{3}\right) - \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(2(0)-1)}{3}\right)\right] \\ &= \boxed{\tfrac{4}{3}\ln 3 + \tfrac{4\sqrt{3}}{3}\pi}. \end{aligned}$$

71. The arc length of the graph of $y = \ln x$ from $x = 1$ to $x = e$ is given by

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^e \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx.$$

To evaluate $\int \frac{\sqrt{x^2+1}}{x}\, dx$, use the substitution $x = \tan\theta$.

Since $x > 0,\ 0 < \theta < \frac{\pi}{2}$.

Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = \sec\theta$ since $0 < \theta < \frac{\pi}{2}$.

The lower limit of integration becomes $\theta = \tan^{-1} 1 = \frac{\pi}{4}$, and the upper limit of integration becomes $\theta = \tan^{-1} e$.

So,

$$\begin{aligned}\int \frac{\sqrt{x^2+1}}{x}\, dx &= \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta \\ &= \int (\tan\theta\sec\theta + \csc\theta)\, d\theta \\ &= \sec\theta - \ln|\cot\theta + \csc\theta| + C.\end{aligned}$$

Since $0 < \theta < \frac{\pi}{2}$ and $\tan\theta = x, \sec\theta = \sqrt{x^2+1}, \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$,
and $\csc\theta = \sqrt{\cot^2\theta + 1} = \sqrt{\frac{1}{x^2}+1} = \frac{\sqrt{x^2+1}}{x}$.

So,

$$\begin{aligned}\int \frac{\sqrt{x^2+1}}{x}\, dx &= \sec\theta - \ln|\cot\theta + \csc\theta| + C \\ &= \sqrt{x^2+1} - \ln\left|\frac{1}{x} + \frac{\sqrt{x^2+1}}{x}\right| + C \\ &= \sqrt{x^2+1} - \ln\left|\frac{1+\sqrt{x^2+1}}{x}\right| + C.\end{aligned}$$

Therefore,

$$\begin{aligned}L = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx &= \left[\sqrt{x^2+1} - \ln\left|\frac{1+\sqrt{x^2+1}}{x}\right|\right]_1^e \\ &= \left[\sqrt{e^2+1} - \ln\left(\frac{1+\sqrt{e^2+1}}{e}\right)\right] - \left[\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right] \\ &= \left[\sqrt{e^2+1} - \ln\left(1+\sqrt{e^2+1}\right) + \ln(e)\right] - \left[\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right] \\ &= \boxed{1 - \sqrt{2} + \sqrt{e^2+1} + \ln\left(\frac{1+\sqrt{2}}{1+\sqrt{e^2+1}}\right)}.\end{aligned}$$

73. (a) Since the rate of change of the number of people with the flu P with respect to time t (in days) is proportional to the product of P and $50 - P$, the differential equation can be written as $\frac{dP}{dt} = CP(50-P)$. The differential equation can also be written in the form $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ where $M = 50$ is the carrying capacity and $k = 0.15$ is the maximum growth rate. The differential equation becomes $\boxed{\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{50}\right) \text{ with } P(0) = 1}$.

(b) The initial number of people with the flu is $P_0 = 1$. The solution to the differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is $P(t) = \frac{M}{1+ae^{-kt}}$ where $k = 0.15$, $M = 50$, and $a = \frac{M-P_0}{P_0} = \frac{50-1}{1} = 49$. Therefore, $\boxed{P(t) = \frac{50}{1+49e^{-0.15t}}}$.

(c) Find t so that $P(t) = 25$.

$$\begin{aligned}\frac{50}{1+49e^{-0.15t}} &= 25\\ e^{-0.15t} &= \frac{\frac{50}{25}-1}{49}\\ -0.15t &= \ln\left(\frac{\frac{50}{25}-1}{49}\right)\\ t &= -\frac{1}{0.15}\ln\left(\frac{1}{49}\right) = \frac{1}{0.15}\ln 49 \approx \boxed{29.945 \text{ days}}.\end{aligned}$$

(d) Find t so that $P(t) = 0.80(50)$.

$$\begin{aligned}\frac{50}{1+49e^{-0.15t}} &= 40\\ e^{-0.15t} &= \frac{\frac{50}{40}-1}{49}\\ -0.15t &= \ln\left(\frac{0.25}{49}\right)\\ t &= -\frac{1}{0.15}\ln\left(\frac{0.25}{49}\right) \approx \boxed{35.187 \text{ days}}\end{aligned}$$

75. (a) We test possible rational roots, and obtain that the zeros of q are $\boxed{-4, -2, \text{ and } 3}$.

(b) From part (a), we obtain $q(x) = (x+4)(x+2)(x-3)$.

(c) We use partial fractions to obtain

$$\begin{aligned}\frac{3x-7}{x^3+3x^2-10x-24} &= \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x+4}\\ 3x-7 &= A(x-3)(x+4) + B(x+2)(x+4) + C(x+2)(x-3).\end{aligned}$$

When $x = -4$ we obtain $C = -19/14$. When $x = 3$ we get $B = \frac{2}{35}$. And when $x = -2$, we have $A = \frac{13}{10}$. We now obtain

$$\begin{aligned}\int \frac{3x-7}{x^3+3x^2-10x-24}\,dx &= \int \left(\frac{13/10}{x+2} + \frac{2/35}{x-3} + \frac{-19/14}{x+4}\right) dx\\ &= \int \frac{13/10}{x+2}\,dx + \int \frac{2/35}{x-3}\,dx + \int \frac{-19/14}{x+4}\,dx\\ &= \boxed{\tfrac{13}{10}\ln|x+2| + \tfrac{2}{35}\ln|x-3| - \tfrac{19}{14}\ln|x+4| + C}.\end{aligned}$$

Challenge Problems

77. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, so $dx = 2u\,du$. We obtain

$$\int \frac{dx}{\sqrt{x}+2} = \int \frac{2u}{u+2}\,du.$$

Let $y = u + 2$, then we have

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}+2} &= 2\int \frac{y-2}{y}\,du \\
&= 2\int \left(1-\frac{2}{y}\right)dy \\
&= 2(y - 2\ln|y|) + C \\
&= 2(u+2) - 4\ln|u+2| + C \\
&= \boxed{2\sqrt{x} - 4\ln\left(\sqrt{x}+2\right) + C}.
\end{aligned}$$

79. Let $u = \sqrt[3]{x}$, $du = \frac{1}{3x^{2/3}}\,dx$, so $dx = 3u^2\,du$. We obtain

$$\begin{aligned}
\int \frac{x\,dx}{\sqrt[3]{x}-1} &= \int \frac{u^3}{u-1}\left(3u^2\right)du \\
&= 3\int \frac{u^5}{u-1}\,du.
\end{aligned}$$

Let $y = u - 1$, then we have

$$\begin{aligned}
\int \frac{x\,dx}{\sqrt[3]{x}-1} &= 3\int \frac{(y+1)^5}{y}\,dy \\
&= 3\int \frac{y^5+5y^4+10y^3+10y^2+5y+1}{y}\,dy \\
&= 3\int \left(y^4+5y^3+10y^2+10y+5+\frac{1}{y}\right)dy \\
&= 3\left(\frac{1}{5}y^5+\frac{5}{4}y^4+\frac{10}{3}y^3+5y^2+5y+\ln|y|\right)+C \\
&= \frac{3}{5}(u-1)^5+\frac{15}{4}(u-1)^4+10(u-1)^3+15(u-1)^2+15(u-1)+3\ln|u-1|+C \\
&= \frac{3}{5}u^5+\frac{3}{4}u^4+u^3+\frac{3}{2}u^2+3u+3\ln|u-1|+C \\
&= \boxed{\tfrac{3}{5}x^{5/3}+\tfrac{3}{4}x^{4/3}+x+\tfrac{3}{2}x^{2/3}+3x^{1/3}+3\ln\left|x^{1/3}-1\right|+C}.
\end{aligned}$$

81. Let $u = x^{1/6}$, $du = \frac{1}{6x^{5/6}}\,dx$, so $dx = 6u^5\,du$. We obtain

$$\begin{aligned}
\int \frac{dx}{3\sqrt{x}-\sqrt[3]{x}} &= \int \frac{6u^5}{3u^3-u^2}\,du \\
&= \int \frac{6u^3}{3u-1}\,du.
\end{aligned}$$

Let $y = 3u - 1$, and substitute to obtain

$$\begin{aligned}
\int \frac{dx}{3\sqrt{x} - \sqrt[3]{x}} &= \int \frac{6\left(\frac{y+1}{3}\right)^3}{y} \frac{1}{3}\, dy \\
&= \int \frac{\frac{2}{9}y^3 + \frac{2}{3}y^2 + \frac{2}{3}y + \frac{2}{9}}{3y}\, dy \\
&= \int \left(\frac{2}{27}y^2 + \frac{2}{9}y + \frac{2}{9} + \frac{2}{27y}\right) dy \\
&= \frac{2}{81}y^3 + \frac{1}{9}y^2 + \frac{2}{9}y + \frac{2}{27}\ln|y| + C \\
&= \frac{2}{81}(3u-1)^3 + \frac{1}{9}(3u-1)^2 + \frac{2}{9}(3u-1) + \frac{2}{27}\ln|3u-1| + C \\
&= \frac{2}{3}u^3 + \frac{1}{3}u^2 + \frac{2}{9}u + \frac{2}{27}\ln|3u-1| + C \\
&= \boxed{\tfrac{2}{3}x^{1/2} + \tfrac{1}{3}x^{1/3} + \tfrac{2}{9}x^{1/6} + \tfrac{2}{27}\ln\left|3x^{1/6} - 1\right| + C}.
\end{aligned}$$

83. Let $u = 1 + 2x$, and substitute to obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt[4]{1+2x}} &= \int \frac{\frac{1}{2}\, du}{\sqrt[4]{u}} \\
&= \frac{1}{2}\int u^{-1/4}\, du \\
&= \frac{1}{2}\left(\frac{4}{3}u^{3/4}\right) + C \\
&= \boxed{\tfrac{2}{3}(1+2x)^{3/4} + C}.
\end{aligned}$$

85. Let $u = 1 + x$, and substitute to obtain

$$\begin{aligned}
\int \frac{dx}{(1+x)^{2/3}} &= \int \frac{du}{u^{2/3}} \\
&= \int u^{-2/3}\, du \\
&= 3u^{1/3} + C \\
&= \boxed{3(1+x)^{1/3} + C}.
\end{aligned}$$

87. Let $u = x^{1/6}$, $du = \frac{1}{6x^{5/6}}\, dx$, so $dx = 6u^5\, du$. We obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}(1+\sqrt[3]{x})^2} &= \int \frac{6u^5}{u^3(1+u^2)^2}\, du \\
&= 6\int \frac{(1+u^2) - 1}{(1+u^2)^2}\, du \\
&= 6\int \left(\frac{1}{1+u^2} - \frac{1}{(1+u^2)^2}\right) du \\
&= 6\tan^{-1} u - 6\int \frac{1}{(1+u^2)^2}\, du.
\end{aligned}$$

In the last integral, let $u = \tan\theta$, and substitute to obtain

$$\begin{aligned}
\int \frac{1}{(1+u^2)^2}\,du &= \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\
&= \int \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int (1+\cos(2\theta))\,d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\tan^{-1}u + \frac{1}{2}\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} + C \\
&= \frac{1}{2}\tan^{-1}u + \frac{u}{2(u^2+1)} + C.
\end{aligned}$$

We now obtain

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}(1+\sqrt[3]{x})^2} &= 6\tan^{-1}u - 6\left(\frac{1}{2}\tan^{-1}u + \frac{u}{2(u^2+1)}\right) + C \\
&= 3\tan^{-1}u - \frac{3u}{u^2+1} + C \\
&= \boxed{3\tan^{-1}\left(x^{1/6}\right) - 3\tfrac{x^{1/6}}{x^{1/3}+1} + C}.
\end{aligned}$$

89. With the substitution $z = \tan\frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, and $dx = \frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int \frac{dx}{1+\sin x} &= \int \frac{1}{1+\frac{2z}{1+z^2}}\frac{2\,dz}{1+z^2} \\
&= 2\int \frac{dz}{1+2z+z^2} \\
&= 2\int (1+z)^{-2}\,dz \\
&= -2(1+z)^{-1} + C \\
&= \boxed{-\tfrac{2}{1+\tan\frac{x}{2}} + C}.
\end{aligned}$$

91. With the substitution $z = \tan\frac{x}{2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $dx = \frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int \frac{dx}{3+2\cos x} &= \int \frac{1}{3+2\frac{1-z^2}{1+z^2}}\frac{2\,dz}{1+z^2} \\
&= 2\int \frac{dz}{z^2+(\sqrt{5})^2} \\
&= \frac{2\sqrt{5}}{5}\tan^{-1}\left(\frac{\sqrt{5}}{5}z\right) + C \\
&= \boxed{\tfrac{2\sqrt{5}}{5}\tan^{-1}\left(\tfrac{\sqrt{5}}{5}\tan\tfrac{x}{2}\right) + C}.
\end{aligned}$$

93. With the substitution $z=\tan\frac{x}{2}$, $\sin x=\frac{2z}{1+z^2}$, $\cos x=\frac{1-z^2}{1+z^2}$, and $dx=\frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}\int\frac{dx}{1-\sin x+\cos x} &= \int\frac{1}{1-\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}}\frac{2\,dz}{1+z^2}\\ &= \int\frac{dz}{1-z}\\ &= -\ln|1-z|\\ &= \boxed{-\ln\left|1-\tan\tfrac{x}{2}\right|+C}.\end{aligned}$$

95. With the substitution $z=\tan\frac{x}{2}$, $\tan x=\frac{2z}{1-z^2}$, and $dx=\frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}\int\frac{dx}{\tan x-1} &= \int\frac{\frac{2}{1+z^2}}{\frac{2z}{1-z^2}-1}\,dz\\ &= \int\frac{2\left(1-z^2\right)}{(1+z^2)(2z-(1-z^2))}\,dz\\ &= \int\frac{2\left(1-z^2\right)}{(z^2+1)(z^2+2z-1)}\,dz.\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}\frac{2\left(1-z^2\right)}{(z^2+1)(z^2+2z-1)} &= \frac{Az+B}{z^2+1}+\frac{Cz+D}{z^2+2z-1}\\ -2z^2+2 &= (Az+B)\left(z^2+2z-1\right)+(Cz+D)\left(z^2+1\right)\\ -2z^2+2 &= (A+C)z^3+(2A+B+D)z^2+(2B-A+C)z+(D-B).\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}A+C &= 0\\ 2A+B+D &= -2\\ 2B-A+C &= 0\\ D-B &= 2.\end{aligned}$$

We solve this system and obtain $A=-1$, $B=-1$, $C=1$, and $D=1$. So

$$\begin{aligned}\int\frac{dx}{\tan x-1} &= \int\left(\frac{z+1}{z^2+2z-1}-\frac{z+1}{z^2+1}\right)dz\\ &= \frac{1}{2}\int\frac{2(z+1)}{z^2+2z-1}-\frac{1}{2}\int\frac{2z}{z^2+1}\,dz-\int\frac{1}{z^2+1}\,dz\\ &= \frac{1}{2}\ln\left|z^2+2z-1\right|-\frac{1}{2}\ln\left(z^2+1\right)-\tan^{-1}z+C\\ &= \boxed{\tfrac{1}{2}\ln\left|\tan^2\left(\tfrac{x}{2}\right)+2\tan\tfrac{x}{2}-1\right|-\tfrac{1}{2}\ln\left(\tan^2\left(\tfrac{x}{2}\right)+1\right)-\tfrac{x}{2}+C}.\end{aligned}$$

97. With the substitution $z = \tan \frac{x}{2}$, $\sec x = \frac{1+z^2}{1-z^2}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int \frac{\sec x\,dx}{\tan x - 2} &= \int \frac{\frac{1+z^2}{1-z^2}\frac{2}{1+z^2}}{\frac{2z}{1-z^2} - 2}\,dz \\
&= \int \frac{1}{z - (1-z^2)}\,dz \\
&= \int \frac{1}{z^2 + z - 1}\,dz \\
&= \int \frac{1}{\left(z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)\left(z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)}\,dz.
\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
\frac{1}{\left(z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)\left(z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)} &= \frac{A}{z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} + \frac{B}{z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} \\
1 &= A\left(z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right) + B\left(z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right).
\end{aligned}$$

When $z = -\frac{1}{2}\sqrt{5} - \frac{1}{2}$ we obtain $B = -\sqrt{5}/5$, and when $z = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ we have $A = \sqrt{5}/5$. So we obtain

$$\begin{aligned}
\int \frac{\sec x\,dx}{\tan x - 2} &= \int \left(\frac{\sqrt{5}/5}{z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} + \frac{-\sqrt{5}/5}{z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)}\right) dz \\
&= \frac{\sqrt{5}}{5} \ln\left|z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right| - \frac{\sqrt{5}}{5} \ln\left|z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right| + C \\
&= \boxed{\frac{\sqrt{5}}{5} \ln\left|\tan \frac{x}{2} - \frac{1}{2}\sqrt{5} + \frac{1}{2}\right| - \frac{\sqrt{5}}{5} \ln\left|\tan \frac{x}{2} + \frac{1}{2}\sqrt{5} + \frac{1}{2}\right| + C}.
\end{aligned}$$

99. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\sec x = \frac{1+z^2}{1-z^2}$, and $dx = \frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int \frac{\sec x\,dx}{1 + \sin x} &= \int \frac{\frac{1+z^2}{1-z^2}\frac{2}{1+z^2}}{1 + \frac{2z}{1+z^2}}\,dz \\
&= \int \frac{-2\left(1+z^2\right)}{(z-1)(z+1)^3}\,dz.
\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
\frac{-2\left(1+z^2\right)}{(z-1)(z+1)^3} &= \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{(z+1)^2} + \frac{D}{(z+1)^3} \\
-2\left(1+z^2\right) &= A(z+1)^3 + B(z-1)(z+1)^2 + C(z-1)(z+1) + D(z-1).
\end{aligned}$$

When $z = -1$, we have $D = 2$, and when $z = 1$, $A = -1/2$. We now obtain

$$\begin{aligned}
-2z^2 - 2 &= (-1/2)(z+1)^3 + B(z-1)(z+1)^2 + C(z-1)(z+1) + 2(z-1) \\
-2z^2 - 2 &= \left(B - \frac{1}{2}\right)z^3 + \left(B + C - \frac{3}{2}\right)z^2 + \left(\frac{1}{2} - B\right)z - \left(B + C + \frac{5}{2}\right).
\end{aligned}$$

Equating coefficients, we obtain $B = 1/2$ and $C = -1$. We now have

$$\begin{aligned}
\int \frac{\sec x\,dx}{1+\sin x} &= \int \left(\frac{-1/2}{z-1} + \frac{1/2}{z+1} + \frac{-1}{(z+1)^2} + \frac{2}{(z+1)^3} \right) dz \\
&= \int \frac{-1/2}{z-1}\,dz + \int \frac{1/2}{z+1}\,dz + \int \frac{-1}{(z+1)^2}\,dz + \int \frac{2}{(z+1)^3}\,dz \\
&= -\frac{1}{2}\ln|z-1| + \frac{1}{2}\ln|z+1| + \frac{1}{z+1} - \frac{1}{(z+1)^2} + C \\
&= \boxed{-\tfrac{1}{2}\ln\left|\tan\tfrac{x}{2} - 1\right| + \tfrac{1}{2}\ln\left|\tan\tfrac{x}{2}+1\right| + \tfrac{1}{\tan\frac{x}{2}+1} - \tfrac{1}{\left(\tan\frac{x}{2}+1\right)^2} + C}.
\end{aligned}$$

101. With the substitution $z = \tan\frac{x}{2}$, $\csc x = \frac{1+z^2}{2z}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2\,dz}{1+z^2}$, we obtain

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \frac{\csc x}{3+4\tan x}\,dx &= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \frac{\frac{1+z^2}{2z}\frac{2}{1+z^2}}{3+4\frac{2z}{1-z^2}}\,dz \\
&= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \frac{\frac{1}{z}\left(1-z^2\right)}{3(1-z^2)+8z}\,dz \\
&= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \frac{z^2-1}{z(3z+1)(z-3)}\,dz.
\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
\frac{z^2-1}{z(3z+1)(z-3)} &= \frac{A}{z} + \frac{B}{3z+1} + \frac{C}{z-3} \\
z^2 - 1 &= A(3z+1)(z-3) + Bz(z-3) + Cz(3z+1).
\end{aligned}$$

When $z = 0$, we have $A = 1/3$, when $z = -1/3$, $B = -4/5$, and when $z = 3$, $C = 4/15$. We now obtain

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \frac{\csc x}{3+4\tan x}\,dx &= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \left(\frac{1/3}{z} + \frac{-4/5}{3z+1} + \frac{4/15}{z-3} \right) dz \\
&= \left[\frac{1}{3}\ln z - \frac{4}{15}\ln(3z+1) + \frac{4}{15}\ln(3-z) \right]_{\tan(\pi/8)}^{\sqrt{3}/3} \\
&= \frac{1}{3}\ln\frac{\sqrt{3}}{3} - \frac{4}{15}\ln\left(3\frac{\sqrt{3}}{3}+1\right) + \frac{4}{15}\ln\left(3 - \frac{\sqrt{3}}{3}\right) \\
&\quad -\left(\frac{1}{3}\ln\left(\tan\frac{\pi}{8}\right) - \frac{4}{15}\ln\left(3\tan\frac{\pi}{8}+1\right) + \frac{4}{15}\ln\left(3-\tan\frac{\pi}{8}\right)\right).
\end{aligned}$$

$$= \boxed{\tfrac{1}{3}\ln\tfrac{\sqrt{3}}{3} - \tfrac{1}{3}\ln\left(\sqrt{2}-1\right) - \tfrac{4}{15}\ln\left(\sqrt{3}+1\right) - \tfrac{4}{15}\ln\left(4-\sqrt{2}\right) + \tfrac{4}{15}\ln\left(3\sqrt{2}-2\right) + \tfrac{4}{15}\ln\left(3-\tfrac{\sqrt{3}}{3}\right)}.$$

103. With the substitution $z = \tan\frac{x}{2}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2\,dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int_0^{\pi/4} \frac{4\,dx}{\tan x + 1} &= \int_0^{\tan(\pi/8)} \frac{4\frac{2}{1+z^2}}{\frac{2z}{1-z^2}+1}\,dz \\
&= \int_0^{\tan(\pi/8)} \frac{8\left(1-z^2\right)}{(1+z^2)(2z+(1-z^2))}\,dz \\
&= \int_0^{\tan(\pi/8)} \frac{8\left(z^2-1\right)}{(z^2+1)(z^2-2z-1)}\,dz.
\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
\frac{8(z^2-1)}{(z^2+1)(z^2-2z-1)} &= \frac{Az+B}{z^2+1}+\frac{Cz+D}{z^2-2z-1} \\
8(z^2-1) &= (Az+B)(z^2-2z-1)+(Cz+D)(z^2+1) \\
8z^2-8 &= (A+C)z^3+(B-2A+D)z^2+(C-2B-A)z+(D-B).
\end{aligned}$$

Equating coefficients, we obtain

$$\begin{aligned}
A+C &= 0 \\
B-2A+D &= 8 \\
-A-2B+C &= 0 \\
-B+D &= -8.
\end{aligned}$$

We solve this system, and obtain $A=-4$, $B=4$, $C=4$, and $D=-4$. We now have

$$\begin{aligned}
\int_0^{\pi/2}\frac{4\,dx}{\tan x+1} &= \int_0^{\tan(\pi/8)}\left(\frac{-4z+4}{z^2+1}+\frac{4z-4}{z^2-2z-1}\right)dz \\
&= -2\int_0^{\tan(\pi/8)}\frac{2z}{z^2+1}\,dz+4\int_0^{\tan(\pi/8)}\frac{1}{z^2+1}\,dz+2\int_0^{\tan(\pi/8)}\frac{2z-2}{z^2-2z-1}\,dz \\
&= -2\left[\ln\left(z^2+1\right)\right]_0^{\tan(\pi/8)}+4\left[\tan^{-1}z\right]_0^{\tan(\pi/8)}+2\left[\ln\left(1+2z-z^2\right)\right]_0^{\tan(\pi/8)} \\
&= -2\ln\left(\tan^2(\pi/8)+1\right)-(-2)\ln\left(0^2+1\right)+4\tan^{-1}(\tan(\pi/8))-4\tan^{-1}0 \\
&\quad +2\ln\left(1+2\tan(\pi/8)-\tan^2(\pi/8)\right)-2\ln\left(1+2(0)-(0)^2\right) \\
&= -2\ln\left(\left(\sqrt{2}-1\right)^2+1\right)+4\left(\frac{1}{8}\pi\right)+2\ln\left(1+2\left(\sqrt{2}-1\right)-\left(\sqrt{2}-1\right)^2\right) \\
&= 2\ln\left(\frac{2\sqrt{2}-2}{2-\sqrt{2}}\right)+\frac{1}{2}\pi \\
&= 2\ln\left(\sqrt{2}\right)+\frac{1}{2}\pi \\
&= \boxed{\ln 2+\tfrac{\pi}{2}}.
\end{aligned}$$

105. We simplify as follows

$$\begin{aligned}
\ln\sqrt{\frac{1-\cos x}{1+\cos x}} &= \ln\sqrt{\frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)}} \\
&= \ln\frac{1-\cos x}{\sqrt{1-\cos^2 x}} \\
&= \ln\frac{1-\cos x}{\sqrt{\sin^2 x}} \\
&= \ln\frac{1-\cos x}{|\sin x|} \\
&= \ln\left|\frac{1-\cos x}{\sin x}\right| \\
&= \ln\left|\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right| \\
&= \ln|\csc x-\cot x|.
\end{aligned}$$

107. We simplify as follows

$$\begin{aligned}
\ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| &= \ln\left|\frac{1+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right| \\
&= \ln\left|\frac{1+\frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}}{1-\frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}}\right| \\
&= \ln\left|\frac{1+\frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}}{1-\frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}}\right| \\
&= \ln\left|\frac{1+\frac{1-\cos x}{\sin x}}{1-\frac{1-\cos x}{\sin x}}\right| \\
&= \ln\left|\frac{\sin x+1-\cos x}{\sin x-1+\cos x}\right| \\
&= \ln\left|\frac{\tan x+\sec x-1}{\tan x-\sec x+1}\frac{\sec x+\tan x}{\sec x+\tan x}\right| \\
&= \ln\left|\frac{(\tan x+\sec x-1)(\sec x+\tan x)}{(\tan x-\sec x+1)(\sec x+\tan x)}\right| \\
&= \ln\left|\frac{(\tan x+\sec x-1)(\sec x+\tan x)}{\tan x\sec x+\tan^2 x-\sec^2 x-\sec x\tan x+\sec x+\tan x}\right| \\
&= \ln\left|\frac{(\tan x+\sec x-1)(\sec x+\tan x)}{-1+\sec x+\tan x}\right| \\
&= \ln\left|\sec x+\tan x\right|.
\end{aligned}$$

AP® Practice Problems

1. Apply polynomial division to $\frac{x^2+6}{x+1}$.

$$\begin{array}{r|l}
 & \quad\quad x-1 \\
\hline
(x+1) & x^2+0x+6 \\
 & -(x^2+\ \ x) \\
\hline
 & \quad -x+6 \\
 & -(-x-1) \\
\hline
 & \quad\quad 7
\end{array}$$

The quotient is $x-1$ and the remainder is 7.

Therefore, $\frac{x^2+6}{x+1} = x-1+\frac{7}{x+1}$ and $\int \frac{x^2+6}{x+1}\,dx = \int\left(x-1+\frac{7}{x+1}\right)dx = \boxed{\frac{x^2}{2}-x+7\ln|x+1|+C}$.

The answer is C.

3. Apply polynomial division to $\frac{x^4+3x^2-2}{x^2+1}$.

$$\begin{array}{r|l}
 & \quad\quad x^2 \quad\quad +2 \\
\hline
(x^2+1) & x^4+0x^3+3x^2+0x-2 \\
 & -(x^4 \quad\quad +x^2) \\
\hline
 & \quad\quad 2x^2+0x-2 \\
 & \quad -(2x^2 \quad\quad +2) \\
\hline
 & \quad\quad\quad\quad -4
\end{array}$$

The quotient is x^2+2 and the remainder is -4.

Therefore, $\frac{x^4+3x^2-2}{x^2+1} = x^2+2-\frac{4}{x^2+1}$ and $\int \frac{x^4+3x^2-2}{x^2+1}\,dx = \int \left(x^2+2-\frac{4}{x^2+1}\right) dx = \boxed{\frac{x^3}{3}+2x-4\tan^{-1}x+C}$.

The answer is D.

5. To evaluate $\int \frac{12}{x^2-9}\,dx$, notice the integrand is a proper rational function in lowest terms. Begin by factoring the denominator: $x^2-9=(x+3)(x-3)$. Since the factors are linear and distinct, this is a Case 1 type integrand and can be written as $\frac{12}{(x+3)(x-3)} = \frac{A}{x+3}+\frac{B}{x-3}$. Clear the fractions by multiplying both sides of the equation by $(x+3)(x-3)$.

   $12 = A(x-3)+B(x+3)$.

   Grouping like terms, $12=(A+B)x+(-3A+3B)$.

   This is an identity in x, so the coefficients of like powers of x must be equal.

   $$A+B=0$$
   $$-3A+3B=12$$

   This is a system of two equations containing two variables.

   After solving this system, the solution is $A=-2$ and $B=2$.

   So, $\frac{12}{(x+3)(x-3)} = \frac{-2}{x+3}+\frac{2}{x-3}$ and $\int \frac{12}{x^2-9}\,dx = \int \left(\frac{-2}{x+3}+\frac{2}{x-3}\right) dx$.

   $= -2\ln|x+3| + 2\ln|x-3| + C = \boxed{2\ln\left|\frac{x-3}{x+3}\right| + C}$.

   The answer is A.

7. To evaluate $\int \frac{x+6}{x(x+2)}\,dx$, notice the integrand is a proper rational function in lowest terms. Since the factors are linear and distinct, this is a Case 1 type integrand and can be written as $\frac{x+6}{x(x+2)} = \frac{A}{x}+\frac{B}{x+2}$.

   Clear the fractions by multiplying both sides of the equation by $x(x+2)$.

   $x+6 = A(x+2)+Bx$

   Grouping like terms, $x+6=(A+B)x+2A$.

   This is an identity in x, so the coefficients of like powers of x must be equal.

   $$A+B=1$$
   $$2A=6$$

   This is a system of two equations containing two variables.

   Since $2A=6$, $A=3$ and $B=1-3=-2$.

   So, $\frac{x+6}{x(x+2)} = \frac{3}{x}-\frac{2}{x+2}$ and $\int \frac{x+6}{x(x+2)}\,dx = \int \left(\frac{3}{x}-\frac{2}{x+2}\right) dx = \boxed{3\ln|x|-2\ln|x+2|+C}$.

   The answer is C.

9. Express the logistic model $\frac{dP}{dt} = P\left(3 - \frac{P}{2000}\right)$ in the form $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ to find the carrying capacity.

$$\frac{dP}{dt} = P\left(3 - \frac{P}{2000}\right) = 3P\left(1 - \frac{P}{6000}\right).$$

Therefore, the carrying capacity is $\boxed{M = 6000}$.

The answer is C.

11. (a) The size of the insect population at time t follows a logistic growth model $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ where k is the maximum growth rate and M is the carrying capacity. We are given the daily maximum growth rate of $k = 0.20$, a carrying capacity of $M = 600{,}000$ insects, and an initial population size of 100 insects. The differential equation becomes $\boxed{\frac{dP}{dt} = 0.20P\left(1 - \frac{P}{600{,}000}\right) \text{ with } P(0) = 100}$ insects.

(b) The initial population size is $P_0 = 100$ insects.

The solution to the differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is $P(t) = \frac{M}{1+ae^{-kt}}$ where $k = 0.20$, $M = 600{,}000$, and $a = \frac{M-P_0}{P_0} = \frac{600{,}000-100}{100} = 5999$. Therefore, $\boxed{P(t) = \frac{600{,}000}{1+5999e^{-0.20t}}}$.

(c) Find t so that $P(t) = 100{,}000$.

$$\begin{aligned}
\frac{600{,}000}{1+5999e^{-0.20t}} &= 100{,}000 \\
e^{-0.20t} &= \frac{\frac{600{,}000}{100{,}000} - 1}{5999} = \frac{5}{5999} \\
-0.20t &= \ln\left(\frac{5}{5999}\right) \\
t &= -\frac{1}{0.20}\ln\left(\frac{5}{5999}\right) = \boxed{35.449 \text{ days}}.
\end{aligned}$$

The population of insects will exceed 100,000 on the 35th day.

7.6 Approximating Integrals: The Trapezoidal Rule, Trapezoidal Sums, Simpson's Rule

Concepts and Vocabulary

1. True

Skill Building

3. When $n = 3$ we have $\Delta x = \frac{6-0}{3} = 2$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}
\int_0^6 f(x)\,dx &\approx \frac{\Delta x}{2}\left[f(0) + 2f(2) + 2f(4) + f(6)\right] \\
&= \frac{2}{2}(6 + 2(3) + 2(3) + 4) \\
&= \boxed{22}.
\end{aligned}$$

When $n = 6$ we have $\Delta x = \frac{6-0}{6} = 1$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^6 f(x)\,dx &\approx \frac{\Delta x}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)\right]\\ &= \frac{1}{2}(6 + 2(3) + 2(3) + 2(4) + 2(3) + 2(2) + 4)\\ &= \boxed{20}.\end{aligned}$$

5. When $n = 2$ we have $\Delta x = \frac{6-0}{2} = 3$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^6 f(x)\,dx &\approx \frac{\Delta x}{3}\left[f(0) + 4f(3) + f(6)\right]\\ &= \frac{3}{3}(6 + 4(4) + 4)\\ &= \boxed{26}.\end{aligned}$$

When $n = 6$ we have $\Delta x = \frac{6-0}{6} = 1$. So Simpson's Rule provides the approximation

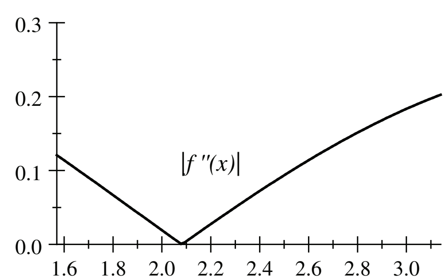
$$\begin{aligned}\int_0^6 f(x)\,dx &\approx \frac{\Delta x}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)\right]\\ &= \frac{1}{3}(6 + 4(3) + 2(3) + 4(4) + 2(3) + 4(2) + 4)\\ &= \boxed{\tfrac{58}{3}}.\end{aligned}$$

7. (a) With $n = 3$ we have $\Delta x = \frac{\pi - \pi/2}{3} = \pi/6$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx &\approx \frac{\Delta x}{2}\left[f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 2f\left(\frac{5\pi}{6}\right) + f(\pi)\right]\\ &= \frac{\pi/6}{2}\left[\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} + 2\frac{\sin\frac{2\pi}{3}}{\frac{2\pi}{3}} + 2\frac{\sin\frac{5\pi}{6}}{\frac{5\pi}{6}} + \frac{\sin\pi}{\pi}\right]\\ &= \frac{\pi}{12}\left[\frac{2}{\pi} + \frac{3}{2}\frac{\sqrt{3}}{\pi} + \frac{6}{5\pi} + 0\right]\\ &\approx \boxed{0.483}.\end{aligned}$$

(b) We have $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = -\frac{1}{x^2}(\sin x - x\cos x)$, and $\frac{d^2}{dx^2}\left(\frac{\sin x}{x}\right) = -\frac{1}{x^3}\left(x^2\sin x - 2\sin x + 2x\cos x\right)$. The graph of $|f''(x)| = \left|-\frac{1}{x^3}\left(x^2\sin x - 2\sin x + 2x\cos x\right)\right|$ shows that the maximum occurs at $x = \pi$, and so $M = \left|-\frac{1}{\pi^3}\left(\pi^2\sin\pi - 2\sin\pi + 2\pi\cos\pi\right)\right| = \frac{2}{\pi^2}$. We obtain

$$\text{Error} \le \frac{(\pi - \pi/2)^3\left(\frac{2}{\pi^2}\right)}{12(3)^2} \approx 0.007$$



(c) We require

$$\begin{aligned} \text{Error} &\le \frac{(\pi - \pi/2)^3\left(\frac{2}{\pi^2}\right)}{12(n)^2} = \frac{\pi}{48n^2} < 0.0001 \\ \frac{\pi}{48(0.0001)} &< n^2 \\ \sqrt{\frac{\pi}{48(0.0001)}} &< n \\ 25.58 &< n \end{aligned}$$

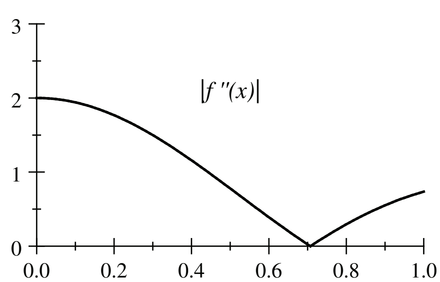
So we need $n = \boxed{26}$.

9. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_0^1 e^{-x^2}\,dx &\approx \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \\ &= \frac{1/4}{2}\left[e^{-(0)^2} + 2e^{-\left(\frac{1}{4}\right)^2} + 2e^{-\left(\frac{1}{2}\right)^2} + 2e^{-\left(\frac{3}{4}\right)^2} + e^{-(1)^2}\right] \\ &= \frac{1}{8}\left[1 + 2e^{-\frac{1}{16}} + 2e^{-\frac{1}{4}} + 2e^{-\frac{9}{16}} + e^{-1}\right] \\ &\approx \boxed{0.743}. \end{aligned}$$

(b) We have $\frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$, and $\frac{d^2}{dx^2}\left(e^{-x^2}\right) = 4x^2e^{-x^2} - 2e^{-x^2}$. The graph of $|f''(x)| = \left|4x^2e^{-x^2} - 2e^{-x^2}\right|$ shows that the maximum $M = 2$. We obtain

$$\text{Error} \le \frac{(1-0)^3(2)}{12(4)^2} = \frac{1}{96}$$



(c) We require

$$\begin{aligned} \text{Error} &\le \frac{(1-0)^3(2)}{12(n)^2} = \frac{1}{6n^2} < 0.0001 \\ \frac{1}{6(0.0001)} &< n^2 \\ \sqrt{\frac{1}{6(0.0001)}} &< n \\ 40.82 &< n \end{aligned}$$

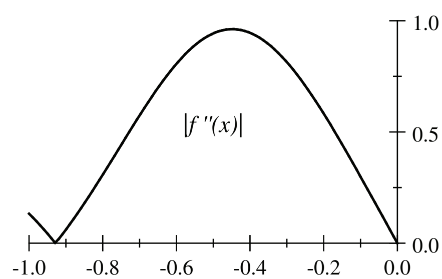
So we need $n = \boxed{41}$.

11. (a) With $n = 4$ we have $\Delta x = \frac{0-(-1)}{4} = \frac{1}{4}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}
\int_{-1}^{0} \frac{dx}{\sqrt{1-x^3}} &\approx \frac{\Delta x}{2}\left[f(-1) + 2f\left(\frac{-3}{4}\right) + 2f\left(\frac{-1}{2}\right) + 2f\left(\frac{-1}{4}\right) + f(0)\right] \\
&= \frac{1/4}{2}\left[\frac{1}{\sqrt{1-(-1)^3}} + 2\frac{1}{\sqrt{1-(-3/4)^3}} + 2\frac{1}{\sqrt{1-(-1/2)^3}} \right. \\
&\qquad \left. + 2\frac{1}{\sqrt{1-(-1/4)^3}} + \frac{1}{\sqrt{1-(0)^3}}\right] \\
&= \frac{1}{8}\left[\frac{1}{2}\sqrt{2} + \frac{16}{91}\sqrt{91} + \frac{4}{3}\sqrt{2} + \frac{16}{65}\sqrt{65} + 1\right] \\
&\approx \boxed{0.907}.
\end{aligned}$$

(b) We have $\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^3}}\right) = \frac{3x^2}{2(1-x^3)^{\frac{3}{2}}}$, and $\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{1-x^3}}\right) = \frac{3x(5x^3+4)}{4(1-x^3)^{\frac{5}{2}}}$. The graph of $|f''(x)| = \left|\frac{3x(5x^3+4)}{4(1-x^3)^{\frac{5}{2}}}\right|$ shows that the maximum $M \le 1$. We obtain

$$\text{Error} \le \frac{(0-(-1))^3(1)}{12(4)^2} = \frac{1}{192}$$



(c) We require

$$\begin{aligned}
\text{Error} &\le \frac{(0-(-1))^3(1)}{12(n)^2} = \frac{1}{12n^2} < 0.0001 \\
\frac{1}{0.0001(12)} &< n^2 \\
\sqrt{\frac{1}{0.0001(12)}} &< n \\
28.87 &< n
\end{aligned}$$

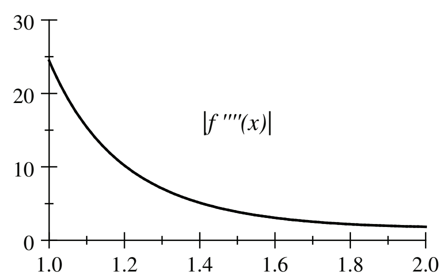
So we need $n = \boxed{29}$.

13. (a) With $n = 4$ we have $\Delta x = \frac{2-1}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_1^2 \frac{e^x}{x}\,dx &\approx \frac{\Delta x}{3}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right] \\ &= \frac{1/4}{3}\left[\frac{e^1}{1} + 4\frac{e^{5/4}}{5/4} + 2\frac{e^{3/2}}{3/2} + 4\frac{e^{7/4}}{7/4} + \frac{e^2}{2}\right] \\ &= \frac{1}{12}\left[e + \frac{16}{5}e^{\frac{5}{4}} + \frac{4}{3}e^{\frac{3}{2}} + \frac{16}{7}e^{\frac{7}{4}} + \frac{1}{2}e^2\right] \\ &\approx \boxed{3.059}.\end{aligned}$$

(b) We have $\frac{d^4}{dx^4}\left(\frac{e^x}{x}\right) = \frac{1}{x^5}e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right)$. The graph of $\left|f^4(x)\right|$ shows $M \le \left|f^{(4)}(1)\right| = 9e$. We obtain

$$\text{Error} \le \frac{(2-1)^5(9e)}{180(4)^4} \approx 5.309 \times 10^{-4}$$



(c) We require

$$\begin{aligned}\text{Error} &\le \frac{(2-1)^5(9e)}{180(n)^4} < 0.0001 \\ \frac{9e}{0.0001(180)} &< n^4 \\ \sqrt[4]{\frac{9e}{0.0001(180)}} &< n \\ 6.072 &< n\end{aligned}$$

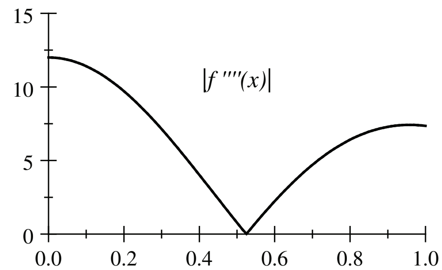
Since n must be even, we need $n = \boxed{8}$.

15. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx \frac{\Delta x}{3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] \\ &= \frac{1/4}{3}\left[e^{-(0)^2} + 4e^{-(1/4)^2} + 2e^{-(1/2)^2} + 4e^{-(3/4)^2} + e^{-(1)^2}\right] \\ &= \frac{1}{12}\left[1 + 4e^{-\frac{1}{16}} + 2e^{-\frac{1}{4}} + 4e^{-\frac{9}{16}} + e^{-1}\right] \\ &\approx \boxed{0.747}.\end{aligned}$$

(b) We have $\frac{d^4}{dx^4}\left(e^{-x^2}\right) = 4e^{-x^2}\left(4x^4 - 12x^2 + 3\right)$. The graph of $\left|f^4(x)\right|$ shows $M \le 12$. We obtain

$$\text{Error} \le \frac{(1-0)^5(12)}{180(4)^4} \approx 2.604 \times 10^{-4}$$



(c) We require

$$\begin{aligned} \text{Error} &\le \frac{(1-0)^5(12)}{180(n)^4} = \frac{1}{15n^4} < 0.0001 \\ \frac{1}{0.0001(15)} &< n^4 \\ \sqrt[4]{\frac{1}{0.0001(15)}} &< n \\ 5.081 &< n \end{aligned}$$

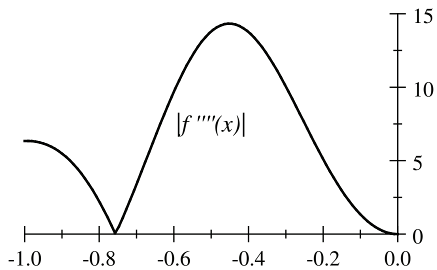
We need $n = \boxed{6}$.

17. (a) With $n = 4$ we have $\Delta x = \frac{0-(-1)}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_{-1}^{0} \frac{dx}{\sqrt{1-x^3}} &\approx \frac{\Delta x}{3}\left[f(-1) + 4f\left(\frac{-3}{4}\right) + 2f\left(\frac{-1}{2}\right) + 4f\left(\frac{-1}{4}\right) + f(0)\right] \\ &= \frac{1/4}{3}\left[\frac{1}{\sqrt{1-(-1)^3}} + 4\frac{1}{\sqrt{1-(-3/4)^3}} + 2\frac{1}{\sqrt{1-(-1/2)^3}}\right. \\ &\quad \left. +4\frac{1}{\sqrt{1-(-1/4)^3}} + \frac{1}{\sqrt{1-(0)^3}}\right] \\ &= \frac{1}{12}\left[\frac{1}{2}\sqrt{2} + \frac{32}{91}\sqrt{91} + \frac{4}{3}\sqrt{2} + \frac{32}{65}\sqrt{65} + 1\right] \\ &\approx \boxed{0.910}. \end{aligned}$$

(b) We have $\frac{d^4}{dx^4}\left(\frac{1}{\sqrt{1-x^3}}\right) = \frac{135x^2\left(7x^6+40x^3+16\right)}{16(1-x^3)^{\frac{9}{2}}}$. The graph of $\left|f^4(x)\right|$ shows $M \le 15$. We obtain

$$\text{Error} \le \frac{(0-(-1))^5(15)}{180(4)^4} \approx 3.255 \times 10^{-4}$$



(c) We require

$$\begin{aligned} \text{Error} &\le \frac{(0-(-1))^5(15)}{180(n)^4} = \frac{1}{12(n)^4} < 0.0001 \\ \frac{1}{0.0001(12)} &< n^4 \\ \sqrt[4]{\frac{1}{0.0001(12)}} &< n \\ 5.373 &< n \end{aligned}$$

We need $n = \boxed{6}$.

19. (a) $\int_1^2 \frac{dx}{x} = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2$.

(b) With $n = 5$ we have $\Delta x = \frac{2-1}{5} = \frac{1}{5}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_1^2 \frac{dx}{x} &\approx \frac{\Delta x}{2}\left[f(1) + 2f\left(\frac{6}{5}\right) + 2f\left(\frac{7}{5}\right) + 2f\left(\frac{8}{5}\right) + 2f\left(\frac{9}{5}\right) + f(2)\right] \\ &= \frac{1/5}{2}\left[\frac{1}{1} + 2\frac{1}{6/5} + 2\frac{1}{7/5} + 2\frac{1}{8/5} + 2\frac{1}{9/5} + \frac{1}{2}\right] \\ &= \frac{1}{10}\left[1 + \frac{5}{3} + \frac{10}{7} + \frac{5}{4} + \frac{10}{9} + \frac{1}{2}\right] \\ &= \frac{1753}{2520} \\ &\approx \boxed{0.6956}. \end{aligned}$$

(c) With $n = 6$ we have $\Delta x = \frac{2-1}{6} = \frac{1}{6}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_1^2 \frac{dx}{x} &\approx \frac{\Delta x}{3}\left[f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{5}{3}\right) + 4f\left(\frac{11}{6}\right) + f(2)\right] \\ &= \frac{1/6}{3}\left[\frac{1}{1} + 4\frac{1}{7/6} + 2\frac{1}{4/3} + 4\frac{1}{3/2} + 2\frac{1}{5/3} + 4\frac{1}{11/6} + \frac{1}{2}\right] \\ &= \frac{1}{18}\left[\frac{1}{1} + \frac{24}{7} + \frac{3}{2} + \frac{8}{3} + \frac{6}{5} + \frac{24}{11} + \frac{1}{2}\right] \\ &= \frac{14\,411}{20\,790} \\ &\approx \boxed{0.6932}. \end{aligned}$$

21. The arc length is given by $\int_0^{\pi/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/2}\sqrt{1+(\cos x)^2}\,dx = \int_0^{\pi/2}\sqrt{1+\cos^2 x}\,dx$.

   (a) With $n=4$ we have $\Delta x = \frac{(\pi/2)-0}{4} = \frac{\pi}{8}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2}\sqrt{1-\cos^2 x}\,dx &\approx \frac{\Delta x}{3}\left[f(0)+4f\left(\frac{\pi}{8}\right)+2f\left(\frac{\pi}{4}\right)+4f\left(\frac{3\pi}{8}\right)+f\left(\frac{\pi}{2}\right)\right]\\ &= \frac{\pi/8}{3}\left[\sqrt{1+\cos^2 0}+4\sqrt{1+\cos^2\frac{\pi}{8}}+2\sqrt{1+\cos^2\frac{\pi}{4}}\right.\\ &\quad\left.+4\sqrt{1+\cos^2\frac{3\pi}{8}}+\sqrt{1+\cos^2\frac{\pi}{2}}\right]\\ &\approx \boxed{1.910}.\end{aligned}$$

   (b) With $n=3$ we have $\Delta x = \frac{(\pi/2)-0}{3} = \frac{\pi}{6}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2}\sqrt{1-\cos^2 x}\,dx &\approx \frac{\Delta x}{2}\left[f(0)+2f\left(\frac{\pi}{6}\right)+2f\left(\frac{\pi}{3}\right)+f\left(\frac{\pi}{2}\right)\right]\\ &= \frac{\pi/6}{2}\left[\sqrt{1+\cos^2 0}+2\sqrt{1+\cos^2\frac{\pi}{6}}+2\sqrt{1+\cos^2\frac{\pi}{3}}+\sqrt{1+\cos^2\frac{\pi}{2}}\right]\\ &\approx \boxed{1.910}.\end{aligned}$$

23. The work is given by the integral $\int_1^{2.5} p\,dV$, which we approximate by Simpson's Rule, with $\Delta V = 0.25$.

$$\begin{aligned}\int_1^{2.5} p\,dV &\approx \frac{\Delta V}{3}[f(1.0)+4f(1.25)+2f(1.5)+4f(1.75)+2f(2.0)+4f(2.25)+f(2.5)]\\ &= \frac{0.25}{3}[68.7+4(55.0)+2(45.8)+4(39.3)+2(34.4)+4(30.5)+27.5]\\ &\approx \boxed{62.983 \text{ inch-pounds}}.\end{aligned}$$

25. The volume is given by the integral $\int_0^{150} S\,dx$, which we approximate by the Trapezoidal Rule, with $\Delta x = 25$.

$$\begin{aligned}\int_0^{150} S\,dx &\approx \frac{\Delta x}{2}[f(0)+2f(25)+2f(50)+2f(75)+2f(100)+2f(125)+f(150)]\\ &= \frac{25}{2}[105+2(118)+2(142)+2(120)+2(110)+2(90)+78]\\ &\approx \boxed{16,787.5 \text{ m}^3}.\end{aligned}$$

27. The volume is given by the integral $\int_0^{25} A\,dx$. We first approximate by the Trapezoidal Rule, with $\Delta x = 2.5$.

$$\begin{aligned}\int_0^{25} A\,dx &\approx \frac{\Delta x}{2}[f(0)+2f(2.5)+2f(5.0)+2f(7.5)+2f(10)+2f(12.5)\\ &\quad +2f(15.0)+2f(17.5)+2f(20)+2f(22.5)+f(25)]\\ &= \frac{2.5}{2}[0+2(2510)+2(3860)+2(4870)+2(5160)+2(5590)\\ &\quad +2(5810)+2(6210)+2(6890)+2(7680)+8270]\\ &\approx \boxed{131,787.5 \text{ m}^3}.\end{aligned}$$

We next approximate by Simpson's Rule.

$$\begin{aligned}\int_0^{25} A\,dx &\approx \frac{\Delta x}{3}\left[f(0)+4f(2.5)+2f(5.0)+4f(7.5)+2f(10)+4f(12.5)\right.\\&\quad\left.+2f(15.0)+4f(17.5)+2f(20)+4f(22.5)+f(25)\right]\\&= \frac{2.5}{3}\left[0+4(2510)+2(3860)+4(4870)+2(5160)+4(5590)\right.\\&\quad\left.+2(5810)+4(6210)+2(6890)+4(7680)+8270\right]\\&\approx \boxed{132,625\text{ m}^3}.\end{aligned}$$

29. The volume is given by the integral $\int_2^8 \pi y^2\,dx$. We approximate by the Trapezoidal Rule, with $\Delta x = 2$.

$$\begin{aligned}\int_2^8 \pi y^2\,dx &\approx \frac{\Delta x}{2}\left[f(2)+2f(4)+2f(6)+f(8)\right]\\&= \frac{2}{2}\left[\pi(1)^2+2\pi(3)^2+2\pi(3.5)^2+\pi(3)^2\right]\\&\approx \boxed{164.934}.\end{aligned}$$

31. The volume is given by $\int_0^1 \pi\left(\sin^{-1} y\right)^2\,dy$.

(a) With $n=4$ we have $\Delta y = \frac{1-0}{4} = \frac{1}{4}$. Using the disk method, Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^1 \pi\left(\sin^{-1} y\right)^2 dy &\approx \frac{\Delta y}{3}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]\\&= \frac{1/4}{3}\left[\pi\left(\sin^{-1}0\right)^2+4\left(\pi\left(\sin^{-1}\frac{1}{4}\right)^2\right)+2\left(\pi\left(\sin^{-1}\frac{1}{2}\right)^2\right)\right.\\&\quad\left.+4\left(\pi\left(\sin^{-1}\frac{3}{4}\right)^2\right)+\pi\left(\sin^{-1}1\right)^2\right]\\&\approx \boxed{1.6095}.\end{aligned}$$

Using the shell method, we have $\Delta x = \frac{\pi/2-0}{4} = \frac{\pi}{8}$. Simpson's Rule provides the approximation

$$\begin{aligned}&\int_0^{\pi/2} 2\pi x\left(1-\sin x\right)\,dx\\&\approx \frac{\Delta x}{3}\left[f(0)+4f\left(\frac{\pi}{8}\right)+2f\left(\frac{\pi}{4}\right)+4f\left(\frac{3\pi}{8}\right)+f\left(\frac{\pi}{2}\right)\right]\\&= \frac{\pi/8}{3}\left[2\pi(0)(1-\sin(0))+4\left(2\pi\left(\frac{\pi}{8}\right)\left(1-\sin\left(\frac{\pi}{8}\right)\right)\right)+2\left(2\pi\left(\frac{\pi}{4}\right)\left(1-\sin\left(\frac{\pi}{4}\right)\right)\right)\right.\\&\quad\left.+4\left(2\pi\left(\frac{3\pi}{8}\right)\left(1-\sin\left(\frac{3\pi}{8}\right)\right)\right)+2\pi\left(\frac{\pi}{2}\right)\left(1-\sin\left(\frac{\pi}{2}\right)\right)\right]\\&\approx \boxed{1.4709}.\end{aligned}$$

(b) With $n = 3$ we have $\Delta y = \frac{1-0}{3} = \frac{1}{3}$. Using the disk method, Trapezoidal Rule provides the approximation

$$\begin{aligned}
&\int_0^1 \pi \left(\sin^{-1} y\right)^2 \, dy \\
\approx\ & \frac{\Delta y}{2}\left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1)\right] \\
=\ & \frac{1/3}{2}\left[\pi\left(\sin^{-1} 0\right) + 2\left(\pi\left(\sin^{-1}\frac{1}{3}\right)^2\right) + 2\left(\pi\left(\sin^{-1}\frac{2}{3}\right)^2\right) + \pi\left(\sin^{-1} 1\right)^2\right] \\
\approx\ & \boxed{1.9705}.
\end{aligned}$$

Using the shell method, we have $\Delta x = \frac{\pi/2-0}{3} = \frac{\pi}{6}$. The Trapezoidal Rule provides the approximation

$$\begin{aligned}
\int_0^{\pi/2} 2\pi x\,(1 - \sin x)\, dx \ &\approx\ \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)\right] \\
&=\ \frac{\pi/6}{2}\left[2\pi\,(0)\,(1 - \sin(0)) + 2\left(2\pi\left(\frac{\pi}{6}\right)\left(1 - \sin\left(\frac{\pi}{6}\right)\right)\right)\right. \\
&\qquad \left. + 2\left(2\pi\left(\frac{\pi}{3}\right)\left(1 - \sin\left(\frac{\pi}{3}\right)\right)\right) + 2\pi\left(\frac{\pi}{2}\right)\left(1 - \sin\left(\frac{\pi}{2}\right)\right)\right] \\
&\approx\ \boxed{1.3228}.
\end{aligned}$$

33. (a) With $n = 6$ we have $\Delta x = \frac{\pi-0}{6} = \frac{\pi}{6}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}
\int_0^{\pi} f(x)\, dx \ &\approx\ \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 2f\left(\frac{5\pi}{6}\right) + f(2\pi)\right] \\
&=\ \frac{\pi/6}{2}\left[1 + 2\frac{\sin(\pi/6)}{\pi/6} + 2\frac{\sin(\pi/3)}{\pi/3} + 2\frac{\sin(\pi/2)}{\pi/2}\right. \\
&\qquad \left. + 2\frac{\sin(2\pi/3)}{2\pi/3} + 2\frac{\sin(5\pi/6)}{5\pi/6} + \frac{\sin(2\pi)}{2\pi}\right] \\
&\approx\ \boxed{1.845}.
\end{aligned}$$

(b) Simpson's Rule provides the approximation

$$\begin{aligned}
\int_0^{\pi} f(x)\, dx \ &\approx\ \frac{\Delta x}{3}\left[f(0) + 4f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + 4f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 4f\left(\frac{5\pi}{6}\right) + f(2\pi)\right] \\
&=\ \frac{\pi/6}{3}\left[1 + 4\frac{\sin(\pi/6)}{\pi/6} + 2\frac{\sin(\pi/3)}{\pi/3} + 4\frac{\sin(\pi/2)}{\pi/2}\right. \\
&\qquad \left. + 2\frac{\sin(2\pi/3)}{2\pi/3} + 4\frac{\sin(5\pi/6)}{5\pi/6} + \frac{\sin(2\pi)}{2\pi}\right] \\
&\approx\ \boxed{1.852}.
\end{aligned}$$

Challenge Problems

35. Since $T_n = \frac{1}{2}(L_n + R_n)$ where L_n is the Riemann Sum using left endpoints, and R_n is the Riemann Sum using right endpoints, we have

$$\begin{aligned}\lim_{n\to\infty} T_n &= \lim_{n\to\infty} \frac{1}{2}(L_n + R_n) \\ &= \frac{1}{2}\lim_{n\to\infty} L_n + \frac{1}{2}\lim_{n\to\infty} R_n \\ &= \frac{1}{2}\int_a^b f(x)\,dx + \frac{1}{2}\int_a^b f(x)\,dx \\ &= \int_a^b f(x)\,dx.\end{aligned}$$

AP® Practice Problems

1. Since $y = x^3$ is nonnegative on $[0,4]$, $\int_0^4 x^3\,dx$ is the area under the graph of $y = x^3$ from $x = 0$ to $x = 4$. Partition $[0,4]$ into four subintervals, each of equal width: $[0,1], [1,2], [2,3]$, and $[3,4]$.

   The width of each subinterval is $\Delta x = 1$.

   Now apply the Trapezoidal Rule:

$$\begin{aligned}\int_0^4 x^3\,dx &\approx \frac{1}{2}[f(0)+f(1)]\Delta x + \frac{1}{2}[f(1)+f(2)]\Delta x + \frac{1}{2}[f(2)+f(3)]\Delta x + \frac{1}{2}[f(3)+f(4)]\Delta x \\ &= \frac{1}{2}[f(0)+2f(1)+2f(2)+2f(3)+f(4)]\Delta x \\ &= \frac{1}{2}\left[0^3 + 2(1^3) + 2(2^3) + 2(3^3) + 4^3\right](1) = \boxed{68}.\end{aligned}$$

   The answer is B.

3. Partition $[-2,6]$ into four subintervals, each of equal width: $[-2,0], [0,2], [2,4]$, and $[4,6]$.

   The width of each subinterval is $\Delta x = 2$.

   Now apply the Trapezoidal Rule:

$$\begin{aligned}\int_{-2}^6 e^{x^2}\,dx &\approx \frac{1}{2}[f(-2)+f(0)]\Delta x + \frac{1}{2}[f(0)+f(2)]\Delta x + \frac{1}{2}[f(2)+f(4)]\Delta x + \frac{1}{2}[f(4)+f(6)]\Delta x \\ &= \frac{1}{2}[f(-2)+2f(0)+2f(2)+2f(4)+f(6)]\Delta x \\ &= \frac{1}{2}\left[e^4 + 2\cdot e^0 + 2\cdot e^4 + 2\cdot e^{16} + e^{36}\right](2) \\ &= \boxed{2 + 3e^4 + 2e^{16} + e^{36}}.\end{aligned}$$

   The answer is A.

5. (a) Partition the interval $[0,4]$ into four subintervals of equal width $\Delta x = \frac{4-0}{4} = 1$.
   The four subintervals are $[0,1], [1,2], [2,3]$, and $[3,4]$.
   In each subinterval, choose u_i as the left endpoint of the ith interval.

Then $u_1 = 0,\ u_2 = 1,\ u_3 = 2$, and $u_4 = 3$.

$$\begin{aligned}\int_0^4 \frac{1}{1+x^3}\,dx &\approx \sum_{i=1}^{4} \frac{1}{1+u_i^3}\Delta x \\ &= [f(0)+f(1)+f(2)+f(3)]\Delta x \\ &= \left(1+\frac{1}{2}+\frac{1}{9}+\frac{1}{28}\right)\cdot 1 \approx \boxed{1.647}\end{aligned}$$

(b) Partition the interval $[0,4]$ into four subintervals of equal width $\Delta x = \frac{4-0}{4} = 1$.
The four subintervals are $[0,1],[1,2],[2,3]$, and $[3,4]$.
In each subinterval, choose u_i as the right endpoint of the ith interval.
Then $u_1 = 1,\ u_2 = 2,\ u_3 = 3$, and $u_4 = 4$.

$$\begin{aligned}\int_0^4 \frac{1}{1+x^3}\,dx &\approx \sum_{i=1}^{4} \frac{1}{1+u_i^3}\Delta x \\ &= [f(1)+f(2)+f(3)+f(4)]\Delta x \\ &= \left(\frac{1}{2}+\frac{1}{9}+\frac{1}{28}+\frac{1}{65}\right)\cdot 1 \approx \boxed{0.662}\end{aligned}$$

(c) Partition the interval $[0,4]$ into four subintervals of equal width $\Delta x = \frac{4-0}{4} = 1$.
The four subintervals are $[0,1],[1,2],[2,3]$, and $[3,4]$.
Now apply the Trapezoidal Rule:

$$\begin{aligned}\int_0^4 \frac{1}{1+x^3}\,dx &\approx \frac{1}{2}[f(0)+f(1)]\Delta x + \frac{1}{2}[f(1)+f(2)]\Delta x + \frac{1}{2}[f(2)+f(3)]\Delta x \\ &\quad + \frac{1}{2}[f(3)+f(4)]\Delta x \\ &= \frac{1}{2}[f(0)+2f(1)+2f(2)+2f(3)+f(4)]\Delta x \\ &= \frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+2\left(\frac{1}{9}\right)+2\left(\frac{1}{28}\right)+\frac{1}{65}\right](1) \\ &\approx \boxed{1.155}.\end{aligned}$$

7.7 Improper Integrals

Concepts and Vocabulary

1. (c), an improper
3. False
5. False

Skill Building

7. The integral is $\boxed{\text{improper}}$, since upper limit of integration is ∞.

9. The integral is $\boxed{\text{not improper}}$, since the function $f(x) = \frac{1}{x-1}$ is continuous on the closed interval $[2,3]$.

11. The integral is $\boxed{\text{improper}}$, since the function $f(x) = \frac{1}{x}$ is undefined, at the endpoint $x = 0$.

13. The integral is $\boxed{\text{improper}}$, since the function $f(x) = \frac{x}{x^2-1}$ is undefined, at the endpoint $x = 1$.

15. We evaluate

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^3} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} \\
&= \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b \\
&= \lim_{b\to\infty} \left[-\frac{1}{2b^2} - \left(-\frac{1}{2(1)^2}\right)\right] \\
&= \lim_{b\to\infty} \left[\frac{1}{2} - \frac{1}{2b^2}\right] \\
&= \frac{1}{2}.
\end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{2}}$.

17. We evaluate

$$\begin{aligned}
\int_0^\infty e^{2x}\,dx &= \lim_{b\to\infty} \int_0^b e^{2x}\,dx \\
&= \lim_{b\to\infty} \left[\frac{1}{2}e^{2x}\right]_0^b \\
&= \lim_{b\to\infty} \left[\frac{1}{2}e^{2(b)} - \frac{1}{2}e^{2(0)}\right] \\
&= \lim_{b\to\infty} \left[\frac{1}{2}e^{2b} - \frac{1}{2}\right] \\
&= \infty.
\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

19. We evaluate

$$\begin{aligned}
\int_{-\infty}^{-1} \frac{4}{x}\,dx &= \lim_{a\to-\infty} \int_a^{-1} \frac{4}{x}\,dx \\
&= \lim_{a\to-\infty} \left[4\ln|x|\right]_a^{-1} \\
&= \lim_{a\to-\infty} \left[4\ln|-1| - 4\ln|a|\right] \\
&= \lim_{a\to-\infty} \left[-4\ln|a|\right] \\
&= -\infty.
\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

21. We evaluate

$$\begin{aligned}
\int_3^\infty \frac{dx}{(x-1)^4} &= \lim_{b\to\infty}\int_3^b \frac{dx}{(x-1)^4} \\
&= \lim_{b\to\infty}\left[-\frac{1}{3(x-1)^3}\right]_3^b \\
&= \lim_{b\to\infty}\left[-\frac{1}{3(b-1)^3} - \left(-\frac{1}{3(3-1)^3}\right)\right] \\
&= \lim_{b\to\infty}\left[\frac{1}{24} - \frac{1}{3(b-1)^3}\right] \\
&= \frac{1}{24}.
\end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{24}}$.

23. We evaluate

$$\begin{aligned}
\int_{-\infty}^\infty \frac{dx}{x^2+4} &= \lim_{a\to-\infty}\int_a^0 \frac{dx}{x^2+4} + \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+4} \\
&= \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_a^0 + \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^b \\
&= \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\frac{0}{2} - \frac{1}{2}\tan^{-1}\frac{a}{2}\right] + \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{b}{2} - \frac{1}{2}\tan^{-1}\frac{0}{2}\right] \\
&= \lim_{a\to-\infty}\left[-\frac{1}{2}\tan^{-1}\frac{a}{2}\right] + \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{b}{2}\right] \\
&= \frac{-1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

The improper integral converges to $\boxed{\frac{\pi}{2}}$.

25. We evaluate

$$\begin{aligned}
\int_0^1 \frac{dx}{x^2} &= \lim_{a\to0^+}\int_a^1 \frac{dx}{x^2} \\
&= \lim_{a\to0^+}\left[-\frac{1}{x}\right]_a^1 \\
&= \lim_{a\to0^+}\left[-\frac{1}{1} - \left(-\frac{1}{a}\right)\right] \\
&= \lim_{a\to0^+}\left[\frac{1}{a} - 1\right] \\
&= \infty.
\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

27. We evaluate

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{a\to 0^+} \int_a^1 \frac{dx}{x} \\ &= \lim_{a\to 0^+} \left[\ln|x|\right]_a^1 \\ &= \lim_{a\to 0^+} \left[\ln|1| - \ln|a|\right] \\ &= \lim_{a\to 0^+} \left[-\ln|a|\right] \\ &= \infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

29. We evaluate

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{4-x}} &= \lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} \\ &= \lim_{b\to 4^-} \left[-2\sqrt{4-x}\right]_0^b \\ &= \lim_{b\to 4^-} \left[-2\sqrt{4-b} - \left(-2\sqrt{4-0}\right)\right] \\ &= \lim_{b\to 4^-} \left[4 - 2\sqrt{4-b}\right] \\ &= 4.\end{aligned}$$

The improper integral converges to $\boxed{4}$.

31. We split the integral into two improper integrals:

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}}.$$

We consider

$$\begin{aligned}\int_{-1}^0 \frac{dx}{\sqrt[3]{x}} &= \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt[3]{x}} \\ &= \lim_{b\to 0^-} \left[\frac{3}{2}x^{\frac{2}{3}}\right]_{-1}^b \\ &= \lim_{b\to 0^-} \left[\frac{3}{2}b^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}}\right] \\ &= \lim_{b\to 0^-} \left[\frac{3}{2}b^{\frac{2}{3}} - \frac{3}{2}\right] \\ &= -\frac{3}{2}.\end{aligned}$$

And

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt[3]{x}} &= \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt[3]{x}} \\
&= \lim_{a\to 0^+} \left[\frac{3}{2}x^{\frac{2}{3}}\right]_a^1 \\
&= \lim_{a\to 0^+} \left[\frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2}(a)^{\frac{2}{3}}\right] \\
&= \lim_{a\to 0^+} \left[\frac{3}{2} - \frac{3}{2}a^{\frac{2}{3}}\right] \\
&= \frac{3}{2}.
\end{aligned}$$

We conclude that the improper integral $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$ converges to $-\frac{3}{2} + \frac{3}{2} = \boxed{0}$.

33. We evaluate

$$\begin{aligned}
\int_0^\infty \cos x\, dx &= \lim_{b\to\infty} \int_0^b \cos x\, dx \\
&= \lim_{b\to\infty} [\sin x]_0^b \\
&= \lim_{b\to\infty} [\sin b - \sin 0] \\
&= \lim_{b\to\infty} [\sin b]\,.
\end{aligned}$$

Since this limit does not exist, the improper integral $\boxed{\text{diverges}}$.

35. We evaluate

$$\begin{aligned}
\int_{-\infty}^0 e^x\, dx &= \lim_{a\to -\infty} \int_a^0 e^x\, dx \\
&= \lim_{a\to -\infty} [e^x]_a^0 \\
&= \lim_{a\to -\infty} \left[e^0 - e^a\right] \\
&= \lim_{a\to -\infty} [1 - e^a] \\
&= 1 - 0 \\
&= 1.
\end{aligned}$$

The improper integral converges to $\boxed{1}$.

37. Let $u = x^2$ and substitute

$$\begin{aligned}
\int_0^{\pi/2} \frac{x\,dx}{\sin x^2} &= \int_0^{\pi^2/4} \frac{\frac{1}{2}du}{\sin u} \\
&= \frac{1}{2}\int_0^{\pi^2/4} \csc u\,du \\
&= \lim_{a\to 0^+} \frac{1}{2}\int_a^{\pi^2/4} \csc u\,du \\
&= \lim_{a\to 0^+} \left[-\frac{1}{2}\ln|\csc u + \cot u|\right]_a^{\pi^2/4} \\
&= \lim_{a\to 0^+} \left[-\frac{1}{2}\ln\left|\csc\frac{\pi^2}{4} + \cot\frac{\pi^2}{4}\right| + \frac{1}{2}\ln|\csc a + \cot a|\right] \\
&= \lim_{a\to 0^+} \left[-\frac{1}{2}\ln\left|\csc\frac{\pi^2}{4} + \cot\frac{\pi^2}{4}\right| + \frac{1}{2}\ln|\csc a + \cot a|\right] \\
&= \infty.
\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

39. We evaluate

$$\begin{aligned}
\int_0^1 \frac{dx}{1-x^2} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(1-x)(1+x)} \\
&= \lim_{b\to 1^-}\int_0^b \left(\frac{1}{2(x+1)} - \frac{1}{2(x-1)}\right)dx \\
&= \lim_{b\to 1^-}\left[\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln|x-1|\right]_0^b \\
&= \lim_{b\to 1^-}\left[\frac{1}{2}\ln(b+1) - \frac{1}{2}\ln|b-1| - \left(\frac{1}{2}\ln(0+1) - \frac{1}{2}\ln|0-1|\right)\right] \\
&= \lim_{b\to 1^-}\left[\frac{1}{2}\ln(b+1) - \frac{1}{2}\ln|b-1|\right] \\
&= \infty.
\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

41. The function $f(x) = \frac{x}{\sqrt{1-x^2}}$ is continuous on $[0,1)$ but is not defined at $x = 1$, so $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ is an improper integral.

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-}\int_0^b \left(1-x^2\right)^{-1/2} x\,dx \\
&= -\frac{1}{2}\lim_{b\to 1^-}\left[\sqrt{1-x^2}\right]_0^b \\
&= -\frac{1}{2}\lim_{b\to 1^-}\left(\sqrt{1-b^2} - \sqrt{1-0}\right) = 1.
\end{aligned}$$

Therefore, $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ converges to $\boxed{1}$.

43. We evaluate

$$\begin{aligned}\int_0^{\pi/4} \tan(2x)\,dx &= \lim_{b\to\frac{\pi}{4}^-}\int_0^b \tan(2x)\,dx \\ &= \lim_{b\to\frac{\pi}{4}^-}\left[\frac{1}{2}\ln|\sec 2x|\right]_0^b \\ &= \lim_{b\to\frac{\pi}{4}^-}\left(\frac{1}{2}\ln|\sec 2b|\right) \\ &= \infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

45. To evaluate $\int_0^\infty \frac{x\,dx}{(x+1)^{5/2}}$, use the substitution $u=\sqrt{x+1}$. Then $x=u^2-1$ and $dx=2u\,du$.

The lower bound becomes $u=\sqrt{0+1}=1$.

For the upper bound, as x approaches ∞, u also approaches ∞.

The improper integral becomes

$$\int_0^\infty \frac{x\,dx}{(x+1)^{5/2}} = \int_1^\infty \frac{(u^2-1)2u\,du}{u^5} = 2\int_1^\infty\left(\frac{u^3-u}{u^5}\right)du = 2\int_1^\infty\left(\frac{1}{u^2}-\frac{1}{u^4}\right)du.$$

By definition,

$$\begin{aligned}2\int_1^\infty\left(\frac{1}{u^2}-\frac{1}{u^4}\right)du &= 2\lim_{b\to\infty}\int_1^b\left(\frac{1}{u^2}-\frac{1}{u^4}\right)du = 2\lim_{b\to\infty}\left[-\frac{1}{u}+\frac{1}{3u^3}\right]_1^b \\ &= 2\lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{3b^3}-\left(-1+\frac{1}{3}\right)\right] = \frac{4}{3}.\end{aligned}$$

Therefore, $\int_0^\infty \frac{x\,dx}{(x+1)^{5/2}}$ converges to $\boxed{\frac{4}{3}}$.

47. We split the integral into two improper integrals:

$$\int_{-\infty}^\infty \frac{dx}{x^2+4x+5} = \int_{-\infty}^0 \frac{dx}{x^2+4x+5} + \int_0^\infty \frac{dx}{x^2+4x+5}.$$

We consider

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2+4x+5} &= \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+4x+5} \\ &= \lim_{b\to\infty}\int_0^b \frac{dx}{(x+2)^2+1} \\ &= \lim_{b\to\infty}\left[\tan^{-1}(x+2)\right]_0^b \\ &= \lim_{b\to\infty}\left[\tan^{-1}(b+2)-\tan^{-1}(0+2)\right] \\ &= \frac{\pi}{2}-\tan^{-1}2.\end{aligned}$$

We now determine

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{x^2+4x+5} &= \lim_{a\to-\infty} \int_a^0 \frac{dx}{x^2+4x+5} \\ &= \lim_{a\to-\infty} \int_a^0 \frac{dx}{(x+2)^2+1} \\ &= \lim_{a\to-\infty} \left[\tan^{-1}(x+2)\right]_a^0 \\ &= \lim_{a\to-\infty} \left[\tan^{-1}(0+2) - \tan^{-1}(a+2)\right] \\ &= \tan^{-1} 2 - \left(-\frac{\pi}{2}\right) \\ &= \tan^{-1} 2 + \frac{\pi}{2}.\end{aligned}$$

We conclude that the improper integral converges, with $\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5} = \left(\frac{\pi}{2} - \tan^{-1} 2\right) + \left(\tan^{-1} 2 + \frac{\pi}{2}\right) = \boxed{\pi}$.

49. We evaluate

$$\begin{aligned}\int_{-\infty}^{2} \frac{dx}{\sqrt{4-x}} &= \lim_{a\to-\infty} \int_a^2 \frac{dx}{\sqrt{4-x}} \\ &= \lim_{a\to-\infty} \left[-2\sqrt{4-x}\right]_a^2 \\ &= \lim_{a\to-\infty} \left[-2\sqrt{4-2} - \left(-2\sqrt{4-a}\right)\right] \\ &= \lim_{a\to-\infty} \left[2\sqrt{4-a} - 2\sqrt{2}\right] \\ &= \infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

51. We evaluate

$$\begin{aligned}\int_{2}^{4} \frac{2x\,dx}{\sqrt[3]{x^2-4}} &= \lim_{a\to2^+} \int_a^4 \frac{2x\,dx}{\sqrt[3]{x^2-4}} \\ &= \lim_{a\to2^+} \left[\frac{3}{2}\left(x^2-4\right)^{2/3}\right]_a^4 \\ &= \lim_{a\to2^+} \left[\frac{3}{2}\left(4^2-4\right)^{2/3} - \frac{3}{2}\left(a^2-4\right)^{2/3}\right] \\ &= \lim_{a\to2^+} \left[3(18)^{1/3} - \frac{3}{2}\left(a^2-4\right)^{2/3}\right] \\ &= \boxed{3\sqrt[3]{18}}.\end{aligned}$$

The improper integral converges to $\boxed{3\sqrt[3]{18}}$.

53. We split the integral into two improper integrals:

$$\int_{-1}^{1} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{1} \frac{dx}{x^3}.$$

We consider

$$\begin{aligned}\int_{-1}^{0}\frac{dx}{x^3} &= \lim_{b\to 0^-}\int_{-1}^{b}\frac{dx}{x^3}\\ &= \lim_{b\to 0^-}\left[-\frac{1}{2x^2}\right]_{-1}^{b}\\ &= \lim_{b\to 0^-}\left[-\frac{1}{2b^2}-\left(-\frac{1}{2(-1)^2}\right)\right]\\ &= \lim_{b\to 0^-}\left[\frac{1}{2}-\frac{1}{2b^2}\right]\\ &= -\infty.\end{aligned}$$

We conclude that the improper integral $\int_{-1}^{1}\frac{dx}{x^3}$ $\boxed{\text{diverges}}$.

55. We split the integral into two improper integrals:

$$\int_0^2\frac{dx}{(x-1)^{1/3}}=\int_0^1\frac{dx}{(x-1)^{1/3}}+\int_1^2\frac{dx}{(x-1)^{1/3}}.$$

We consider

$$\begin{aligned}\int_0^1\frac{dx}{(x-1)^{1/3}} &= \lim_{b\to 1^-}\int_0^b\frac{dx}{(x-1)^{1/3}}\\ &= \lim_{b\to 1^-}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_0^b\\ &= \lim_{b\to 1^-}\left[\frac{3}{2}(b-1)^{\frac{2}{3}}-\frac{3}{2}(0-1)^{\frac{2}{3}}\right]\\ &= \lim_{b\to 1^-}\left[\frac{3}{2}(b-1)^{\frac{2}{3}}-\frac{3}{2}\right]\\ &= -\frac{3}{2}.\end{aligned}$$

And

$$\begin{aligned}\int_1^2\frac{dx}{(x-1)^{1/3}} &= \lim_{a\to 1^+}\int_a^2\frac{dx}{(x-1)^{1/3}}\\ &= \lim_{a\to 1^+}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_a^2\\ &= \lim_{a\to 1^+}\left[\frac{3}{2}(2-1)^{\frac{2}{3}}-\frac{3}{2}(a-1)^{\frac{2}{3}}\right]\\ &= \lim_{a\to 1^+}\left[\frac{3}{2}-\frac{3}{2}(a-1)^{\frac{2}{3}}\right]\\ &= \frac{3}{2}.\end{aligned}$$

We conclude that the improper integral $\int_0^2\frac{dx}{(x-1)^{1/3}}$ converges to $-\frac{3}{2}+\frac{3}{2}=\boxed{0}$.

57. We evaluate

$$\begin{aligned}
\int_1^2 \frac{dx}{(2-x)^{3/4}} &= \lim_{b\to 2^-} \int_1^b \frac{dx}{(2-x)^{3/4}} \\
&= \lim_{b\to 2^-} \left[-4(2-x)^{1/4}\right]_1^b \\
&= \lim_{b\to 2^-} \left[-4(2-b)^{1/4} - \left(-4(2-1)^{1/4}\right)\right] \\
&= \lim_{b\to 2^-} \left[4 - 4\sqrt[4]{2-b}\right] \\
&= 4.
\end{aligned}$$

The improper integral converges to $\boxed{4}$.

59. The function $f(x) = \frac{2x}{(x^2-1)^{3/2}}$ is continuous on $(1,3]$ but is not defined at $x = 1$, so $\int_1^3 \frac{2x\,dx}{(x^2-1)^{3/2}}$ is an improper integral.

$$\begin{aligned}
\int_1^3 \frac{2x\,dx}{(x^2-1)^{3/2}} &= \lim_{b\to 1^+} \int_b^3 \left(x^2-1\right)^{-3/2}(2x\,dx) \\
&= -2\lim_{b\to 1^+} \left[\left(x^2-1\right)^{-1/2}\right]_b^3 \\
&= -2\lim_{b\to 1^+} \left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{b^2-1}}\right)
\end{aligned}$$

Since

$$\lim_{b\to 1^+} \frac{1}{\sqrt{b^2-1}} = \infty, \int_1^3 \frac{2x\,dx}{(x^2-1)^{3/2}} = -2\lim_{b\to 1^+}\left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{b^2-1}}\right) = \infty.$$

Therefore,

$$\int_1^3 \frac{2x\,dx}{(x^2-1)^{3/2}} \boxed{\text{diverges}}.$$

61. We evaluate

$$\begin{aligned}
\int_0^\infty xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_0^b \\
&= \lim_{b\to\infty} \left[-\frac{1}{2}e^{-b^2} - \left(-\frac{1}{2}e^{-0^2}\right)\right] \\
&= \lim_{b\to\infty} \left[\frac{1}{2} - \frac{1}{2}e^{-b^2}\right] \\
&= \frac{1}{2}.
\end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{2}}$.

63. (a) We have

$$\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x}.$$

By the comparison test, since $p = 1$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges, we conclude that $\int_1^\infty \frac{1}{\sqrt{x^2-1}}\,dx$ $\boxed{\text{diverges}}$.

65. (a) We have

$$\frac{1+e^{-x}}{x} \geq \frac{1}{x}.$$

By the comparison test, since $p = 1$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges, we conclude that $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ $\boxed{\text{diverges}}$.

67. (a) We have

$$\sin^2 x \leq 1$$

so

$$\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}.$$

By the comparison test, since $p = 2$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, we conclude that $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ $\boxed{\text{converges}}$.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \approx \boxed{0.673}.$$

69. (a) We have

$$\frac{1}{(x+1)\sqrt{x}} \leq \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}.$$

By the comparison test, since $p = \frac{3}{2}$ and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, we conclude that $\int_1^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$ $\boxed{\text{converges}}$.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{1}{(x+1)\sqrt{x}}\,dx = \boxed{\frac{\pi}{2}}.$$

Applications and Extensions

71. The area is given by the improper integral $\int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx$. We evaluate

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx &= \lim_{b\to\infty} \int_0^b \left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx \\
&= \lim_{b\to\infty} \left[\ln|x+1| - \ln|x+2|\right]_0^b \\
&= \lim_{b\to\infty} \left[\ln|b+1| - \ln|b+2| - (\ln|0+1| - \ln|0+2|)\right] \\
&= \lim_{b\to\infty} \left[\ln 2 + \ln|b+1| - \ln|b+2|\right] \\
&= \lim_{b\to\infty} \left[\ln 2 + \ln\left|\frac{b+1}{b+2}\right|\right] \\
&= \boxed{\ln 2}.
\end{aligned}$$

73. The volume is given by the improper integral $\int_0^\infty \pi\left(e^{-x}\right)^2 dx$. We evaluate

$$\begin{aligned}
\int_0^\infty \pi\left(e^{-x}\right)^2 dx &= \lim_{b\to\infty} \int_0^b \pi e^{-2x}\, dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{2}\pi e^{-2x}\right]_0^b \\
&= \lim_{b\to\infty} \left[-\frac{1}{2}\pi e^{-2b} - \left(-\frac{1}{2}\pi e^{-2(0)}\right)\right] \\
&= \lim_{b\to\infty} \left[\frac{1}{2}\pi - \frac{1}{2}\pi e^{-2b}\right] \\
&= \boxed{\tfrac{\pi}{2}}.
\end{aligned}$$

75. The area is given by the improper integral $\int_{-\infty}^\infty \frac{8a^3}{x^2+4a^2}\, dx = \int_{-\infty}^0 \frac{8a^3}{x^2+4a^2}\, dx + \int_0^\infty \frac{8a^3}{x^2+4a^2}\, dx$. We evaluate first

$$\begin{aligned}
\int_{-\infty}^0 \frac{8a^3}{x^2+4a^2}\, dx &= \lim_{t\to-\infty} \int_t^0 \frac{8a^3}{x^2+4a^2}\, dx \\
&= \lim_{t\to-\infty} \left[\left(8a^3\right)\frac{1}{2a}\tan^{-1}\left(\frac{x}{2a}\right)\right]_t^0 \\
&= \lim_{t\to-\infty} \left[\left(8a^3\right)\frac{1}{2a}\tan^{-1}\left(\frac{0}{2a}\right) - \left(8a^3\right)\frac{1}{2a}\tan^{-1}\left(\frac{t}{2a}\right)\right] \\
&= \lim_{t\to-\infty} \left[-4a^2\tan^{-1}\left(\frac{t}{2a}\right)\right] \\
&= \left(-4a^2\right)\left(-\frac{\pi}{2}\right) \\
&= 2\pi a^2.
\end{aligned}$$

We next evaluate

$$\begin{aligned}
\int_0^\infty \frac{8a^3}{x^2+4a^2}\, dx &= \lim_{t\to\infty} \int_0^t \frac{8a^3}{x^2+4a^2}\, dx \\
&= \lim_{t\to\infty} \left[\left(8a^3\right)\frac{1}{2a}\tan^{-1}\left(\frac{x}{2a}\right)\right]_0^t \\
&= \lim_{t\to\infty} \left[\left(8a^3\right)\frac{1}{2a}\tan^{-1}\left(\frac{t}{2a}\right) - \left(8a^3\right)\frac{1}{2a}\tan^{-1}\left(\frac{0}{2a}\right)\right] \\
&= \lim_{t\to\infty} \left[4a^2\tan^{-1}\left(\frac{t}{2a}\right)\right] \\
&= \left(4a^2\right)\left(\frac{\pi}{2}\right) \\
&= 2\pi a^2.
\end{aligned}$$

So the area under the graph is $2\pi a^2 + 2\pi a^2 = \boxed{4\pi a^2}$.

77. (a) We have $R(t) = 100$, and $r = 0.08$. So the present value of the asset is given by the improper integral

$$\begin{aligned}\int_0^{\infty} R(t)e^{-rt}\,dt &= \lim_{b\to\infty}\int_0^b 100e^{-(0.08)t}\,dt\\ &= \lim_{b\to\infty}\left[\frac{100}{-0.08}e^{-0.08t}\right]_0^b\\ &= \lim_{b\to\infty}\left[\frac{100}{-0.08}e^{-0.08b}-\left(\frac{100}{-0.08}e^{-0.08(0)}\right)\right]\\ &= \lim_{b\to\infty}\left[1250-1250e^{-0.08b}\right]\\ &= 1250.\end{aligned}$$

The present value of the asset is \$1250.00.

(b) Now with $R(t) = 1000 + 80t$, and $r = 0.07$, the present value of the asset is given by the improper integral

$$\begin{aligned}\int_0^{\infty} R(t)e^{-rt}\,dt &= \lim_{b\to\infty}\int_0^b (1000+80t)e^{-(0.07)t}\,dt\\ &= \lim_{b\to\infty}\left[1000\int_0^b e^{-(0.07)t}\,dt + 80\int_0^b te^{-at}\,dt\right]\\ &= \lim_{b\to\infty}\left[1000\frac{1}{-0.07}e^{-0.07t}+80\left(-\frac{1}{(0.07)^2}e^{-(0.07)t}((0.07)t+1)\right)\right]_0^b\\ &= \lim_{b\to\infty}\left[1000\frac{1}{-0.07}e^{-0.07b}+80\left(-\frac{1}{(0.07)^2}e^{-(0.07)b}((0.07)b+1)\right)\right.\\ &\qquad\left.-\left(1000\frac{1}{-0.07}e^{-0.07(0)}+80\left(-\frac{1}{(0.07)^2}e^{-(0.07)(0)}((0.07)(0)+1)\right)\right)\right]\\ &= \lim_{b\to\infty}\left[\frac{1000}{-0.07}e^{-0.07b}+80\left(-\frac{1}{(0.07)^2}e^{-(0.07)b}((0.07)b+1)\right)\right.\\ &\qquad\left.-\left(\frac{1000}{-0.07}-\frac{80}{(0.07)^2}\right)\right]\\ &= -\left(\frac{1000}{-0.07}-\frac{80}{(0.07)^2}\right)\\ &\approx 30,612.\end{aligned}$$

The present value of the asset is approximately $\boxed{\$30,612}$.

79. We evaluate the improper integral

$$\begin{aligned}
\frac{2\pi NIr}{10}\int_x^\infty \frac{dy}{(r^2+y^2)^{3/2}} &= \frac{2\pi NIr}{10}\lim_{b\to\infty}\int_x^b \frac{dy}{(r^2+y^2)^{3/2}} \\
&= \frac{2\pi NIr}{10}\lim_{b\to\infty}\left[\frac{1}{r^2}\frac{y}{\sqrt{r^2+y^2}}\right]_x^b \quad \text{(by Formula 61)} \\
&= \frac{2\pi NIr}{10}\lim_{b\to\infty}\left[\frac{1}{r^2}\frac{b}{\sqrt{r^2+b^2}} - \frac{1}{r^2}\frac{x}{\sqrt{r^2+x^2}}\right] \\
&= \frac{2\pi NIr}{10}\lim_{b\to\infty}\left[\frac{1}{r^2}\frac{1}{\sqrt{(r/b)^2+1}} - \frac{1}{r^2}\frac{x}{\sqrt{r^2+x^2}}\right] \\
&= \frac{2\pi NIr}{10}\frac{1}{r^2}\left(1-\frac{x}{\sqrt{r^2+x^2}}\right) \\
&= \boxed{\tfrac{\pi NI}{5r}\left(1-\tfrac{x}{\sqrt{r^2+x^2}}\right)}.
\end{aligned}$$

81. We have $W = \int_1^\infty F(r)\,dr$, where F is the force acting on an object. So we evaluate the improper integral

$$\begin{aligned}
\int_1^\infty F(r)\,dr &= \lim_{b\to\infty}\int_1^b \frac{GmM}{r^2}\,dr \\
&= \lim_{b\to\infty}\left[-\frac{GmM}{r}\right]_1^b \\
&= \lim_{b\to\infty}\left[-\frac{GmM}{b} - \left(-\frac{GmM}{1}\right)\right] \\
&= \lim_{b\to\infty}\left[GmM - \frac{GmM}{b}\right] \\
&= \boxed{GmM}.
\end{aligned}$$

83. We evaluate the improper integral

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx.$$

Let $u = x$, $du = dx$, and $dv = e^{-x}dx$, $v = -e^{-x}$, and use integration by parts.

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\left[\left[x\left(-e^{-x}\right)\right]_0^b - \int_0^b \left(-e^{-x}\right)dx\right] \\
&= \lim_{b\to\infty}\left[-be^{-b} + \int_0^b e^{-x}\,dx\right] \\
&= \lim_{b\to\infty}\left[-be^{-b} + \left[-e^{-x}\right]_0^b\right] \\
&= \lim_{b\to\infty}\left[-be^{-b} + \left[-e^{-b} - \left(-e^{-0}\right)\right]\right] \\
&= \lim_{b\to\infty}\left[-be^{-b} + 1 - e^{-b}\right] \\
&= \boxed{1}.
\end{aligned}$$

85. We evaluate the improper integral

$$\int_0^{\infty} e^{-x}\cos x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos x\,dx.$$

We use integration by parts to obtain formula 126, and so

$$\begin{aligned}\int_0^{\infty} e^{-x}\cos x\,dx &= \lim_{b\to\infty}\left[\frac{1}{2}e^{-x}(-\cos x+\sin x)\right]_0^b\\ &= \lim_{b\to\infty}\left[\frac{1}{2}e^{-b}(-\cos b+\sin b)-\left(\frac{1}{2}e^{-0}(-\cos 0+\sin 0)\right)\right]\\ &= \lim_{b\to\infty}\left[\frac{1}{2}-\frac{1}{2}e^{-b}(\cos b-\sin b)\right]\\ &= \boxed{\tfrac{1}{2}}.\end{aligned}$$

87. By definition, $\int_0^{\infty}\frac{dx}{(x^2+4)^{3/2}} = \lim_{b\to\infty}\int_0^b \frac{dx}{(x^2+4)^{3/2}}$.

To evaluate $\int \frac{dx}{(x^2+4)^{3/2}}$, use the substitution $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\sec^2\theta} = 2\sec\theta$ since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

When $x = 0$, $2\tan\theta = 0$ and so $\theta = 0$. As $x\to\infty, \tan\theta\to\infty$ and so $\theta\to\frac{\pi}{2}^{-}$.

So,

$$\begin{aligned}\int_0^{\infty}\frac{dx}{(x^2+4)^{3/2}} &= \lim_{b\to\infty}\int_0^b\frac{dx}{(x^2+4)^{3/2}} = \lim_{b\to\pi/2}\int_0^b\frac{1}{8\sec^3\theta}2\sec^2\theta\,d\theta = \frac{1}{4}\lim_{b\to\pi/2}\int_0^b\cos\theta\,d\theta\\ &= \frac{1}{4}\lim_{b\to\pi/2}[\sin\theta]_0^b = \frac{1}{4}\lim_{b\to\pi/2}(\sin b-0) = \frac{1}{4}(1-0) = \frac{1}{4}.\end{aligned}$$

Therefore, $\int_0^{\infty}\frac{dx}{(x^2+4)^{3/2}}$ converges to $\boxed{\frac{1}{4}}$.

89. By definition, $\int_1^{\infty}\frac{dx}{(x+1)\sqrt{2x+x^2}} = \lim_{b\to\infty}\int_1^b\frac{dx}{(x+1)\sqrt{2x+x^2}}$.

To evaluate $\int\frac{dx}{(x+1)\sqrt{2x+x^2}}$, begin by completing the square on the expression $2x+x^2$.

$$2x+x^2 = x^2+2x+1-1 = (x+1)^2-1.$$

The integral becomes $\int\frac{dx}{(x+1)\sqrt{2x+x^2}} = \int\frac{dx}{(x+1)\sqrt{(x+1)^2-1}}$.

Use the substitution $u = x+1$. Then $du = dx$ and $\int\frac{dx}{(x+1)\sqrt{2x+x^2}} = \int\frac{dx}{(x+1)\sqrt{(x+1)^2-1}} = \int\frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec} u + C = \operatorname{arcsec}(x+1)+C$.

So,

$$\begin{aligned}\int_1^{\infty}\frac{dx}{(x+1)\sqrt{2x+x^2}} &= \lim_{b\to\infty}\int_1^b\frac{dx}{(x+1)\sqrt{2x+x^2}} = \lim_{b\to\infty}[\operatorname{arcsec}(x+1)]_1^b\\ &= \lim_{b\to\infty}[\operatorname{arcsec}(b+1)-\operatorname{arcsec}(2)] = \frac{\pi}{2}-\frac{\pi}{3} = \frac{\pi}{6}.\end{aligned}$$

Therefore, $\int_1^{\infty}\frac{dx}{(x+1)\sqrt{2x+x^2}}$ converges to $\boxed{\frac{\pi}{6}}$.

91. We evaluate the improper integral

$$\begin{aligned}\int_0^{\infty} \sin x\,dx &= \lim_{b\to\infty}\int_0^b \sin x\,dx\\ &= \lim_{b\to\infty}\left[-\cos x\right]_0^b\\ &= \lim_{b\to\infty}\left[-\cos b-(-\cos 0)\right]\\ &= \lim_{b\to\infty}\left[1-\cos b\right].\end{aligned}$$

Since this limit does not exist, the improper integral $\int_0^{\infty}\sin x\,dx$ $\boxed{\text{diverges}}$. We now evaluate the improper integral

$$\begin{aligned}\int_{-\infty}^{0} \sin x\,dx &= \lim_{a\to-\infty}\int_a^0 \sin x\,dx\\ &= \lim_{a\to-\infty}\left[-\cos x\right]_a^0\\ &= \lim_{a\to-\infty}\left[-\cos 0-(-\cos a)\right]\\ &= \lim_{a\to-\infty}\left[\cos a-1\right].\end{aligned}$$

Since this limit does not exist, the improper integral $\int_{-\infty}^{0}\sin x\,dx$ $\boxed{\text{diverges}}$. We now determine

$$\begin{aligned}\lim_{t\to\infty}\int_{-t}^{t} \sin x\,dx &= \lim_{t\to\infty}\left[\cos x\right]_{-t}^{t}\\ &= \lim_{t\to\infty}\left[\cos t-(\cos(-t))\right]\\ &= \lim_{t\to\infty}\left[\cos t-\cos(t)\right]\\ &= \lim_{t\to\infty}\left[0\right]\\ &= \boxed{0}.\end{aligned}$$

93. Since

$$\frac{1}{\sqrt{2+\sin x}} \geq \frac{1}{\sqrt{2+1}} = \frac{1}{\sqrt{3}}$$

for all x, and since $\int_0^{\infty}\frac{1}{3}\,dx$ diverges, by the Comparison Test, $\int_0^{\infty}\frac{1}{\sqrt{2+\sin x}}\,dx$ diverges.

95. (a) We evaluate the improper integral, and let $u = x^n$, $du = nx^{n-1}\,dx$, $dv = e^{-x}\,dx$, and $v = -e^{-x}$. We obtain

$$\begin{aligned}\int_0^\infty x^n e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b x^n e^{-x}\,dx\\ &= \lim_{b\to\infty}\left[\left[x^n\left(-e^{-x}\right)\right]_0^b - \int_0^b nx^{n-1}\left(-e^{-x}\right)dx\right]\\ &= \lim_{b\to\infty}\left[-b^n e^{-b} + n\int_0^b x^{n-1}e^{-x}\,dx\right]\\ &= \lim_{b\to\infty}\left[-b^n e^{-b}\right] + \lim_{b\to\infty}\left[n\int_0^b x^{n-1}e^{-x}\,dx\right]\\ &= 0 + n\lim_{b\to\infty}\left[\int_0^b x^{n-1}e^{-x}\,dx\right]\\ &= n\int_0^\infty x^{n-1}e^{-x}\,dx.\end{aligned}$$

(b) Using part (a) repeatedly, we obtain

$$\begin{aligned}\int_0^\infty x^n e^{-x}\,dx &= n\int_0^\infty x^{n-1}e^{-x}\,dx\\ &= n(n-1)\int_0^\infty x^{n-2}e^{-x}\,dx\\ &= \cdots\\ &= [n(n-1)\cdots 1]\int_0^\infty e^{-x}\,dx\\ &= (n!)\lim_{b\to\infty}\int_0^b e^{-x}\,dx\\ &= (n!)\lim_{b\to\infty}\left[-e^{-x}\right]_0^b\\ &= (n!)\lim_{b\to\infty}\left[-e^{-b} - \left(-e^{-0}\right)\right]\\ &= (n!)\lim_{b\to\infty}\left[1 - e^{-b}\right]\\ &= n!.\end{aligned}$$

97. We evaluate the improper integral, and obtain for $p \neq 1$

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^p} &= \lim_{t\to a^+}\int_t^b \frac{dx}{(x-a)^p}\\ &= \lim_{t\to a^+}\left[\frac{(x-a)^{1-p}}{1-p}\right]_t^b\\ &= \lim_{t\to a^+}\left[\frac{(b-a)^{1-p}}{1-p} - \frac{(t-a)^{1-p}}{1-p}\right].\end{aligned}$$

If $p > 1$, then $1 - p < 0$, and

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{t\to a^+}\left[\frac{(b-a)^{1-p}}{1-p} - \frac{(t-a)^{1-p}}{1-p}\right] = \infty.$$

If $0 < p < 1$, then $1 - p > 0$, and

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{t\to a^+}\left[\frac{(b-a)^{1-p}}{1-p} - \frac{(t-a)^{1-p}}{1-p}\right] = \frac{(b-a)^{1-p}}{1-p}.$$

And if $p = 1$, we have

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^p} &= \lim_{t\to a^+}\int_t^b \frac{dx}{x-a} \\ &= \lim_{t\to a^+}\left[\ln|x-a|\right]_t^b \\ &= \lim_{t\to a^+}\left[\ln|b-a| - \ln|t-a|\right] \\ &= \infty.\end{aligned}$$

So $\int_a^b \frac{dx}{(x-a)^p}$ converges if $0 < p < 1$, and diverges if $p \geq 1$.

99. If $p = 0$, then $\frac{1}{(x-a)^p} = 1$ for all x in $(a, b]$, and so $\int_a^b \frac{dx}{(x-a)^p} = b - a$. Likewise, $\frac{1}{(b-a)^p} = 1$ for all x in $[a, b)$, so $\int_a^b \frac{dx}{(b-x)^p} = 1$. And if $p < 0$, then $\frac{1}{(x-a)^p}$ and $\frac{1}{(b-a)^p}$ are continuous on $[a, b]$, so the integrals $\int_a^b \frac{dx}{(x-a)^p}$ and $\int_a^b \frac{dx}{(b-x)^p}$ are ordinary definite integrals, and hence converge. For example, $\int_1^2 \frac{dx}{(x-1)^{-1}} = \int_1^2 (x-1)\,dx = \left[\frac{1}{2}(x-1)^2\right]_1^2 = \frac{1}{2}$.

101. We evaluate, for $s > 0$,

$$\begin{aligned}L\{f(x)\} &= \int_0^\infty e^{-sx} f(x)\,dx \\ &= \int_0^\infty e^{-sx} x\,dx \\ &= \lim_{b\to\infty}\int_0^b e^{-sx} x\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{s^2}e^{-sx}(sx+1)\right]_0^b \\ &= \lim_{b\to\infty}\left[-\frac{1}{s^2}e^{-bs}(bs+1) - \left(-\frac{1}{s^2}e^{-(0)s}((0)s+1)\right)\right] \\ &= \lim_{b\to\infty}\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-bs}(bs+1)\right] \\ &= \boxed{\frac{1}{s^2}}.\end{aligned}$$

103. We evaluate, for $s > 0$,

$$\begin{aligned} L\{f(x)\} &= \int_0^\infty e^{-sx} f(x)\,dx \\ &= \int_0^\infty e^{-sx} \sin x\,dx \\ &= \lim_{b\to\infty} \int_0^b e^{-sx} \sin x\,dx \\ &= \lim_{b\to\infty} \left[\frac{-e^{-sx}(\cos x + s\sin x)}{s^2+1}\right]_0^b \\ &= \lim_{b\to\infty} \left[\frac{-e^{-bs}(\cos b + s\sin b)}{s^2+1} - \frac{-e^{-(0)s}(\cos(0) + s\sin(0))}{s^2+1}\right] \\ &= \lim_{b\to\infty} \left[\frac{1}{s^2+1} - e^{-bs}\frac{\cos b + s\sin b}{s^2+1}\right] \\ &= \boxed{\tfrac{1}{s^2+1}}. \end{aligned}$$

105. We evaluate, for $s > a$,

$$\begin{aligned} L\{f(x)\} &= \int_0^\infty e^{-sx} f(x)\,dx \\ &= \int_0^\infty e^{-sx} e^{ax}\,dx \\ &= \lim_{b\to\infty} \int_0^b e^{x(a-s)}\,dx \\ &= \lim_{b\to\infty} \left[\frac{e^{x(a-s)}}{a-s}\right]_0^b \\ &= \lim_{b\to\infty} \left[\frac{e^{b(a-s)}}{a-s} - \frac{e^{0(a-s)}}{a-s}\right] \\ &= \lim_{b\to\infty} \left[\frac{e^{-b(s-a)}}{a-s} - \frac{1}{a-s}\right] \\ &= \boxed{\tfrac{1}{s-a}}. \end{aligned}$$

But for $s \le a$, $\int_0^\infty e^{-sx} e^x\,dx$ diverges.

Challenge Problems

107. The arc length is given by $\int_0^1 \sqrt{1 + (dy/dx)^2}\,dx$. We differentiate, and obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\sqrt{x - x^2} - \sin^{-1}\sqrt{x}\right) \\ &= \frac{1}{2}\left(x - x^2\right)^{-1/2}(1-2x) - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{1}{2\sqrt{x}} \\ &= \frac{1-2x}{2\sqrt{x-x^2}} - \frac{1}{2\sqrt{x-x^2}} \\ &= \frac{-x}{\sqrt{x-x^2}}. \end{aligned}$$

So

$$\begin{aligned}\sqrt{1+(dy/dx)^2} &= \sqrt{1+\left(\frac{-x}{\sqrt{x-x^2}}\right)^2}\\ &= \sqrt{1+\frac{x^2}{x-x^2}}\\ &= \frac{1}{\sqrt{1-x}}.\end{aligned}$$

We now evaluate the improper integral.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x}}\,dx &= \lim_{b\to 1^-}\int_0^b (1-x)^{-1/2}\,dx\\ &= \lim_{b\to 1^-}\left[-2\sqrt{1-x}\right]_0^b\\ &= \lim_{b\to 1^-}\left[-2\sqrt{1-b}-\left(-2\sqrt{1-0}\right)\right]\\ &= \lim_{b\to 1^-}\left[2-2\sqrt{1-b}\right]\\ &= \boxed{2}.\end{aligned}$$

109. We split the integral into two improper integrals:

$$\int_{-\infty}^{\infty} e^{(x-e^x)}\,dx = \lim_{x\to-\infty}\int_x^0 e^x e^{-e^x}\,dx + \lim_{x\to\infty}\int_0^x e^x e^{-e^x}\,dx.$$

For the first integral, let $u = e^x$, then $du = e^x\,dx$, and we obtain

$$\begin{aligned}\lim_{x\to-\infty}\int_x^0 e^x e^{-e^x}\,dx &= \lim_{x\to-\infty}\int_{e^x}^1 e^{-u}\,du\\ &= \lim_{x\to-\infty}\left[-e^{-u}\right]_{e^x}^1\\ &= \lim_{x\to-\infty}\left[-e^{-1}-\left(-e^{-e^x}\right)\right]\\ &= \lim_{x\to-\infty}\left[e^{-e^x}-e^{-1}\right]\\ &= 1-\frac{1}{e}.\end{aligned}$$

For the second integral, let $u = e^x$, then $du = e^x\,dx$, and we obtain

$$\begin{aligned}\lim_{x\to\infty}\int_0^x e^x e^{-e^x}\,dx &= \lim_{x\to\infty}\int_1^{e^x} e^{-u}\,du\\ &= \lim_{x\to\infty}\left[-e^{-u}\right]_1^{e^x}\\ &= \lim_{x\to\infty}\left[-e^{-e^x}-\left(-e^{-1}\right)\right]\\ &= \lim_{x\to\infty}\left[e^{-1}-e^{-e^x}\right]\\ &= \frac{1}{e}.\end{aligned}$$

So we have

$$\int_{-\infty}^{\infty} e^{(x-e^x)}\,dx = \left(1-\frac{1}{e}\right)+\frac{1}{e} = \boxed{1}.$$

111. Since $a < b$, $f(x) \geq 0$ for all x. The integral of f is 0 outside of the interval $[a, b]$, and so we have
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}(b-a) = 1.$$
We conclude that f is a probability density function.

113. We need to evaluate
$$\begin{aligned}\int_{-\infty}^{\infty} xf(x)\,dx &= \int_a^b x\frac{1}{b-a}\,dx \\ &= \frac{1}{b-a}\int_a^b x\,dx \\ &= \frac{1}{b-a}\left[\frac{1}{2}b^2 - \frac{1}{2}a^2\right] \\ &= \frac{(b-a)(b+a)}{2(b-a)} \\ &= \frac{a+b}{2}.\end{aligned}$$
The mean of f is $\mu = \boxed{\frac{a+b}{2}}$.

115. We need to evaluate
$$\begin{aligned}\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx &= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx \\ &= \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\ &= \frac{1}{b-a}\left[\frac{1}{3}\left(x - \frac{a+b}{2}\right)^3\right]_a^b \\ &= \frac{1}{b-a}\left[\frac{1}{3}\left(b - \frac{a+b}{2}\right)^3 - \frac{1}{3}\left(a - \frac{a+b}{2}\right)^3\right] \\ &= \frac{1}{b-a}\left[\frac{1}{3}\left(\frac{1}{2}b - \frac{1}{2}a\right)^3 - \frac{1}{3}\left(\frac{1}{2}a - \frac{1}{2}b\right)^3\right] \\ &= \frac{1}{24(b-a)}\left[2(b-a)^3\right] \\ &= \frac{(b-a)^2}{12}.\end{aligned}$$
The variance of f is $\sigma^2 = \boxed{\frac{(b-a)^2}{12}}$. The standard deviation is $\sigma = \frac{b-a}{2\sqrt{3}}$.

AP® Practice Problems

1. The function $f(x) = \frac{x+2}{x^2+4x-12} = \frac{x+2}{(x+6)(x-2)}$ is continuous on $[0, 2)$ but is not defined at $x = 2$, so $\int_0^2 \frac{x+2}{x^2+4x-12}\,dx$ is an improper integral.
$$\int_0^2 \frac{x+2}{x^2+4x-12}\,dx = \lim_{b\to 2^-}\int_0^b \frac{x+2}{x^2+4x-12}\,dx.$$

Now use the substitution $u = x^2+4x-12$. So, $du = (2x+4)\,dx = 2(x+2)\,dx$, $(x+2)\,dx = \frac{du}{2}$, and $\int \frac{x+2}{x^2+4x-12}\,dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+4x-12\right| + C$.

$$\text{Therefore, } \int_0^2 \frac{x+2}{x^2+4x-12}\,dx = \lim_{b\to 2^-}\int_0^b \frac{x+2}{x^2+4x-12}\,dx$$
$$= \lim_{b\to 2^-}\left[\frac{1}{2}\ln\left|x^2+4x-12\right|\right]_0^b$$
$$= \frac{1}{2}\lim_{b\to 2^-}\left[\ln\left|b^2+4b-12\right| - \ln 12\right].$$

Since $\lim\limits_{b\to 2^-}\left|b^2+4b-12\right| = 0^+$, $\lim\limits_{b\to 2^-}\ln\left|b^2+4b-12\right| = \infty$ and so $\int_0^2 \frac{x+2}{x^2+4x-12}\,dx$ $\boxed{\text{diverges}}$.

The answer is D.

3. The function $f(x) = \frac{8x}{\sqrt[3]{8-x^2}}$ is continuous for $x \geq 3$.

By definition, $\int_3^\infty \frac{8x}{\sqrt[3]{8-x^2}}\,dx = \lim\limits_{b\to\infty}\int_3^b \frac{8x}{\sqrt[3]{8-x^2}}\,dx$.

$$\lim_{b\to\infty}\int_3^b \frac{8x}{\sqrt[3]{8-x^2}} = -4\lim_{b\to\infty}\int_3^b \left(8-x^2\right)^{-1/3}(-2x\,dx)$$
$$= -4\lim_{b\to\infty}\left[\frac{3}{2}(8-x^2)^{2/3}\right]_3^b$$
$$= -6\lim_{b\to\infty}\left[(8-b^2)^{2/3} - 1\right].$$

Since $\lim\limits_{b\to\infty}(8-b^2)^{2/3} = \sqrt[3]{\left[\lim\limits_{b\to\infty}(8-b^2)\right]^2} = \infty$, $-6\lim\limits_{b\to\infty}\left[\left(8-b^2\right)^{2/3} - 1\right] = -\infty$ and so $\int_3^\infty \frac{8x}{\sqrt[3]{8-x^2}}\,dx$ $\boxed{\text{diverges}}$.

The answer is D.

5. The function $f(x) = \frac{1}{x\,(\ln x)^2}$ is continuous for $x \geq 2$.

By definition, $\int_2^\infty \frac{1}{x\,(\ln x)^2}\,dx = \lim\limits_{b\to\infty}\int_2^b \frac{1}{x\,(\ln x)^2}\,dx$.

To evaluate $\int \frac{1}{x\,(\ln x)^2}\,dx$, use the substitution $u = \ln x$. Then $du = \frac{1}{x}\,dx$, $dx = x\,du$, and $\int \frac{1}{x\,(\ln x)^2}\,dx = \int \frac{1}{xu^2}(x\,du) = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{\ln x} + C$.

$$\text{So, } \int_2^\infty \frac{1}{x\,(\ln x)^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\,(\ln x)^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_2^b = -\lim_{b\to\infty}\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right)$$
$$= -\left(0 - \frac{1}{\ln 2}\right) = \boxed{\frac{1}{\ln 2}}.$$

The answer is A.

7. Using the method of disks, the volume is given by $V = \pi \int_1^\infty (y)^2\,dx = \pi \int_1^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^4}\,dx$.

By definition, $\int_1^\infty \frac{1}{x^4}\,dx = \lim\limits_{b\to\infty} \int_1^b \frac{1}{x^4}\,dx$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^4}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^4}\,dx \\
&= \lim_{b\to\infty} \int_1^b x^{-4}\,dx \\
&= \lim_{b\to\infty} \left[\frac{x^{-3}}{-3}\right]_1^b \\
&= -\frac{1}{3} \lim_{b\to\infty} \left(\frac{1}{b^3} - 1\right) \\
&= -\frac{1}{3}(0-1) = \frac{1}{3}.
\end{aligned}$$

The volume is $V = \pi \int_1^\infty \frac{1}{x^4}\,dx = \pi \cdot \frac{1}{3} = \boxed{\frac{\pi}{3}}$.

7.8 Integration Using Tables and Computer Algebra Systems

Skill Building

1. We use Integral 123,

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C,$$

with $a = 2$ and $b = 1$. We obtain

$$\begin{aligned}
\int e^{2x} \cos x\,dx &= \frac{e^{2x}}{2^2+1^2}(2\cos x + \sin x) + C \\
&= \boxed{\tfrac{1}{5} e^{2x}(2\cos x + \sin x) + C}.
\end{aligned}$$

3. Use Integral 31 with $a = 2$, $b = 3$, and $n = 4$.

$$\begin{aligned}
\int x(2+3x)^4\,dx &= \frac{(2+3x)^5}{3^2}\left(\frac{2+3x}{4+2} - \frac{2}{4+1}\right) + C \\
&= \boxed{\tfrac{(2+3x)^5}{9}\left(\tfrac{2+3x}{6} - \tfrac{2}{5}\right) + C}
\end{aligned}$$

5. Use Integral 41 with $a = 6$ and $b = 3$.

$$\int \frac{x^2\,dx}{\sqrt{6+3x}} = \frac{2}{15 \cdot 3^2}\left[8 \cdot 6^2 - (4 \cdot 6 \cdot 3)x + \left(3 \cdot 3^2\right)x^2\right]\sqrt{6+3x} + C$$
$$= \frac{2}{135}\left(288 - 72x + 27x^2\right)\sqrt{6+3x} + C$$
$$= \boxed{\tfrac{2}{45}\left(32 - 8x + 3x^2\right)\sqrt{6+3x} + C}$$

7. Use Integral 50 with $a = 2$. Then $a^2 = 4$ and

$$\int \frac{\sqrt{x^2+4}}{x}\,dx = \boxed{\sqrt{x^2+4} - 2\ln\left|\tfrac{2+\sqrt{x^2+4}}{x}\right| + C}$$

9. Use Integral 59 with $a = 2$. Then $a^2 = 4$ and

$$\int \frac{dx}{\left(x^2-4\right)^{3/2}} = \boxed{-\tfrac{x}{4\sqrt{x^2-4}} + C}$$

11. Use Integral 120 with $m = 3$ and $n = 2$.

$$\int x^3(\ln x)^2\,dx = \frac{x^{3+1}(\ln x)^2}{3+1} - \frac{2}{3+1}\int x^3(\ln x)^{2-1}\,dx = \frac{x^4(\ln x)^2}{4} - \frac{1}{2}\int x^3\ln x\,dx$$

Use Integral 117 with $n = 3$ to evaluate $\int x^3 \ln x\,dx$.

$$\int x^3\ln x\,dx = \left(\frac{x^{3+1}}{3+1}\right)\left(\ln x - \frac{1}{3+1}\right) + C = \frac{x^4}{4}\left(\ln x - \frac{1}{4}\right) + C$$

So,

$$\int x^3(\ln x)^2\,dx = \frac{x^4(\ln x)^2}{4} - \frac{1}{2}\left[\frac{x^4}{4}\left(\ln x - \frac{1}{4}\right)\right] + C = \boxed{\tfrac{x^4(\ln x)^2}{4} - \tfrac{x^4}{8}\left(\ln x - \tfrac{1}{4}\right) + C}$$

13. Use Integral 47 with $a = 4$. Then $a^2 = 16$ and

$$\int \sqrt{x^2-16}\,dx = \frac{x}{2}\sqrt{x^2-16} - \frac{16}{2}\ln\left|x + \sqrt{x^2-16}\right| + C$$
$$= \boxed{\tfrac{x}{2}\sqrt{x^2-16} - 8\ln\left|x + \sqrt{x^2-16}\right| + C}$$

15. Use Integral 67 with $a = \sqrt{6}$. Then $a^2 = 6$ and

$$\int \left(6-x^2\right)^{3/2}\,dx = \frac{x}{4}\left[6-x^2\right]^{3/2} + \frac{(3 \cdot 6)x}{8}\sqrt{6-x^2} + \frac{3(\sqrt{6})^4}{8}\sin^{-1}\frac{x}{\sqrt{6}} + C$$
$$= \boxed{\tfrac{x}{4}\left(6-x^2\right)^{3/2} + \tfrac{9}{4}x\sqrt{6-x^2} + \tfrac{27}{2}\sin^{-1}\tfrac{x\sqrt{6}}{6} + C}$$

17. Use Integral 69 with $a = 5$.

$$\int \sqrt{10x - x^2}\,dx = \frac{x-5}{2}\sqrt{(2 \cdot 5)x - x^2} + \frac{5^2}{2}\cos^{-1}\left(\frac{5-x}{5}\right) + C$$
$$= \boxed{\tfrac{x-5}{2}\sqrt{10x-x^2} + \tfrac{25}{2}\cos^{-1}\left(\tfrac{5-x}{5}\right) + C}$$

19. Use Integral 95 with $a = 3$ and $b = 8$.

$$\int \cos(3x)\cos(8x)\,dx = \frac{\sin[(3+8)x]}{2(3+8)} + \frac{\sin[(3-8)x]}{2(3-8)} + C = \frac{\sin(11x)}{22} + \frac{\sin(-5x)}{(-10)} + C$$

Since $\sin(-5x) = -\sin(5x)$, $\frac{\sin(-5x)}{(-10)} = \frac{\sin(5x)}{10}$ and the integral becomes

$$\int \cos(3x)\cos(8x)\,dx = \boxed{\tfrac{\sin(11x)}{22} + \tfrac{\sin(5x)}{10} + C}.$$

21. Use Integral 109.

$$\int x\tan^{-1} x\,dx = \boxed{\tfrac{x^2+1}{2}\tan^{-1} x - \tfrac{x}{2} + C}$$

23. Use Integral 117 with $n = 4$.

$$\int x^4 \ln x\,dx = \left(\frac{x^{4+1}}{4+1}\right)\left(\ln x - \frac{1}{4+1}\right) + C = \boxed{\tfrac{x^5}{5}\left(\ln x - \tfrac{1}{5}\right) + C}$$

25. Use Integral 128.

$$\int \sinh^2 x\,dx = \boxed{\tfrac{\sinh(2x)}{4} - \tfrac{x}{2} + C}$$

27. Use Integral 79 with $n = 2$ and $a = 4$.

$$\int \frac{dx}{x^2\sqrt{8x-x^2}} = \frac{\sqrt{(2\cdot 4)x - x^2}}{4(1-2\cdot 2)x^2} + \frac{2-1}{(2\cdot 2-1)\cdot 4}\int \frac{dx}{x^{2-1}\sqrt{8x-x^2}} + C$$
$$= -\frac{\sqrt{8x-x^2}}{12x^2} + \frac{1}{12}\int \frac{dx}{x\sqrt{8x-x^2}} + C$$

Use Integral 76 with $a = 4$ to evaluate $\int \frac{dx}{x\sqrt{8x-x^2}}$.

$$\int \frac{dx}{x\sqrt{8x-x^2}} = -\frac{\sqrt{(2\cdot 4)x - x^2}}{4x} + C = -\frac{\sqrt{8x-x^2}}{4x}$$

So, the integral becomes

$$\int \frac{dx}{x^2\sqrt{8x-x^2}} = -\frac{\sqrt{8x-x^2}}{12x^2} + \frac{1}{12}\left[-\frac{\sqrt{8x-x^2}}{4x}\right] + C = \boxed{-\tfrac{\sqrt{8x-x^2}}{12x^2} - \tfrac{\sqrt{8x-x^2}}{48x} + C}$$

29. Use Integral 36 with $n = 2$ and $a = 2$. Then $a^2 = 4$ and

$$\int \frac{dx}{(4+x^2)^2} = \frac{1}{2(2-1)\cdot 4}\left[\frac{x}{(4+x^2)^{2-1}} + (2\cdot 2 - 3)\int \frac{dx}{(4+x^2)^{2-1}}\right] + C$$
$$= \frac{1}{8}\left(\frac{x}{4+x^2} + \int \frac{dx}{4+x^2}\right) + C$$

Use Integral 17 with $a = 2$ to evaluate $\int \frac{dx}{4+x^2}$.

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

So,

$$\int \frac{dx}{(4+x^2)^2} = \boxed{\tfrac{1}{8}\left(\tfrac{x}{4+x^2} + \tfrac{1}{2}\tan^{-1}\tfrac{x}{2}\right) + C}.$$

31. Use Integral 132.

$$\int x \sinh x\, dx = \boxed{x \cosh x - \sinh x + C}$$

33. Begin with the substitution $u = x + 1$. Then $x = u - 1, dx = du$, and $\sqrt{4x+5} = \sqrt{4(u-1)+5} = \sqrt{4u+1}$.

The integral becomes

$$\int (x+1)\sqrt{4x+5}\, dx = \int u\sqrt{4u+1}\, du.$$

Use Integral 38 with $a = 1$ and $b = 4$.

$$\int u\sqrt{4u+1}\, du = \frac{2}{15 \cdot 16}(12u-2)(1+4u)^{3/2} + C = \frac{1}{60}(6u-1)(1+4u)^{3/2} + C$$

So, using $u = x + 1$,

$$\begin{aligned}\int (x+1)\sqrt{4x+5}\, dx &= \frac{1}{60}[6(x+1)-1][1+4(x+1)]^{3/2} + C \\ &= \boxed{\tfrac{1}{60}(6x+5)(4x+5)^{3/2} + C}\end{aligned}$$

35. We complete the square, and write $3x - x^2 = \frac{9}{4} - \left(x - \frac{3}{2}\right)^2$. Then substitute $u = x - \frac{3}{2}$, split the integral, and use symmetry to obtain

$$\begin{aligned}\int_1^2 \frac{x^3}{\sqrt{3x-x^2}}\, dx &= \int_1^2 \frac{x^3}{\sqrt{\frac{9}{4} - \left(x-\frac{3}{2}\right)^2}}\, dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\left(u+\frac{3}{2}\right)^3}{\sqrt{\frac{9}{4}-u^2}}\, du \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u^3 + \frac{9}{2}u^2 + \frac{27}{4}u + \frac{27}{8}}{\sqrt{\frac{9}{4}-u^2}}\, du \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u^3 + \frac{27}{4}u}{\sqrt{\frac{9}{4}-u^2}}\, du + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{9}{2}u^2 + \frac{27}{8}}{\sqrt{\frac{9}{4}-u^2}}\, du \\ &= 0 + 2\int_0^{\frac{1}{2}} \frac{\left(-\frac{9}{2}\right)\left(\frac{9}{4}-u^2\right) + \frac{27}{2}}{\sqrt{\frac{9}{4}-u^2}}\, du \\ &= -9\int_0^{\frac{1}{2}} \sqrt{\frac{9}{4}-u^2}\, du + 27\int_0^{\frac{1}{2}} \frac{1}{\sqrt{\frac{9}{4}-u^2}}\, du.\end{aligned}$$

Now use Integrals 60 and 16 to get

$$\begin{aligned}
\int_1^2 \frac{x^3}{\sqrt{3x-x^2}}\,dx &= -9\left[\frac{u}{2}\sqrt{\frac{9}{4}-u^2}+\frac{\frac{9}{4}}{2}\sin^{-1}\frac{u}{\frac{3}{2}}\right]_0^{1/2}+27\left[\sin^{-1}\frac{u}{\frac{3}{2}}\right]_0^{1/2} \\
&= -9\left(\frac{\frac{1}{2}}{2}\sqrt{\frac{9}{4}-\left(\frac{1}{2}\right)^2}+\frac{9}{8}\sin^{-1}\frac{2\left(\frac{1}{2}\right)}{3}-\left(\frac{0}{2}\sqrt{\frac{9}{4}-0^2}+\frac{9}{4}\sin^{-1}\frac{2\,(0)}{3}\right)\right) \\
&\quad +27\left(\sin^{-1}\frac{2\left(\frac{1}{2}\right)}{3}-\sin^{-1}\frac{2\,(0)}{3}\right) \\
&= -9\left(\frac{\sqrt{2}}{4}+\frac{9}{8}\sin^{-1}\frac{1}{3}-0\right)+27\left(\sin^{-1}\frac{1}{3}-0\right) \\
&= \boxed{\tfrac{135}{8}\sin^{-1}\tfrac{1}{3}-\tfrac{9\sqrt{2}}{4}}.
\end{aligned}$$

37. (a) Using a CAS we obtain

$$\int e^{2x}\cos x\,dx = \boxed{\tfrac{e^{2x}}{5}\left(\sin x+2\cos x\right)+C}.$$

(b) We obtained in problem 1,

$$\int e^{2x}\sin x\,dx = \boxed{\tfrac{1}{5}e^{2x}(2\cos x+\sin x)+C}.$$

(c) Since

$$\frac{e^{2x}}{5}\left(\sin x+2\cos x\right) = \frac{1}{5}e^{2x}(2\cos x+\sin x)$$

we see that the results are equivalent.

39. Refer to the integral in exercise 3. Using WoLframAlpha,

$$\int x\,(2+3x)^4\,dx = \boxed{\tfrac{27}{2}x^6+\tfrac{216}{5}x^5+54x^4+32x^3+8x^2+\text{constant}}$$

A screenshot is shown below.

$$\int x\,(2+3\,x)^4\,dx = \frac{27\,x^6}{2}+\frac{216\,x^5}{5}+54\,x^4+32\,x^3+8\,x^2+\text{constant}$$

41. Refer to the integral in exercise 5. Using WolframAlpha,

$$\int \frac{x^2}{\sqrt{6+3x}}\,dx = \boxed{\tfrac{2\sqrt{x+2}}{15\sqrt{3}}\left(3x^2-8x+32\right)+\text{constant}}$$

A screenshot is shown below.

$$\int \frac{x^2}{\sqrt{6+3\,x}}\,dx = \frac{2\sqrt{x+2}\,\left(3\,x^2-8\,x+32\right)}{15\sqrt{3}}+\text{constant}$$

43. Refer to the integral in exercise 7. Using WolframAlpha,

$$\int \frac{\sqrt{4+x^2}}{x}\,dx = \boxed{\sqrt{x^2+4} - 2\log\left(\sqrt{x^2+4}+2\right) + 2\log(x) + \text{constant}}$$

where $\log(x)$ is the natural logarithm.

A screenshot is shown below.

$$\int \frac{\sqrt{4+x^2}}{x}\,dx = \sqrt{x^2+4} - 2\log\left(\sqrt{x^2+4}+2\right) + 2\log(x) + \text{constant}$$

45. Refer to the integral in exercise 9. Using WolframAlpha,

$$\int \frac{dx}{(-4+x^2)^{3/2}} = \boxed{-\frac{x}{4\sqrt{x^2-4}} + \text{constant}}$$

A screenshot is shown below.

$$\int \frac{1}{(-4+x^2)^{3/2}}\,dx = -\frac{x}{4\sqrt{x^2-4}} + \text{constant}$$

47. Refer to the integral in exercise 11. Using WolframAlpha,

$$\int x^3\log^2(x)\,dx = \boxed{\tfrac{1}{32}x^4\left[8\log^2(x) - 4\log(x) + 1\right] + \text{constant}}$$

where $\log(x)$ is the natural logarithm.

A screenshot is shown below.

$$\int x^3\log^2(x)\,dx = \frac{1}{32}x^4\left(8\log^2(x) - 4\log(x) + 1\right) + \text{constant}$$

49. Refer to the integral in exercise 13. Using WolframAlpha,

$$\int \sqrt{-16+x^2}\,dx = \boxed{\tfrac{1}{2}x\sqrt{x^2-16} - 8\log\left(\sqrt{x^2-16}+x\right) + \text{constant}}$$

where $\log(x)$ is the natural logarithm.

A screenshot is shown below.

$$\int \sqrt{-16+x^2}\,dx = \frac{1}{2}x\sqrt{x^2-16} - 8\log\left(\sqrt{x^2-16}+x\right) + \text{constant}$$

51. Refer to the integral in exercise 15. Using WolframAlpha,

$$\int (6-x^2)^{3/2}\,dx = \boxed{\tfrac{27}{2}\sin^{-1}\left(\tfrac{x}{\sqrt{6}}\right) - \tfrac{1}{4}x\sqrt{6-x^2}(x^2-15) + \text{constant}}$$

A screenshot is shown below.

$$\int (6-x^2)^{3/2}\,dx = \frac{27}{2}\sin^{-1}\left(\frac{x}{\sqrt{6}}\right) - \frac{1}{4}x\sqrt{6-x^2}\,(x^2-15) + \text{constant}$$

53. Refer to the integral in exercise 17. Using WolframAlpha,

$$\int \sqrt{10x - x^2}\, dx = \boxed{\frac{\sqrt{-(x-10)x}\left[\sqrt{x-10}(x-5)\sqrt{x} - 50\log\left(\sqrt{x-10}+\sqrt{x}\right)\right]}{2\sqrt{x-10}\sqrt{x}} + \text{constant}}$$

where $\log(x)$ is the natural logarithm.

A screenshot is shown below.

$$\int \sqrt{10\,x - x^2}\, dx = \frac{\sqrt{-(x-10)\,x}\,\left(\sqrt{x-10}\,(x-5)\sqrt{x} - 50\log\left(\sqrt{x-10} + \sqrt{x}\right)\right)}{2\sqrt{x-10}\sqrt{x}} + \text{constant}$$

55. Refer to the integral in exercise 19. Using WolframAlpha,

$$\int \cos(3x)\cos(8x)\, dx = \boxed{\tfrac{1}{10}\sin(5x) + \tfrac{1}{22}\sin(11x) + \text{constant}}.$$

A screenshot is shown below.

$$\int \cos(3\,x)\cos(8\,x)\, dx = \frac{1}{10}\sin(5\,x) + \frac{1}{22}\sin(11\,x) + \text{constant}$$

57. Refer to the integral in exercise 21. Using WolframAlpha,

$$\int x\tan^{-1}x\, dx = \boxed{\tfrac{1}{2}\left[\left(x^2+1\right)\tan^{-1}x - x\right] + \text{constant}}$$

A screenshot is shown below.

$$\int x\tan^{-1}(x)\, dx = \frac{1}{2}\left(\left(x^2+1\right)\tan^{-1}(x) - x\right) + \text{constant}$$

59. Refer to the integral in exercise 23. Using WolframAlpha,

$$\int x^4\log(x)\, dx = \boxed{\tfrac{1}{25}x^5[5\log(x) - 1] + \text{constant}}$$

where $\log(x)$ is the natural logarithm.

A screenshot is shown below.

$$\int x^4\log(x)\, dx = \frac{1}{25}x^5(5\log(x) - 1) + \text{constant}$$

61. Refer to the integral in exercise 25. Using WolframAlpha,

$$\int \sinh^2(x)\, dx = \boxed{\tfrac{1}{4}[\sinh(2x) - 2x] + \text{constant}}$$

A screenshot is shown below.

$$\int \sinh^2(x)\, dx = \frac{1}{4}(\sinh(2\,x) - 2\,x) + \text{constant}$$

63. Refer to the integral in exercise 27. Using WolframAlpha,

$$\int \frac{1}{x^2\sqrt{8x-x^2}}\,dx = \boxed{-\frac{\sqrt{-(x-8)x}(x+4)}{48x^2} + \text{constant}}$$

A screenshot is shown below.

$$\int \frac{1}{x^2\sqrt{8x-x^2}}\,dx = -\frac{\sqrt{-(x-8)x}\,(x+4)}{48x^2} + \text{constant}$$

65. Refer to the integral in exercise 29. Using WolframAlpha,

$$\int \frac{1}{(4+x^2)^2}\,dx = \boxed{\frac{1}{16}\left[\frac{2x}{x^2+4} + \tan^{-1}\left(\frac{x}{2}\right)\right] + \text{constant}}$$

A screenshot is shown below.

$$\int \frac{1}{(4+x^2)^2}\,dx = \frac{1}{16}\left(\frac{2x}{x^2+4} + \tan^{-1}\left(\frac{x}{2}\right)\right) + \text{constant}$$

67. Refer to the integral in exercise 31. Using WolframAlpha,

$$\int x\sinh(x)\,dx = \boxed{x\cosh(x) - \sinh(x) + \text{constant}}$$

A screenshot is shown below.

$$\int x\sinh(x)\,dx = x\cosh(x) - \sinh(x) + \text{constant}$$

69. Refer to the integral in exercise 33. Using WolframAlpha,

$$\int (x+1)\sqrt{4x+5}\,dx = \boxed{\frac{1}{60}(4x+5)^{3/2}(6x+5) + \text{constant}}$$

A screenshot is shown below.

$$\int (1+x)\sqrt{5+4x}\,dx = \frac{1}{60}(4x+5)^{3/2}(6x+5) + \text{constant}$$

71. (a) Using a CAS we obtain

$$\int_1^2 \frac{x^3}{\sqrt{3x-x^2}}\,dx = \boxed{\tfrac{135}{8}\sin^{-1}\tfrac{1}{3} - \tfrac{9}{4}\sqrt{2}}.$$

(b) We obtained in problem 15,

$$\int_1^e \frac{dx}{x^2\sqrt{x^2+2}} = \boxed{\tfrac{135}{8}\sin^{-1}\tfrac{1}{3} - \tfrac{9}{4}\sqrt{2}}.$$

(c) Since

$$\frac{135}{8}\sin^{-1}\frac{1}{3} - \frac{9}{4}\sqrt{2} = \frac{135}{8}\sin^{-1}\frac{1}{3} - \frac{9}{4}\sqrt{2}$$

we see that the results are equivalent.

73. Using a CAS we obtain
$$\int \sqrt{1+x^3}\,dx = \int \sqrt{x^3+1}\,dx$$
and we conclude that this integral cannot be expressed using elementary functions.

75. Using a CAS we obtain
$$\int e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\operatorname{erf}(x) + C$$
and we conclude that this integral cannot be expressed using elementary functions.

77. Using a CAS we obtain
$$\int x\tan x\,dx = \frac{1}{2}ix^2 - i\operatorname{dilog}\left(ie^{ix}\right) - i\pi\tan^{-1}\left(e^{ix}\right) - i\operatorname{dilog}\left(-ie^{ix}\right) + C$$
and we conclude that this integral cannot be expressed using elementary functions.

Chapter 7 Review Exercises

1. We complete the square: $x^2+4x+20 = (x+2)^2+16$. Then we let $u = x+2$ to obtain
$$\begin{aligned}\int \frac{dx}{x^2+4x+20} &= \int \frac{dx}{(x+2)^2+4^2} \\ &= \int \frac{du}{u^2+4^2} \\ &= \frac{1}{4}\tan^{-1}\frac{u}{4} + C \\ &= \boxed{\tfrac{1}{4}\tan^{-1}\left(\tfrac{x+2}{4}\right) + C}.\end{aligned}$$

3. Factor out $\sec\phi\tan\phi$. Then let $u = \sec\phi$, so $du = \sec\phi\tan\phi\,d\phi$. We substitute and obtain
$$\begin{aligned}\int \sec^3\phi\tan\phi\,d\phi &= \int \sec^2\phi\,\sec\phi\tan\phi\,d\phi \\ &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C \\ &= \boxed{\tfrac{1}{3}\sec^3\phi + C}.\end{aligned}$$

5. Factor out $\sin\phi$ and use the identity $\sin^2\phi = 1-\cos^2\phi$.
$$\begin{aligned}\int \sin^3\phi\,d\phi &= \int \sin^2\phi\,\sin\phi\,d\phi \\ &= \int \left(1-\cos^2\phi\right)\sin\phi\,d\phi.\end{aligned}$$
Let $u = \cos\phi$, then $du = -\sin\phi\,d\phi$, so $\sin\phi\,d\phi = -du$. We substitute and obtain
$$\begin{aligned}\int \sin^3\phi\,d\phi &= \int \left(1-u^2\right)(-du) \\ &= \int \left(u^2-1\right)du \\ &= \frac{1}{3}u^3 - u + C \\ &= \boxed{\tfrac{1}{3}\cos^3\phi - \cos\phi + C}.\end{aligned}$$

7. Let $u = x + 2$, so $du = dx$, then substitute.

$$\int \frac{dx}{\sqrt{(x+2)^2 - 1}} = \int \frac{dx}{\sqrt{u^2 - 1}}$$

Let $u = \sec\theta$, then $du = \sec\theta\tan\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{(x+2)^2-1}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\sec^2\theta - 1}} \\ &= \int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} \\ &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

We have $\sec\theta = u = x + 2$, so $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{u^2-1} = \sqrt{(x+2)^2 - 1}$. We obtain

$$\int \frac{dx}{\sqrt{(x+2)^2-1}} = \boxed{\ln\left|x + 2 + \sqrt{(x+2)^2 - 1}\right| + C}.$$

9. We use integration by parts with $u = v$ and $dw = \csc^2 w\, dw$. Then $du = dv$ and $w = -\cot v$. We obtain

$$\begin{aligned}\int v\csc^2 v\, dv &= v(-\cot v) - \int(-\cot v)\, dv \\ &= -v\cot v + \int \cot v\, dv \\ &= \boxed{-v\cot v + \ln|\sin v| + C}.\end{aligned}$$

11. Let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$. We substitute and obtain

$$\begin{aligned}\int \left(4 - x^2\right)^{3/2} dx &= \int \left(4 - (2\sin\theta)^2\right)^{3/2} (2\cos\theta)\, d\theta \\ &= 2\int (2\cos\theta)^3\, \cos\theta\, d\theta \\ &= 16\int \cos^4\theta\, d\theta \\ &= 16\int \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta \\ &= 4\int \left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right) d\theta \\ &= 4(\theta + \sin(2\theta)) + 4\int \frac{1+\cos(4\theta)}{2}\, d\theta \\ &= 4\theta + 4\sin(2\theta) + 2\theta + \frac{1}{2}\sin(4\theta) + C \\ &= 6\theta + 8\sin\theta\cos\theta + \sin(2\theta)\cos(2\theta) + C \\ &= 6\theta + 8\sin\theta\cos\theta + 2\sin\theta\cos\theta\left(1 - 2\sin^2\theta\right) + C.\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. We obtain

$$\begin{aligned}\int \left(4-x^2\right)^{3/2} dx &= 6\sin^{-1}\left(\frac{x}{2}\right) + 8\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right) \\ &\quad +2\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right)\left(1-2\left(\frac{x}{2}\right)^2\right) + C \\ &= \boxed{6\sin^{-1}\left(\tfrac{x}{2}\right) + 2x\sqrt{4-x^2} + \tfrac{1}{4}x\sqrt{4-x^2}\left(2-x^2\right) + C}.\end{aligned}$$

13. Let $u = e^t$, so $du = e^t\,dt$. We substitute to obtain

$$\begin{aligned}\int \frac{e^{2t}}{e^t-2}\,dt &= \int \frac{e^t}{e^t-2} e^t\,dt \\ &= \int \frac{u}{u-2}\,du \\ &= \int \frac{u-2+2}{u-2}\,du \\ &= \int \left(1+\frac{2}{u-2}\right) du \\ &= u + 2\ln|u-2| + C \\ &= \boxed{e^t + 2\ln|e^t-2| + C}.\end{aligned}$$

15. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{x^4-16} &= \frac{x}{(x^2-4)(x^2+4)} \\ &= \frac{x}{(x-2)(x+2)(x^2+4)} \\ &= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}\end{aligned}$$

so

$$x = A(x+2)\left(x^2+4\right) + B(x-2)\left(x^2+4\right) + (Cx+D)(x-2)(x+2).$$

Let $x = 2$ to obtain $A = 1/16$, and $x = -2$ to get $B = 1/16$. We now have

$$\begin{aligned}x &= \frac{1}{16}(x+2)\left(x^2+4\right) + \frac{1}{16}(x-2)\left(x^2+4\right) + (Cx+D)(x-2)(x+2) \\ x &= \left(C+\frac{1}{8}\right)x^3 + Dx^2 + \left(\frac{1}{2}-4C\right)x - 4D.\end{aligned}$$

Equating coefficients, we obtain $C = -1/8$ and $D = 0$. So

$$\begin{aligned}\int \frac{x\,dx}{x^4-16} &= \int \left(\frac{1/16}{x-2} + \frac{1/16}{x+2} + \frac{(-1/8)\,x}{x^2+4}\right) dx \\ &= \frac{1}{16}\ln|x-2| + \frac{1}{16}\ln|x+2| - \frac{1}{16}\ln\left(x^2+4\right) + C \\ &= \frac{1}{16}\ln\left|x^2-4\right| - \frac{1}{16}\ln\left(x^2+4\right) + C \\ &= \boxed{\tfrac{1}{16}\ln\left|\tfrac{x^2-4}{x^2+4}\right| + C}.\end{aligned}$$

17. Let $u = y + 1$, so $du = dy$ and $y = u - 1$. We substitute and obtain

$$\begin{aligned}
\int \frac{y^2\,dy}{(y+1)^3} &= \int \frac{(u-1)^2\,du}{u^3} \\
&= \int \frac{u^2 - 2u + 1}{u^3}\,du \\
&= \int \left(\frac{1}{u} - 2u^{-2} + u^{-3}\right) du \\
&= \ln|u| + \frac{2}{u} - \frac{1}{2u^2} + C \\
&= \boxed{\ln|y+1| + \tfrac{2}{y+1} - \tfrac{1}{2(y+1)^2} + C}.
\end{aligned}$$

19. We use integration by parts with $u = x$ and $dv = \sec^2 x\,dx$. Then $du = dx$ and $v = \tan x$. We obtain

$$\begin{aligned}
\int x\sec^2 x\,dx &= x\tan x - \int \tan x\,dx \\
&= \boxed{x\tan x - \ln|\sec x| + C}.
\end{aligned}$$

21. We use integration by parts with $u = \ln(1-y)$, $du = \frac{-1}{1-y}\,dy$, $dv = dy$, and $v = y$. We obtain

$$\begin{aligned}
\int \ln(1-y)\,dy &= y\ln(1-y) - \int y\left(\frac{-1}{1-y}\right) dy \\
&= y\ln(1-y) - \int \frac{-y}{1-y}\,dy \\
&= y\ln(1-y) - \int \frac{(1-y)-1}{1-y}\,dy \\
&= y\ln(1-y) - \int \left(1 - \frac{1}{1-y}\right) dy \\
&= \boxed{y\ln(1-y) - y - \ln(1-y) + C}.
\end{aligned}$$

23. We use partial fractions to obtain

$$\begin{aligned}
\frac{3x^2+2}{x^3-x^2} &= \frac{3x^2+2}{x^2(x-1)} \\
&= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \\
3x^2 + 2 &= Ax(x-1) + B(x-1) + Cx^2.
\end{aligned}$$

Let $x = 1$ to obtain $C = 5$. Let $x = 0$ to get $B = -2$. Then

$$\begin{aligned}
3x^2 + 2 &= Ax(x-1) + (-2)(x-1) + 5x^2 \\
3x^2 + 2 &= (A+5)x^2 + (-A-2)x + 2.
\end{aligned}$$

We equate coefficients and determine $A = -2$. We now have

$$\begin{aligned}
\int \frac{3x^2+2}{x^3-x^2}\,dx &= \int \left(\frac{-2}{x} + \frac{-2}{x^2} + \frac{5}{x-1}\right) dx \\
&= \boxed{-2\ln|x| + \tfrac{2}{x} + 5\ln|x-1| + C}.
\end{aligned}$$

25. We use integration by parts with $u = \sin^{-1} x$, $du = \frac{1}{\sqrt{1-x^2}}\,dx$, $dv = x^2 dx$, and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2 \sin^{-1} x\,dx &= \frac{1}{3}x^3 \sin^{-1} x - \int \left(\frac{1}{3}x^3\right)\frac{1}{\sqrt{1-x^2}}\,dx \\ &= \frac{1}{3}x^3 \sin^{-1} x - \frac{1}{3}\int \frac{x^2}{\sqrt{1-x^2}} x\,dx.\end{aligned}$$

Now let $u = 1 - x^2$, so $du = -2x\,dx$, $x\,dx = -\frac{1}{2}\,du$, and $x^2 = 1 - u$. We substitute and obtain

$$\begin{aligned}\int x^2 \sin^{-1} x\,dx &= \frac{1}{3}x^3 \sin^{-1} x - \frac{1}{3}\int \frac{1-u}{\sqrt{u}}\left(-\frac{1}{2}\right)du \\ &= \frac{1}{3}x^3 \sin^{-1} x + \frac{1}{6}\int \left(u^{-1/2} - u^{1/2}\right)du \\ &= \frac{1}{3}x^3 \sin^{-1} x + \frac{1}{6}\left(2\sqrt{u} - \frac{2}{3}u^{3/2}\right) + C \\ &= \boxed{\tfrac{1}{3}x^3 \sin^{-1} x + \tfrac{1}{3}\sqrt{1-x^2} - \tfrac{1}{9}\left(1-x^2\right)^{3/2} + C}.\end{aligned}$$

27. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x^2+2x} &= \frac{1}{2(x+2)} \\ &= \frac{A}{x} + \frac{B}{x+2} \\ 1 &= A(x+2) + Bx.\end{aligned}$$

When $x = -2$ we obtain $B = -1/2$, and when $x = 0$, we have $A = 1/2$. So

$$\begin{aligned}\int \frac{dx}{x^2+2x} &= \int \left(\frac{1/2}{x} + \frac{-1/2}{x+2}\right)dx \\ &= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C \\ &= \boxed{\tfrac{1}{2}\ln\left|\tfrac{x}{x+2}\right| + C}.\end{aligned}$$

29. We use partial fractions to obtain

$$\begin{aligned}\frac{w-2}{1-w^2} &= \frac{w-2}{(1-w)(1+w)} \\ &= \frac{A}{1-w} + \frac{B}{1+w} \\ w-2 &= A(1+w) + B(1-w).\end{aligned}$$

Let $w = -1$ to obtain $B = -3/2$, and let $w = 1$ to obtain $A = -1/2$. We now have

$$\begin{aligned}\int \frac{w-2}{1-w^2}\,dw &= \int \left(\frac{-1/2}{1-w} + \frac{-3/2}{1+w}\right)dw \\ &= \boxed{\tfrac{1}{2}\ln|1-w| - \tfrac{3}{2}\ln|1+w| + C}.\end{aligned}$$

31. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx$ and $\frac{1}{\sqrt{x}}\,dx = 2\,du$. We substitute and obtain

$$\begin{aligned}
\int \frac{1}{\sqrt{x}} \cos^2 \sqrt{x}\,dx &= \int \left(\cos^2 u\right)(2)\,du \\
&= 2\int \frac{1+\cos(2u)}{2}\,du \\
&= \int (1+\cos(2u))\,du \\
&= u + \frac{1}{2}\sin(2u) + C \\
&= u + \sin u \cos u + C \\
&= \boxed{\sqrt{x} + \sin\sqrt{x}\cos\sqrt{x} + C}.
\end{aligned}$$

33. We use formula 98, and obtain

$$\begin{aligned}
\int \sin x \cos(2x)\,dx &= -\frac{\cos(1+2)x}{2(1+2)} - \frac{\cos(1-2)x}{2(1-2)} + C \\
&= \boxed{\tfrac{1}{2}\cos x - \tfrac{1}{6}\cos(3x) + C}.
\end{aligned}$$

35. Let $u = 1 + x^2$, then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. The lower limit of integration is $u = 1 + 0^2 = 1$, and the upper limit becomes $u = 1 + \left(\sqrt{3}\right)^2 = 4$. We substitute to obtain

$$\begin{aligned}
\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}} &= \int_1^4 \frac{\frac{1}{2}\,du}{\sqrt{u}} \\
&= \int_1^4 \frac{1}{2}u^{-1/2}\,du \\
&= \left[u^{1/2}\right]_1^4 \\
&= 4^{1/2} - 1^{1/2} \\
&= \boxed{1}.
\end{aligned}$$

37. We integrate by parts, with $u = \tan^{-1} x$, $du = \frac{1}{1+x^2}\,dx$, $dv = x^n\,dx$, and $v = \frac{1}{n+1}x^{n+1}$. We obtain

$$\begin{aligned}
\int x^n \tan^{-1} x\,dx &= \left(\frac{1}{n+1}x^{n+1}\right)\tan^{-1} x - \int \left(\frac{1}{n+1}x^{n+1}\right)\frac{1}{1+x^2}\,dx \\
&= \frac{x^{n+1}}{n+1}\tan^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}\,dx.
\end{aligned}$$

39. We evaluate

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/2}e^{-x^{1/2}}\,dx.$$

Let $u = x^{1/2}$, then $du = \frac{1}{2}x^{-1/2}\,dx$. We substitute to obtain

$$\begin{aligned}
\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \lim_{b\to\infty} \int_1^{b^{1/2}} 2e^{-u}\,du \\
&= \lim_{b\to\infty} \left[-2e^{-u}\right]_1^{b^{1/2}} \\
&= \lim_{b\to\infty} \left[-2e^{-b^{1/2}} - \left(-2e^{-1}\right)\right] \\
&= \lim_{b\to\infty} \left[-2e^{-b^{1/2}} + 2e^{-1}\right] \\
&= \frac{2}{e}.
\end{aligned}$$

The improper integral converges to $\boxed{\frac{2}{e}}$.

41. We evaluate

$$\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \int_0^b \frac{x\,dx}{\sqrt{1-x^2}}.$$

Let $u = 1 - x^2$ and substitute to obtain

$$\begin{aligned}
\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} &= \lim_{b\to 1^-} \int_1^{1-b^2} \frac{-\frac{1}{2}\,du}{\sqrt{u}} \\
&= \lim_{b\to 1^-} \left(-\frac{1}{2}\right) \int_1^{1-b^2} u^{-1/2}\,du \\
&= \lim_{b\to 1^-} \left[\left(-\frac{1}{2}\right)2\sqrt{u}\right]_1^{1-b^2} \\
&= \lim_{b\to 1^-} \left[-\sqrt{u}\right]_1^{1-b^2} \\
&= \lim_{b\to 1^-} \left[-\sqrt{1-b^2} - \left(-\sqrt{1}\right)\right] \\
&= 1.
\end{aligned}$$

The improper integral converges to $\boxed{1}$.

43. We evaluate

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin x}{\cos x}\,dx &= \lim_{b\to(\pi/2)^-} \int_0^b \tan x\,dx \\
&= \lim_{b\to(\pi/2)^-} \left[\ln|\sec x|\right]_0^b \\
&= \lim_{b\to(\pi/2)^-} \left[\ln|\sec b| - \ln|\sec 0|\right] \\
&= \lim_{b\to(\pi/2)^-} \left[\ln|\sec b|\right] \\
&= \infty.
\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

45. Since

$$\frac{1+e^{-x}}{x} \geq \frac{1}{x}$$

for all x, and since $\int_1^{\infty} \frac{1}{x}\,dx$ diverges, by the Comparison Test, $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$ $\boxed{\text{diverges}}$.

47. We use integration by parts with $u = x^2$ and $dv = \cos x\,dx$. Then

$$\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx.$$

Then $f(x) = \boxed{x^2 \sin x}$.

49. The arc length is given by $\int_0^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/2} \sqrt{1 + (\cos x)^2}\,dx = \int_0^{\pi/2} \sqrt{1 + \cos^2 x}\,dx$.

(a) With $n = 3$ we have $\Delta x = \frac{(\pi/2) - 0}{3} = \frac{\pi}{6}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1 + \cos^2 x}\,dx &\approx \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)\right] \\ &= \frac{\pi/6}{2}\left[\sqrt{1 + \cos^2 0} + 2\sqrt{1 + \cos^2 \frac{\pi}{6}} + 2\sqrt{1 + \cos^2 \frac{\pi}{3}} + \sqrt{1 + \cos^2 \frac{\pi}{2}}\right] \\ &\approx \boxed{1.910}.\end{aligned}$$

(b) With $n = 4$ we have $\Delta x = \frac{(\pi/2) - 0}{4} = \frac{\pi}{8}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1 + \cos^2 x}\,dx &\approx \frac{\Delta x}{3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{4}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right] \\ &= \frac{\pi/8}{3}\left[\sqrt{1 + \cos^2 0} + 4\sqrt{1 + \cos^2 \frac{\pi}{8}} + 2\sqrt{1 + \cos^2 \frac{\pi}{4}}\right. \\ &\qquad \left. + 4\sqrt{1 + \cos^2 \frac{3\pi}{8}} + \sqrt{1 + \cos^2 \frac{\pi}{2}}\right] \\ &\approx \boxed{1.910}.\end{aligned}$$

51. The area is given by the improper integral

$$\begin{aligned}\int_0^1 x^{-2/3}\,dx &= \lim_{a \to 0^+} \int_a^1 x^{-2/3}\,dx \\ &= \lim_{a \to 0^+} \left[3x^{1/3}\right]_a^1 \\ &= \lim_{a \to 0^+} \left[3(1)^{1/3} - 3(a)^{1/3}\right] \\ &= 3.\end{aligned}$$

The area of the region is $\boxed{3}$.

53. Express the logistic model $\frac{dP}{dt} = 0.0024P(100 - P)$ in the form $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ where M is the carrying capacity and k is the maximum population growth rate.

$$\frac{dP}{dt} = 0.0024P(100 - P) = 0.0024 \cdot 100P\left(1 - \frac{P}{100}\right) = 0.24P\left(1 - \frac{P}{100}\right)$$

(a) The carrying capacity is $M = \boxed{100}$.

(b) The maximum population growth rate is $k = \boxed{0.24}$.

(c) At the inflection point, the size of the population is given by one-half the carrying capacity. Therefore, the size of the population is $\boxed{50}$ at the inflection point.

AP® Review Problems

1. Use the identity $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$.

$$\int \cos^2 x\,dx = \int \frac{1}{2}[1+\cos(2x)]\,dx = \frac{1}{2}\left[x+\frac{1}{2}\sin(2x)\right]+C = \boxed{\tfrac{1}{2}x+\tfrac{1}{4}\sin(2x)+C}$$

The answer is B.

3. Partition $[0,2]$ into five subintervals, each of equal width: $[0,0.4]$, $[0.4,0.8]$, $[0.8,1.2]$, $[1.2,1.6]$ and $[1.6,2.0]$.

The width of each subinterval is $\Delta x = 0.4$.

Now apply the Trapezoidal Rule:

$$\begin{aligned}\int_0^2 f(x)\,dx &\approx \frac{1}{2}[f(0)+f(0.4)]\Delta x + \frac{1}{2}[f(0.4)+f(0.8)]\Delta x + \frac{1}{2}[f(0.8)+f(1.2)]\Delta x \\ &\quad + \frac{1}{2}[f(1.2)+f(1.6)]\Delta x + \frac{1}{2}[f(1.6)+f(2.0)]\Delta x \\ &= \frac{1}{2}[f(0)+2f(0.4)+2f(0.8)+2f(1.2)+2f(1.6)+f(2.0)]\Delta x \\ &= \frac{1}{2}[3+2(4)+2(4)+2(6)+2(8)+10](0.4) = \boxed{\tfrac{57}{5}}.\end{aligned}$$

The answer is C.

5. Partition $[0,10]$ into four subintervals $[0,1]$, $[1,4]$, $[4,8]$, and $[8,10]$.
The widths of the four intervals are

$\Delta x_1 = 1-0=1$, $\Delta x_2 = 4-1=3$, $\Delta x_3 = 8-4=4$, and $\Delta x_4 = 10-8=2$.

Now apply the Trapezoidal Rule:

$$\begin{aligned}\int_0^{10} f(x)\,dx &\approx \frac{1}{2}[f(0)+f(1)]\Delta x_1 + \frac{1}{2}[f(1)+f(4)]\Delta x_2 + \frac{1}{2}[f(4)+f(8)]\Delta x_3 \\ &\quad + \frac{1}{2}[f(8)+f(10)]\Delta x_4 \\ &= \frac{1}{2}[4+5](1) + \frac{1}{2}[5+10](3) + \frac{1}{2}[10+12](4) + \frac{1}{2}[12+8](2) \\ &= \frac{9}{2}+\frac{45}{2}+\frac{88}{2}+\frac{40}{2} = \boxed{91}.\end{aligned}$$

The answer is D.

7. Evaluate $\int x\csc^2 x\,dx$ using integration by parts.

Let $u=x$ and $dv=\csc^2 x\,dx$.

Then $du=dx$ and $v=\int \csc^2 x\,dx = -\cot x$.

Now $\displaystyle\int x\csc^2 x\,dx = x(-\cot x) - \int(-\cot x)(dx) = -x\cot x + \int \cot x\,dx$.

To evaluate $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$, use the substitution $u=\sin x$. Then $du=\cos x\,dx$, and $\int\cot x\,dx = \int\frac{\cos x}{\sin x}\,dx = \int \frac{1}{\sin x}(\cos x\,dx) = \int \frac{1}{u}\,du = \ln|u|+C = \ln|\sin x|+C$.

So, $\displaystyle\int x\csc^2 x\,dx = -x\cot x + \int \cot x\,dx = \boxed{-x\cot x+\ln|\sin x|+C}$.

The answer is D.

9. The function $f(x) = \frac{\ln x}{x}$ is continuous on $(0, 10]$ but is not defined at $x = 0$.

So, $\int_0^{10} \frac{\ln x}{x}\, dx$ is an improper integral.

$$\int_0^{10} \frac{\ln x}{x}\, dx = \lim_{b \to 0^+} \int_b^{10} \frac{\ln x}{x}\, dx$$

To evaluate $\int \frac{\ln x}{x}\, dx$, use the substitution $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{\ln x}{x}\, dx = \int \ln x \left(\frac{1}{x}\, dx\right) = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

$$\begin{aligned} \int_0^{10} \frac{\ln x}{x}\, dx &= \lim_{b \to 0^+} \int_b^{10} \frac{\ln x}{x}\, dx \\ &= \lim_{b \to 0^+} \left[\frac{(\ln x)^2}{2}\right]_b^{10} \\ &= \frac{1}{2} \lim_{b \to 0^+} \left[(\ln 10)^2 - (\ln b)^2\right]. \end{aligned}$$

Since $\lim\limits_{b \to 0^+} (\ln b)^2 = \left(\lim\limits_{b \to 0^+} \ln b\right)^2 = \infty$, $\int_0^{10} \frac{\ln x}{x}\, dx$ $\boxed{\text{diverges}}$.

The answer is D.

11. To evaluate $\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx$, use the substitution $x = 4\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Then $dx = 4\cos\theta\, d\theta$ and $\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = 4\sqrt{1 - \sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\cos\theta$ since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

The integral becomes

$$\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx = \int \frac{1}{(4\sin\theta)^2 4\cos\theta}(4\cos\theta\, d\theta) = \frac{1}{16}\int \csc^2\theta\, d\theta = -\frac{1}{16}\cot\theta + C.$$

Since $\sqrt{16 - x^2} = 4\cos\theta$, $\sec\theta = \frac{4}{\sqrt{16-x^2}}$, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{4}{\sqrt{16-x^2}}\right)^2 - 1} = \pm\frac{x}{\sqrt{16-x^2}}$, and $\cot\theta = \pm\frac{\sqrt{16-x^2}}{x}$.

Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cot\theta = \frac{\sqrt{16-x^2}}{x}$.

Therefore, $\displaystyle\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx = -\frac{1}{16}\cot\theta + C = -\frac{1}{16}\frac{\sqrt{16 - x^2}}{x} + C = \boxed{-\frac{1}{16x}\sqrt{16 - x^2} + C}$.

The answer is C.

Practice AP® Exam, Part I

1. For $f(x) = e^{4x} + \sin(2x)$, $f'(x) = e^{4x}\left[\frac{d}{dx}(4x)\right] + \cos(2x)\left[\frac{d}{dx}(2x)\right] = 4e^{4x} + 2\cos(2x)$.

So, $f'(0) = 4e^0 + 2\cos(0) = 4(1) + 2(1) = \boxed{6}$.

The answer is D.

2. Note that $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} (x+1) = 2.$

   Therefore, $\lim_{x\to 1} \frac{(x^2-1)f(x)}{x-1} = \left(\lim_{x\to 1} \frac{x^2-1}{x-1}\right) \cdot \left[\lim_{x\to 1} f(x)\right] = (2)(3) = \boxed{6}.$

   The answer is C.

3. Since $\lim_{x\to 3} f(x) \neq f(3)$, function f is discontinuous at $x = 3$. Statement (B) is TRUE.

   Since $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$, $\lim_{x\to 4} f(x)$ exists. Statement (C) is TRUE.

   Since $\lim_{x\to 5} f(x) = f(5)$, function f is continuous at $x = 5$. Statement (D) is TRUE.

   Since $\lim_{x\to 2^-} f(x) = 1$ and $1 < f(2) < 2$, $\lim_{x\to 2^-} f(x) \neq f(2)$. Statement (A) is FALSE.

   The answer is A.

4. For $f(x) = x^3 - 15x^2 - 1800x + 2000$, $f'(x) = 3x^2 - 30x - 1800 = 3(x-30)(x+20)$.

   Note that $f'(x) < 0$ on the interval $-20 < x < 30$, and defined at $x = -20$ and $x = 30$.

   Therefore, f is decreasing on the interval $-20 \leq x \leq 30$.

   Use $f''(x) = 6x - 30 = 6(x-5)$ to examine concavity.

   Note that $f''(x) < 0$ for $x < 5$. Therefore, f is concave down for $x < 5$.

   So, f is both decreasing and concave down for $\boxed{-20 \leq x < 5}$.

   The answer is B.

5. For $y = e^{2x} \cos(3x)$,

   $$f'(x) = e^{2x}\left[\frac{d}{dx}\cos(3x)\right] + \cos(3x)\left(\frac{d}{dx}e^{2x}\right) = e^{2x}[-3\sin(3x)] + \cos(3x)\big(2e^{2x}\big)$$
   $$= e^{2x}[-3\sin(3x) + 2\cos(3x)] = \boxed{e^{2x}[2\cos(3x) - 3\sin(3x)]}.$$

   The answer is B.

6. Note that $\lim_{x\to 0^-} \frac{\sin(7x)}{2x} = \frac{7}{2} \lim_{x\to 0^-} \frac{\sin(7x)}{7x}$.

   Let $t = 7x$. As $x \to 0^-$, $t \to 0^-$.

   Then $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(7x)}{2x} = \frac{7}{2} \lim_{x\to 0^-} \frac{\sin(7x)}{7x} = \frac{7}{2} \lim_{t\to 0^-} \frac{\sin t}{t} = \frac{7}{2}(1) = \frac{7}{2}$.

   Since $k + 2\ln(x + e^{x+1})$ is continuous for all $x \geq 0$,

   $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left[k + 2\ln(x + e^{x+1})\right] = f(0) = k + 2\ln(0 + e^1) = k + 2.$

   For f to be continuous at $x = 0$, choose k so that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$.

   So, $\frac{7}{2} = k + 2$. Therefore, $k = \frac{3}{2}$ and $\lim_{x\to 0} f(x) = \frac{7}{2}$.

   Note that for $k = \frac{3}{2}$, $f(0) = \frac{7}{2} = \lim_{x\to 0} f(x)$.

   Therefore, f is continuous at $x = 0$ for $k = \boxed{\frac{3}{2}}$.

   The answer is C.

7. Note that $\lim\limits_{x\to 3} \sqrt[5]{(x-3)^4} = \left[\lim\limits_{x\to 3}(x-3)\right]^{4/5} = 0$ and $f(3) = \sqrt[5]{(3-3)^4} = 0$.

   Since $\lim\limits_{x\to 3} \sqrt[5]{(x-3)^4}$ exists and is equal to $f(3)$, function f is continuous at $x = 3$.

   Note that $f'(x) = \frac{4}{5}(x-3)^{-1/5} = \frac{4}{5\sqrt[5]{x-3}}$. So, $f'(3)$ is not defined.

   Therefore, function f is not differentiable at $x = \boxed{3}$.

   The answer is B.

8. $\int (\sec x + \tan x) \tan x \, dx = \int \left(\sec x \tan x + \tan^2 x\right) dx$

   Use the identity $\tan^2 x = \sec^2 x - 1$.

   $$\int (\sec x + \tan x) \tan x \, dx = \int \left(\sec x \tan x + \sec^2 x - 1\right) dx = \boxed{\sec x + \tan x - x + C}$$

   The answer is C.

9. $$\begin{aligned} y' &= \frac{d}{dx}\left(4x + 6e^{\tan x}\right)^{1/2} \\ &= \frac{1}{2}\left(4x + 6e^{\tan x}\right)^{-1/2} \frac{d}{dx}\left(4x + 6e^{\tan x}\right) \\ &= \frac{1}{2}\left(4x + 6e^{\tan x}\right)^{-1/2}\left(4 + 6e^{\tan x} \cdot \sec^2 x\right) \\ &= \frac{4 + 6e^{\tan x} \sec^2 x}{2\sqrt{4x + 6e^{\tan x}}} \\ &= \boxed{\frac{2 + 3e^{\tan x}\sec^2 x}{\sqrt{4x + 6e^{\tan x}}}} \end{aligned}$$

   The answer is D.

10. For a function f that is continuous on the closed interval $a \le x \le b$ and differentiable on the open interval $a < x < b$, the Mean Value Theorem guarantees that $f'(c) = \frac{f(b)-f(a)}{b-a}$ for at least one c between a and b. Applying the Mean Value Theorem to function f that is continuous on $1 \le x \le 5$ and differentiable on $1 < x < 5$ with $f(1) = 10$ and $f(5) = 50$, $f'(c) = \frac{f(5)-f(1)}{5-1} = \frac{50-10}{4} = 10$ for at least one c between 1 and 5.

    The answer is B.

11. The ith term of the sum is $e^{i/20} \cdot \frac{1}{20} = f(u_i)\Delta x$ for $f(x) = e^x$, $u_i = \frac{i}{20}$, and $\Delta x = \frac{1}{20}$.

    The quantity u_i is the right endpoint of the ith subinterval $\left[\frac{i-1}{20}, \frac{i}{20}\right]$ for $i = 1, 2, \ldots, n$.

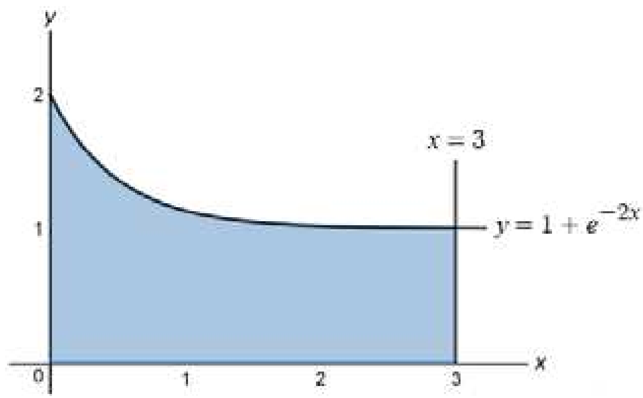
    Since the last term of the Riemann sum is e^2 and the right endpoint of the nth subinterval is $\frac{n}{20}$, therefore $f\left(\frac{n}{20}\right) = e^{n/20} = e^2$, so $\frac{n}{20} = 2$, so $n = 40$. So, the interval $[0, 2]$ is partitioned into 40 subintervals of equal length

    $$\Delta x = \frac{2}{40} = \frac{1}{20}.$$

    Therefore, $\frac{1}{20}\left[e^{1/20} + e^{2/20} + \cdots + e^2\right] = \sum_{i=1}^{40} f(u_i)\Delta x = \boxed{\int_0^2 e^x \, dx.}$

    The answer is B.

12. The region in the first quadrant bounded by the graph of $y = 1 + e^{-2x}$ and the line $x = 3$ is pictured below.



Since $y = 1 + e^{-2x}$ is nonnegative on $[0, 3]$, $\int_0^3 \left(1 + e^{-2x}\right) dx$ is the area under the graph of $y = 1 + e^{-2x}$ from $x = 0$ to $x = 3$.

$$A = \int_0^3 \left(1 + e^{-2x}\right) dx = \left[x + \left(-\frac{1}{2}\right) e^{-2x}\right]_0^3 = \left(3 - \frac{1}{2}e^{-6}\right) - \left(0 - \frac{1}{2}e^0\right) = \boxed{\tfrac{7}{2} - \tfrac{1}{2}e^{-6}}$$

The answer is C.

13. Use the substitution $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u\,du$. The lower limit of integration becomes $u = \sqrt{1} = 1$, and the upper limit of integration becomes $u = \sqrt{4} = 2$. Therefore,

$$\int_1^4 \frac{1 + x}{1 + \sqrt{x}}\, dx = \int_1^2 \frac{1 + u^2}{1 + u} 2u\, du = \boxed{2 \int_1^2 \tfrac{u + u^3}{1 + u}\, du}.$$

The answer is D.

14. Using the Chain Rule, $\frac{d}{dx} f(f(x)) = f'(f(x)) \cdot f'(x)$.

For $f(x) = \sqrt{25 - x^2}$, $f(-3) = \sqrt{25 - (-3)^2} = 4$ and

$$\frac{d}{dx} f(f(-3)) = f'(f(-3)) \cdot f'(-3) = f'(4) \cdot f'(-3).$$

Since $f'(x) = \frac{-x}{\sqrt{25 - x^2}}$, $\frac{d}{dx} f(f(-3)) = f'(4) \cdot f'(-3) = \left(\frac{-4}{\sqrt{25 - 4^2}}\right)\left[\frac{-(-3)}{\sqrt{25 - (-3)^2}}\right] = \left(\frac{-4}{3}\right)\left(\frac{3}{4}\right) = \boxed{-1}$.

The answer is B.

15. $$\int_{-1}^1 f(x)\, dx = \int_{-1}^0 f(x)\, dx + \int_0^1 f(x)\, dx = \int_{-1}^0 e^{2x}\, dx + \int_0^1 e^{-4x}\, dx$$

$$= \left[\frac{1}{2}e^{2x}\right]_{-1}^0 + \left[-\frac{1}{4}e^{-4x}\right]_0^1 = \frac{1}{2}\left(1 - e^{-2}\right) - \frac{1}{4}\left(e^{-4} - 1\right) = \boxed{\tfrac{3}{4} - \tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-4}}$$

The answer is D.

16. Use the limit definition of the derivative of function f at $x = x_0$, $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

    Apply the definition to $f(x) = \sin(2x)$ at $x_0 = \frac{\pi}{4}$.

    $$f'\left(\frac{\pi}{4}\right) = \lim_{h\to 0} \frac{\sin\left[2(\frac{\pi}{4}+h)\right] - \sin(2\cdot\frac{\pi}{4})}{h} = \boxed{\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{2}+2h\right) - \sin\left(\frac{\pi}{2}\right)}{h}}.$$

    The answer is B.

17. At $x = 3$, the graph of f crosses the x-axis. So, $f(3) = 0$.

    At $x = 3$, the graph of f is decreasing. So, $f'(3) < 0$.

    At $x = 3$, the graph of f is concave up. So, $f''(3) > 0$.

    Therefore, $\boxed{f'(3) < f(3) < f''(3)}$.

    The answer is C.

18. $$\lim_{x\to\infty} \frac{4x^3+3x^2+5}{5x^3+x^2-x} = \lim_{x\to\infty} \frac{4x^3+3x^2+5}{5x^3+x^2-x} \cdot \left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right) = \lim_{x\to\infty} \frac{4+\frac{3}{x}+\frac{5}{x^3}}{5+\frac{1}{x}-\frac{1}{x^2}} = \frac{4+0+0}{5+0-0} = \boxed{\frac{4}{5}}.$$

    The answer is B.

19. Given $v(3) = -3$, the velocity of the particle at time t is given by

    $$v(t) = v(3) + \int_3^t a(x)\,dx = -3 + \int_3^t 3\,dx = -3 + [3x]_3^t = -3 + (3t-9) = 3t - 12.$$

    The particle is at rest when $v(t) = 3t - 12 = 0$. So, the particle is at rest at $t = 4$.

    Given $x(2) = 5$, the position of the particle at time t is given by

    $$x(t) = x(2) + \int_2^t v(x)\,dx = 5 + \int_2^t (3x-12)\,dx.$$

    $$\text{So, } x(4) = 5 + \int_2^4 (3x-12)\,dx = 5 + \left[\frac{3}{2}x^2 - 12x\right]_2^4$$
    $$= 5 + [(24-48) - (6-24)] = \boxed{-1}.$$

    The answer is C.

20. For the rectangle with two sides along the x-axis and y-axis and inscribed using the function $f(x) = e^{-0.2x}$, the width of the rectangle is x and the height is $f(x) = e^{-0.2x}$. So, the area as a function of x is given by $A(x) = x \cdot e^{-0.2x}$. To find x so that the area of the rectangle is a maximum, set $A'(x) = 0$ and solve for x.

    Applying the Product Rule,

    $$A'(x) = x \cdot \left(\frac{d}{dx}e^{-0.2x}\right) + e^{-0.2x} \cdot \left(\frac{d}{dx}x\right) = x \cdot \left(-0.2e^{-0.2x}\right) + e^{-0.2x} \cdot (1) = e^{-0.2x}(1-0.2x).$$

    Since $e^{-0.2x} > 0$ for all x, $A'(x) = 0$ when $1 - 0.2x = 0$. So, $A'(x) = 0$ when $x = 5$.

    Note that $A'(x) > 0$ for $x < 5$ and $A'(x) < 0$ for $x > 5$. So, the area is a maximum for $\boxed{x = 5}$.

    The answer is D.

21. For $y = f(x) = \frac{x+2}{2x-6}$, find $f'(x)$ by applying the Quotient Rule.

$$f'(x) = \frac{d}{dx}\left(\frac{x+2}{2x-6}\right) = \frac{(2x-6)\left[\frac{d}{dx}(x+2)\right] - (x+2)\left[\frac{d}{dx}(2x-6)\right]}{(2x-6)^2}$$
$$= \frac{(2x-6)(1)-(x+2)(2)}{(2x-6)^2} = \frac{-10}{(2x-6)^2}.$$

The slope of the tangent line to the graph of y at the point $(4, 3)$ is $m_{\tan} = f'(4) = \frac{-10}{(8-6)^2} = -\frac{5}{2}$.

The slope of the normal line to the graph of y at the point $(4, 3)$ is $m_{\text{norm}} = \frac{-1}{f'(4)} = \frac{2}{5}$.

The equation of the normal line to the graph of y at the point $(4, 3)$ is $y - 3 = \frac{2}{5}(x-4)$.

Multiplying both sides by 5, $5y - 15 = 2(x-4)$ or $\boxed{5y - 2x = 7}$.

The answer is A.

22. For $f(x) = e^{kx^2}$, $f'(x) = e^{kx^2}\left[\frac{d}{dx}\left(kx^2\right)\right] = 2kxe^{kx^2}$

and $f''(x) = 2kx\left(\frac{d}{dx}e^{kx^2}\right) + e^{kx^2}\left[\frac{d}{dx}(2kx)\right] = (2kx)^2 e^{kx^2} + 2ke^{kx^2} = 2ke^{kx^2}\left(2kx^2+1\right)$.

Since f has a point of inflection at $x = \pm 2$, $f''(\pm 2) = 2ke^{4k}(8k+1) = 0$.

Since k is nonzero, and $e^{4k} > 0$ for all k, $8k + 1 = 0$. Therefore, $k = \boxed{-\frac{1}{8}}$.

The answer is B.

23. Apply the Product Rule.

$$\frac{d}{dt}\left[t^2 \int_2^t \ln(u+3)\,du\right] = \left(t^2\right)\left[\frac{d}{dt}\int_2^t \ln(u+3)\,du\right] + \left[\int_2^t \ln(u+3)\,du\right]\left(\frac{d}{dt}t^2\right)$$
$$= t^2 \ln(t+3) + \left[\int_2^t \ln(u+3)\,du\right](2t)$$
$$= \boxed{t^2 \ln(t+3) + 2t\int_2^t \ln(u+3)\,du}$$

The answer is D.

24. Rewrite the differential equation $\frac{dy}{dx} = y\cos(2x)$ as $\frac{dy}{y} = \cos(2x)\,dx$.

Integrate both sides to obtain $\int \frac{dy}{y} = \int \cos(2x)\,dx$

or $\ln|y| = 0.5\sin(2x) + C$ for some constant C.

Applying the condition that $y = 3$ when $x = 0$ yields $\ln 3 = 0.5\sin(0) + C$.

Thus, $C = \ln 3$ and $\ln|y| = 0.5\sin(2x) + \ln(3)$.

Solving for y yields $y = \pm e^{0.5\sin(2x)+\ln(3)} = \pm e^{0.5\sin(2x)}e^{\ln(3)} = \pm 3e^{0.5\sin(2x)}$.

The condition $y = 3$ when $x = 0$ requires the positive sign. Therefore, $\boxed{y = 3e^{0.5\sin(2x)}}$.

The answer is B.

25. $$\int_1^3 [3f'(x) + g(x)g'(x) - 6]\,dx = 3\int_1^3 f'(x)\,dx + \int_1^3 g(x)g'(x)\,dx - 6\int_1^3 dx$$
$$= 3[f(x)]_1^3 + \frac{1}{2}\left\{[g(x)]^2\right\}_1^3 - 6[x]_1^3$$
$$= 3[f(3) - f(1)] + \frac{1}{2}\left\{[g(3)]^2 - [g(1)]^2\right\} - 6(3-1)$$
$$= 3[4-2] + \frac{1}{2}\left\{5^2 - 1^2\right\} + 6(3-1)$$
$$= 6 + 12 - 12 = \boxed{6}$$

The answer is A.

26. If x represents the number of words memorized at time t, then expression $\frac{dx}{dt}$ represents the rate at which a list of M words are memorized. We are given that the rate is proportion to the product of the number of words memorized, M, and the number of words that have not been memorized, $M - x$. Therefore, the differential equation $\boxed{\frac{dx}{dt} = kx(M-x)}$ models this situation.

The answer is C.

27. For $f(x) = (x-3)^2(x-5)$,

$$f'(x) = (x-3)^2\left[\frac{d}{dx}(x-5)\right] + (x-5)\left[\frac{d}{dx}(x-3)^2\right]$$
$$= (x-3)^2 + 2(x-5)(x-3)$$
$$= (x-3)(3x-13).$$

Note that $f'(x) = 0$ for $x = 3$ and $x = \frac{13}{3}$.

For $x < 3$, $f'(x) > 0$. So, the graph of f is increasing for $x < 3$.

For $3 < x < \frac{13}{3}$, $f'(x) < 0$. So, the graph of f is decreasing for $3 < x < \frac{13}{3}$.

For $x > \frac{13}{3}$, $f'(x) > 0$. So, the graph of f is increasing for $x > \frac{13}{3}$.

At $x = \frac{13}{3}$, the graph of f changes from decreasing to increasing.

Therefore, the function has a relative minimum at $\boxed{x = \frac{13}{3}}$.

The answer is D.

28. Use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{d}{dx}\left(x^2 + 3y^2\right) = \frac{d}{dx}(12)$$
$$2x + 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2x}{6y} = -\frac{x}{3y}.$$

So $\left.\frac{dy}{dx}\right|_{(x,y)=(3,1)} = -\frac{(3)}{3(1)} = -1.$

Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{3y}\right) = -\frac{3y\frac{d}{dx}(x) - x\frac{d}{dx}(3y)}{(3y)^2} = -\frac{3y(1) - x\left(3\frac{dy}{dx}\right)}{9y^2} = -\frac{3y - 3x\frac{dy}{dx}}{9y^2} = \frac{3x\frac{dy}{dx} - 3y}{9y^2}.$

So $\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(3,1)} = \frac{3(3)(-1)-3(1)}{9(1)^2} = \frac{-9-3}{9} = -\frac{4}{3}$.

The answer is B.

29. Since $\lim_{x\to 0}[3x-\sin(3x)] = 0$ and $\lim_{x\to 0}[1-\cos(2x)] = 0$, $\lim_{x\to 0}\frac{3x-\sin(3x)}{1-\cos(2x)}$ is of indeterminate form $\frac{0}{0}$. Apply L'Hôpital's Rule to evaluate $\lim_{x\to 0}\frac{3x-\sin(3x)}{1-\cos(2x)}$.

$$\lim_{x\to 0}\frac{3x-\sin(3x)}{1-\cos(2x)} = \lim_{x\to 0}\frac{\frac{d}{dx}[3x-\sin(3x)]}{\frac{d}{dx}[1-\cos(2x)]} = \lim_{x\to 0}\frac{3-3\cos(3x)}{-2\sin(2x)} = -\frac{3}{2}\lim_{x\to\infty}\frac{1-\cos(3x)}{\sin(2x)}.$$

Since $\lim_{x\to 0}[1-\cos(3x)] = 0$ and $\lim_{x\to 0}\sin(2x) = 0$, $\lim_{x\to 0}\frac{1-\cos(3x)}{\sin(2x)}$ is of indeterminate form $\frac{0}{0}$.

Apply L'Hôpital's Rule to evaluate $-\frac{3}{2}\lim_{x\to 0}\frac{1-\cos(3x)}{\sin(2x)}$.

$$-\frac{3}{2}\lim_{x\to 0}\frac{1-\cos(3x)}{\sin(2x)} = -\frac{3}{2}\lim_{x\to 0}\frac{\frac{d}{dx}[1-\cos(3x)]}{\frac{d}{dx}[\sin(2x)]} = -\frac{3}{2}\lim_{x\to 0}\frac{3\sin(3x)}{2\cos(2x)} = -\frac{3}{2}\left(\frac{3\cdot 0}{2\cdot 1}\right) = \boxed{0}$$

The answer is B.

30. For graph I, the graph of f is concave down for $1 \le x \le 4$. So, $f''(x) < 0$ for $1 \le x \le 4$. The graph on the right could represent $f''(x)$ for function f on the left.

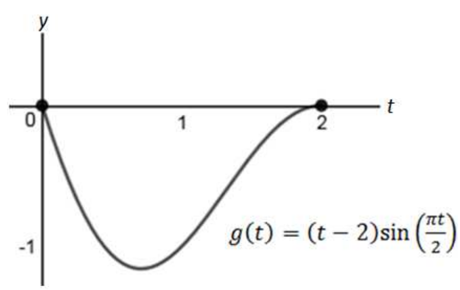
For graph II, the graph of f is concave up for $1 \le x \le 4$. So, $f''(x) > 0$ for $1 \le x \le 4$. The graph on the right could represent $f''(x)$ for function f on the left.

For graph III, the graph of f is concave down for $1 \le x < \frac{5}{2}$ and concave up for $\frac{5}{2} < x \le 4$. So, $f''(x) < 0$ for $1 \le x \le \frac{5}{2}$ and $f''(x) > 0$ for $\frac{5}{2} < x \le 4$. The graph on the right could represent $f''(x)$ for function f on the left.

The answer is D.

Practice AP® Exam, Part II

31. Use a graphing calculator to examine the graph of $g(t) = (t-2)\sin\left(\frac{\pi t}{2}\right)$ on the interval $0 \le t \le 2$.



Since $g(t) < 0$ for all $0 \le t \le 2$, $f(x) = \int_0^x (t-2)\sin\left(\frac{\pi t}{2}\right)dt < 0$ for all $0 \le x \le 2$.

Using the Fundamental Theorem,

$$f'(x) = \frac{d}{dx}\int_0^x (t-2)\sin\left(\frac{\pi t}{2}\right)dt = (x-2)\sin\left(\frac{\pi x}{2}\right) < 0 \text{ for all } 0 < x < 2.$$

Finally, since the graph of g is increasing on the interval $k < x < 2$ for some k, $f''(x) = g'(t) > 0$ on the interval $k < x < 2$. Therefore, only statements I and II are true.

The answer is B.

32. For function $f(x) = 2x^2 + \sqrt[3]{x^4 - 8}$ defined for $x \geq 0$, find x so that $f(x) = 10$.

    Using a graphing calculator, $f(x) = 10$ for $x = 2$.

    Using a graphing calculator, $f'(2) = \frac{32}{3}$.

    Since g is the inverse of f and $f'(2) = \frac{32}{3} \neq 0$, then $g'(10) = \frac{1}{f'(2)} = \frac{1}{\left(\frac{32}{3}\right)} = \frac{3}{32} \approx \boxed{0.0937}$.

    The answer is D.

33. For $f(x) = e^x - x^e - e$, find the point where the graph of f crosses the x-axis.

    Using a graphing calculator to solve$f(x) = 0$, $x \approx 3.4711$.

    Using a graphing calculator, $f'(3.4711) \approx 9.1063$.

    The slope of the normal line to the graph of f at $x \approx 3.4711$ is

    $m_{\text{norm}} \approx -\frac{1}{f'(3.4711)} \approx -\frac{1}{9.1063} \approx \boxed{-0.110}$.

    The answer is B.

34. Functions f and g have perpendicular tangents for all x such that $f'(x) \cdot g'(x) = -1$.

    Since $f'(x) = \frac{1}{x+1}$ and $g'(x) = -\frac{1}{2x\sqrt{x}}$, find all x such that $\left(\frac{1}{x+1}\right) \cdot \left(-\frac{1}{2x\sqrt{x}}\right) = -1$.

    Equivalently, find all x such that $(x+1)(2x\sqrt{x}) = 1$.

    Using a graphing calculator, $x \approx \boxed{0.484}$.

    The answer is A.

35. Partition $[0, 2]$ into five subintervals, each of equal width: $[0, 0.4]$, $[0.4, 0.8]$, $[0.8, 1.2]$, $[1.2, 1.6]$, and $[1.6, 2.0]$.

    The width of each subinterval is $\Delta x = 0.4$.

    Now apply the Trapezoidal Rule:

$$\begin{aligned}\int_0^2 f(x)\,dx &\approx \frac{1}{2}[f(0) + f(0.4)]\Delta x + \frac{1}{2}[f(0.4) + f(0.8)]\Delta x \\ &\quad + \frac{1}{2}[f(0.8) + f(1.2)]\Delta x + \frac{1}{2}[f(1.2) + f(1.6)]\Delta x + \frac{1}{2}[f(1.6) + f(2.0)]\Delta x \\ &= \frac{1}{2}[f(0) + 2f(0.4) + 2f(0.8) + 2f(1.2) + 2f(1.6) + f(2.0)](0.4) \\ &= \frac{1}{2}[10 + 2(12) + 2(13) + 2(16) + 2(19) + 20](0.4) \\ &= \boxed{30}.\end{aligned}$$

    The answer is B.

36. Given $v(0) = 2$, the velocity of the particle at time t is given by

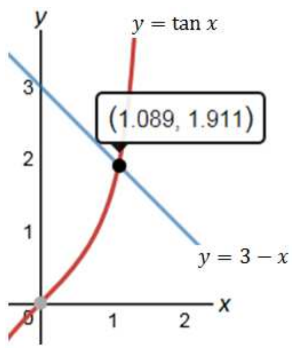
$$v(t) = v(0) + \int_0^t \left(3 - \sqrt{x}\right) dx = 2 + \int_0^t \left(3 - \sqrt{x}\right) dx.$$

    The velocity at time $t = 4$ is $v(4) = 2 + \int_0^4 (3 - \sqrt{x})\,dx = 2 + \frac{20}{3} = \boxed{\frac{26}{3}}$.

    The answer is C.

37. Using a graphing calculator, the region in the first quadrant enclosed by the graph of $y = \tan x$, $y = 3 - x$, and the y-axis is pictured below.

Using a graphing calculator, the graphs of $y = \tan x$ and $y = 3 - x$ intersect at $x \approx 1.089$.



The area of the region in the first quadrant enclosed by the graph of $y = \tan x$, $y = 3 - x$, and the y-axis is given by $A \approx \int_0^{1.089} [(3 - x) - \tan x]\, dx$.

Using a graphing calculator, $A \approx \int_0^{1.089} [(3 - x) - \tan x]\, dx \approx \boxed{1.905}$.

The answer is A.

38. Let $w(t)$ denote the weight of the animal after t days.

Then $w(t) = w(0) + \int_0^t r(x)\, dx = w(0) + \int_0^t 0.16e^{\sqrt{x}}\, dx$.

The number of pounds gained from $t = 0$ to $t = 5$ days is $w(5) - w(0) = \int_0^5 0.16e^{\sqrt{x}}\, dx$.

Using a graphing calculator, $w(5) - w(0) = \int_0^5 0.16e^{\sqrt{x}}\, dx \approx \boxed{4.0 \text{ pounds}}$.

The answer is B.

39. The function $g(x) = \frac{x \ln x - x}{x+1}$ is defined for all $x > 0$.

For $g(x) = \frac{x \ln x - x}{x+1}$, find $g'(x)$ using the Quotient Rule.

$$g'(x) = \frac{(x+1)\left\{\frac{d}{dx}(x \ln x - x)\right\} - (x \ln x - x)\left[\frac{d}{dx}(x+1)\right]}{(x+1)^2}$$
$$= \frac{(x+1)\left(x \cdot \frac{1}{x} + \ln x - 1\right) - (x \ln x - x)(1)}{(x+1)^2} = \frac{\ln x + x}{(x+1)^2}.$$

To find a critical number for $g(x)$, set $g'(x) = \frac{\ln x + x}{(x+1)^2} = 0$.

Using a graphing calculator to solve $\ln x + x = 0$, $x \approx \boxed{0.567}$.

The answer is C.

40. The distance between a point (x, y) on the graph $y = x^2 + x$ and the point $(3, 4)$ is given by

$d = \sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-3)^2 + (x^2 + x - 4)^2}$.

Since $d > 0$, finding x so that d is a minimum is equivalent to finding x so that d^2 is a minimum.

Let $f(x) = d^2 = (x-3)^2 + (x^2 + x - 4)^2$.

Then $f'(x) = 2(x-3) + 2(x^2+x-4)(2x+1)$.

To find the critical numbers of f, solve $f'(x) = 2(x-3) + 2(x^2+x-4)(2x+1) = 0$.

Using a graphing calculator to solve $2(x-3) + 2(x^2+x-4)(2x+1) = 0$, $x \approx -2.137, -1$, and 1.637.

Note that $f(-2.137) \approx 28.854$, $f(-1) \approx 32$, and $f(1.64) \approx 1.958$.

Therefore, the minimum distance is $d \approx \sqrt{1.958} \approx \boxed{1.399}$.

The answer is B.

41. The perimeter of a regular hexagon is given by $P = 6s$, where s is the length of each side.

When the perimeter is $P = 120$ cm, the length of each side is $s = \frac{1}{6}P = \frac{1}{6}(120 \text{ cm}) = 20$ cm.

The perimeter is increasing at a constant rate of 30 cm/min. So, $\frac{dP}{dt} = 30\frac{\text{cm}}{\text{min}}$.

The rate of increase in the area of the hexagon is

$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{3\sqrt{3}}{2}s^2\right) = \frac{3\sqrt{3}}{2}\left(\frac{d}{dt}s^2\right) = \frac{3\sqrt{3}}{2}\left(2s\frac{ds}{dt}\right) = 3\sqrt{3}\left(s\frac{ds}{dt}\right).$$

From $P = 6s$, $\frac{dP}{dt} = \frac{d}{dt}(6s) = 6\frac{ds}{dt}$.

So, $\frac{ds}{dt} = \frac{1}{6}\frac{dP}{dt} = \frac{1}{6}\left(30\frac{\text{cm}}{\text{min}}\right) = 5\frac{\text{cm}}{\text{min}}$.

Therefore, $\frac{dA}{dt} = 3\sqrt{3}\left(s\frac{ds}{dt}\right) = 3\sqrt{3}(20 \text{ cm})\left(5\frac{\text{cm}}{\text{min}}\right) \approx \boxed{519.615\frac{\text{cm}^2}{\text{min}}}$.

The answer is B.

42. For $g(x) = \int_0^x \left[3f(t) + \sqrt{t^3+1}\right] dt$,

$$g(2) = \int_0^2 \left[3f(t) + \sqrt{t^3+1}\right] dt = 3\int_0^2 f(t)\,dt + \int_0^2 \sqrt{t^3+1}\,dt.$$

Using a graphing calculator, $\int_0^2 \sqrt{t^3+1}\,dt \approx 3.241$.

Since $\int_0^2 f(t)\,dt = 10$, $g(2) = 3\int_0^2 f(t)\,dt + \int_0^2 \sqrt{t^3+1}\,dt \approx 3(10) + 3.241 = 33.241$.

By Part 1 of the Fundamental Theorem of Calculus,

$$g'(x) = \frac{d}{dx}\int_0^x \left[3f(t) + \sqrt{t^3+1}\right] dt = 3f(x) + \sqrt{x^3+1},$$

and $g'(2) = 3f(2) + \sqrt{2^3+1} = 3(4) + 3 = 15$.

Therefore, $g(2) + g'(2) \approx 33.241 + 15 = \boxed{48.241}$.

The answer is C.

43. The average density across the length of the rod is $\bar{f} = \frac{1}{2-0}\int_0^2 \sqrt{10-x^3}\,dx$.

Using a graphing calculator, $\int_0^2 \sqrt{10-x^3}\,dx \approx 5.584$.

Therefore, the average density is $\bar{f} = \frac{1}{2-0}\int_0^2 \sqrt{10-x^3}\,dx \approx \frac{1}{2}(5.584) = \boxed{2.792}$.

The answer is C.

44. The distance traveled by the object from $t = 0$ to $t = 10$ is given by $\int_0^{10} |v(t)|\, dt$.

    Since $v(t) = e^{0.2t} - 3$, we find that $v(t) \le 0$ if $0 \le t \le 5 \ln 3$ and $v(t) \ge 0$ if $5 \ln 3 \le t \le 10$. Therefore,

    $$\int_0^{10} |v(t)|\, dt = \int_0^{5\ln 3} [-v(t)]\, dt + \int_{5\ln 3}^{10} v(t)\, dt = -\int_0^{5\ln 3} \left(e^{0.2t} - 3\right) dt + \int_{5\ln 3}^{10} \left(e^{0.2t} - 3\right) dt.$$

    Using a graphing calculator, $\int_0^{5\ln 3} \left(e^{0.2t} - 3\right) dt \approx -6.4792$ and $\int_{5\ln 3}^{10} \left(e^{0.2t} - 3\right) dt \approx 8.4245$.

    Therefore, the distance is

    $$-\int_0^{5\ln 3} \left(e^{0.2t} - 3\right) dt + \int_{5\ln 3}^{10} \left(e^{0.2t} - 3\right) dt \approx -(-6.4792) + 8.4245 \approx \boxed{14.904}.$$

    The answer is D.

45. Consider cross sections perpendicular to the x-axis of thickness dx.

    The base of each triangle is $[\sin x - (1 - \sin x)] = 2 \sin x - 1$.

    The height of each triangle is $\frac{1}{2}(2 \sin x - 1)$.

    The area of each triangle is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\left[\frac{1}{2}(2 \sin x - 1)\right](2 \sin x - 1) = \frac{1}{4}(2 \sin x - 1)^2$.

    The volume of each cross section is $dV = (\text{Area})\, dx = \frac{1}{4}(2 \sin x - 1)^2\, dx$.

    To find the range for x, find the intersection of $y = \sin x$ and $y = 1 - \sin x$.

    $$\begin{aligned} \sin x &= 1 - \sin x \\ 2 \sin x &= 1 \\ \sin x &= \frac{1}{2} \end{aligned}$$

    So, $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

    Therefore, the volume of the solid is $V = \int_{\pi/6}^{5\pi/6} \frac{1}{4}(2 \sin x - 1)^2\, dx = \frac{1}{4} \int_{\pi/6}^{5\pi/6} (2 \sin x - 1)^2\, dx$.

    Using a graphing calculator, $V = \frac{1}{4} \int_{\pi/6}^{5\pi/6} (2 \sin x - 1)^2\, dx \approx \boxed{0.272}$.

    The answer is B.

AP® Practice Exam

Part 2, Free Response

1. (a) Partition $[0, 6]$ into four subintervals: $[0, 1], [1, 3], [3, 5]$, and $[5, 6]$.

    $\Delta x_1 = 1 - 0 = 1,\ \Delta x_2 = 3 - 1 = 2,\ \Delta x_3 = 5 - 3 = 2,$ and $\Delta x_4 = 6 - 5 = 1.$

Now apply the Trapezoidal Rule:

$$\frac{1}{6}\int_0^6 E(x)\,dx \approx \frac{1}{6}\left\{\frac{1}{2}[E(0)+E(1)]\Delta x_1 + \frac{1}{2}[E(1)+E(3)]\Delta x_2 + \frac{1}{2}[E(3) + E(5)]\Delta x_3 + \frac{1}{2}[E(5)+E(6)]\Delta x_4\right\}$$

$$= \frac{1}{12}\{[E(0)+E(1)](1)+[E(1)+E(3)](2)+[E(3)+E(5)](2) + [E(5)+E(6)](1)\}$$

$$= \frac{1}{12}[E(0)+3E(1)+4E(3)+3E(5)+E(6)]$$

$$= \frac{1}{12}[1.0+3(1.3)+4(2.1)+3(4.1)+5.6]$$

$$= \boxed{2.6 \text{ thousand feet}}.$$

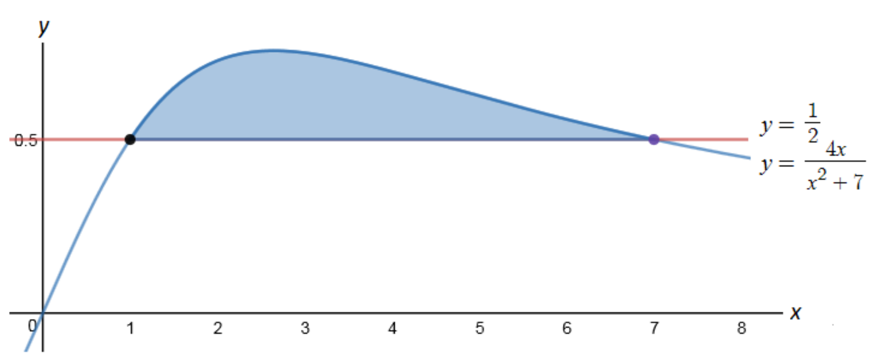
(b) $\frac{1}{6}\int_0^6 E(x)\,dx = 2.6$ thousand feet is the average elevation of the trail along the entire length of the trail.

(c) For a function f that is continuous on the closed interval $a \le x \le b$ and differentiable on the open interval $a < x < b$, the Mean Value Theorem guarantees that $f'(c) = \frac{f(b)-f(a)}{b-a}$ for at least one c between a and b.

Since function E is differentiable on the interval $0 < x < 6$, it is also continuous on the interval. For the interval $3 < x < 5$, the conditions of the Mean Value Theorem are satisfied. Since $E(3) = 2.1$ and $E(5) = 4.1$, there $E'(c) = \frac{E(5)-E(3)}{5-3} = \frac{4.1-2.1}{2} = 1\frac{\text{ft}}{\text{mile}}$ for at least one c between 3 and 5. Therefore, there must be at least one distance x for which the elevation increases at a rate of 1000 feet per mile.

(d) Since $E'(x) > 0$ on the interval $0 \le x \le 6$, E is increasing on the interval $0 \le x \le 6$. Therefore, the Left Riemann sum is the lower sum and underapproximates the value of $\frac{1}{6}\int_0^6 E(x)\,dx$.

2. (a) The region is pictured below.



To find the points of intersection of the curve $y = \frac{4x}{x^2+7}$ with the horizontal line $y = \frac{1}{2}$, set $\frac{4x}{x^2+7} = \frac{1}{2}$ and solve for x. Multiply both sides by $2(x^2+7)$ to obtain

$8x = x^2 + 7$

$x^2 - 8x + 7 = 0$

$(x-1)(x-7) = 0$

The curve and the line intersect at $x = 1$ and 7.

The area bounded by the two curves is $\int_1^7 \left(\frac{4x}{x^2+7} - \frac{1}{2}\right) dx$.

Using a graphing calculator, the area is $\int_1^7 \left(\frac{4x}{x^2+7} - \frac{1}{2}\right) dx \approx \boxed{0.892}$.

(b) Use the method of rings to find the volume of the resulting solid when the region in (a) is rotated about the line $y = -2$.

$$\begin{aligned}\text{Volume} &= \pi \int_1^7 \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dx \\ &= \pi \int_1^7 \left[\left(2 + \frac{4x}{x^2+7}\right)^2 - \left(2 + \frac{1}{2}\right)^2\right] dx \approx \boxed{14.551}\end{aligned}$$

(c) Each section perpendicular to the x-axis is a rectangle with a height equal to five times the length of the base. The base is $\frac{4x}{x^2+7} - \frac{1}{2}$. The height is $5\left(\frac{4x}{x^2+7} - \frac{1}{2}\right)$.

The area of the rectangle is $5\left(\frac{4x}{x^2+7} - \frac{1}{2}\right)^2$.

Multiply the area by the thickness of the section dx.

The total volume is given by $V = 5\int_1^7 \left(\frac{4x}{x^2+7} - \frac{1}{2}\right)^2 dx \approx \boxed{0.862}$.

3. (a) The acceleration of the object is given by $a(t) = v'(t)$.

Use the product rule to find $v'(t)$.

$$\begin{aligned}a(t) = v'(t) &= 10\left\{e^{-0.1t}\left[\frac{d}{dt}\cos\left(\frac{\pi t}{4}\right)\right] + \cos\left(\frac{\pi t}{4}\right)\left[\frac{d}{dt}e^{-0.1t}\right]\right\} \\ &= 10\left\{e^{-0.1t}\left[-\sin\left(\frac{\pi t}{4}\right)\cdot\frac{\pi}{4}\right] + \cos\left(\frac{\pi t}{4}\right)\left[e^{-0.1t}(-0.1)\right]\right\} \\ &= -10e^{-0.1t}\left[\frac{\pi}{4}\sin\left(\frac{\pi t}{4}\right) + 0.1\cos\left(\frac{\pi t}{4}\right)\right] \\ &= \boxed{-e^{-0.1t}\left[\frac{5\pi}{2}\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{4}\right)\right]}\end{aligned}$$

(b) The position of the object at time $t = 5$ is given by

$$\begin{aligned}x(5) &= x(0) + \int_0^5 v(t)\,dt \\ &= \boxed{4 + \int_0^5 10e^{-0.1t}\cos\left(\frac{\pi t}{4}\right) dt}.\end{aligned}$$

(c) Since both the velocity $v(t)$ and acceleration $a(t)$ are negative for $2 < t < 3.839$, the speed of the object is $\boxed{\text{increasing on the interval } 2 < t < 3.839}$.

Since $v(t) < 0$ for $2 < t < 6$, the object is $\boxed{\text{moving to the left on the interval } 2 < t < 6}$.

(d) The distance traveled by the object on the interval $0 \le t \le 6$ is

$$d = \int_0^6 |v(t)|\,dt = \boxed{10\int_0^6 \left|e^{-0.1t}\cos\left(\frac{\pi t}{4}\right)\right| dt}.$$

4. (a) $g(2) = \int_0^2 f(t)\,dt$ is the area of the triangular region between f and the x-axis from $x = 0$ to $x = 2$. So, $g(2) = \int_0^2 f(t)\,dt = \frac{1}{2}(2)(4) = \boxed{4}$.

Using Part 1 of the Fundamental Theorem, $g'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$. So $g'(2) = f(2) = \boxed{4}$.

Since $g'(x) = f(x), g''(x) = f'(x)$. The graph of f has a cusp at $x = 2$. The slope of the line tangent to the graph of f at $x = 2$ is not defined. So, $g''(2)$ $\boxed{\text{does not exist}}$.

(b) Since $g''(x) = f'(x) > 0$ for $x < 2$ and $g''(x) = f'(x) < 0$ for $x > 2$, function g has a point of inflection at $\boxed{x = 2}$.

(c) For $h(x) = \int_{-2}^x f(t)\,dt$, $h'(x) = \frac{d}{dx}\int_{-2}^x f(t)\,dt = f(x) = 0$ when $x = 5$. So, h has a critical number at $x = 5$. Since $h'(x) = f(x) > 0$ for $x < 5$ and $h'(x) = f(x) < 0$ for $x > 5$, function h has a relative maximum at $\boxed{x = 5}$.

(d) Using properties of integrals, break up $\int_{-2}^6 f(t)\,dt$.

$$\int_{-2}^6 f(t)\,dt = \int_{-2}^0 f(t)\,dt + \int_0^2 f(t)\,dt + \int_2^4 f(t)\,dt + \int_4^6 f(t)\,dt.$$

$$h(6) = h(0) + \int_0^2 f(t)\,dt + \int_2^4 f(t)\,dt + \int_4^6 f(t)\,dt.$$

From the graph, $\int_0^2 f(t)\,dt = 4$ and $\int_4^6 f(t)\,dt = 0$. Also, $h(6) = \frac{26}{3}$ and $h(0) = -2$.

So, $\frac{26}{3} = -2 + 4 + \int_2^4 f(t)\,dt + 0$ and $\int_2^4 f(t)\,dt = \boxed{\frac{20}{3}}$.

5. (a) Since $f(x) = \int_0^{4x} \sqrt{100 - t^3}\,dt$, $g(0) = \int_0^0 \sqrt{100 - t^3}\,dt = 0$.

Using Part 1 of the Fundamental Theorem,

$$g'(x) = \frac{d}{dx}\int_0^{4x} \sqrt{100 - t^3}\,dt = \sqrt{100 - (4x)^3}\left[\frac{d}{dx}(4x)\right] = 4\sqrt{100 - 64x^3}.$$

So, the slope of the line tangent to the graph of g at $x = 0$ is $g'(0) = 40$.

Using the point-slope form of the line, $y - 0 = 40(x - 0)$.

The equation of the line tangent to the graph of g at $x = 0$ is $\boxed{y = 40x}$.

(b) Since $g'(x) = 4\sqrt{100 - 64x^3} > 0$ on the interval $0 \le x \le 1$, function g is increasing on the interval $0 \le x \le 1$. So, g is one-to-one on the interval $0 \le x \le 1$ and therefore has an inverse on the interval $0 \le x \le 1$.

(c) Since $g(1) = A$ and h is the inverse of g, $h'(A) = \frac{1}{g'(1)} = \frac{1}{4\sqrt{100 - 64(1^3)}} = \boxed{\frac{1}{24}}$.

(d)
$$\begin{aligned} g''(x) &= \frac{d}{dx}g'(x) = \frac{d}{dx}4\left(100 - 64x^3\right)^{1/2} \\ &= 4\left(\frac{1}{2}\right)\left(100 - 64x^3\right)^{-1/2}\left[\frac{d}{dx}(100 - 64x^3)\right] \\ &= \frac{-384x^2}{\sqrt{100 - 64x^3}} \end{aligned}$$

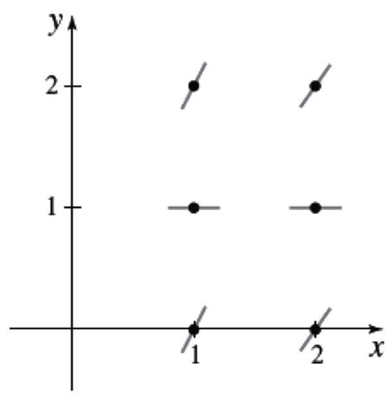
Since $g''(x) < 0$ on the interval $0 \le x \le 1$, the graph of g is concave down on the interval $0 \le x \le 1$. Therefore, a trapezoidal sum used to estimate $g(1) = \int_0^4 \sqrt{100 - t^3}\,dt$ is an $\boxed{\text{underestimation}}$.

6. (a) The right side of the differential equation $\frac{dy}{dx} = \frac{2(y-1)^2}{\sqrt{x}}$ gives the slope of the line tangent to the graph of $y(x)$ at (x, y). For example, the slope of the line tangent to the graph of $y(x)$ at $(1, 2)$ is $\frac{2(2-1)^2}{\sqrt{1}} = 2$. Continuing in this manner, the slope is calculated for each point indicated on the graph.

(x, y)	Slope
$(1, 2)$	2
$(1, 1)$	0
$(1, 0)$	2

(x, y)	Slope
$(2, 2)$	$\frac{2}{\sqrt{2}}$
$(2, 1)$	0
$(2, 0)$	$\frac{2}{\sqrt{2}}$

The slope at each point is plotted on the graph below to produce the slope field.



(b) The line tangent to the graph of f at the point $(1, 2)$ has slope $m = \frac{2(2-1)^2}{\sqrt{1}} = 2$.

Use the point-slope form of a line to find the equation of the line.

$y - 2 = 2(x - 1)$

$\boxed{y - 2 = 2x - 2}$ or $\boxed{y = 2x}$

(c) Rewrite the differential equation $\frac{dy}{dx} = \frac{2(y-1)^2}{\sqrt{x}}$ as $\frac{dy}{(y-1)^2} = \frac{2}{\sqrt{x}}\,dx$.

Integrate both sides to obtain $\int \frac{dy}{(y-1)^2} = \int \frac{2}{\sqrt{x}}\,dx$ or $-\frac{1}{y-1} = 4\sqrt{x} + C$ for some constant C.

Applying the condition that $y = 2$ when $x = 1$ yields $-1 = 4 + C$.

Thus, $C = -5$ and $-\frac{1}{y-1} = 4\sqrt{x} - 5$.

Solving for y yields $\boxed{y = \frac{1}{5-4\sqrt{x}} + 1}$.