Chapter 9 Parametric Equations; Polar Equations

9.1 Parametric Equations

Concepts and Vocabulary

1. Let $x = x(t)$ and $y = y(t)$ be two functions whose common domain is some interval I. The collection of points defined by $(x, y) = (x(t), y(t))$ is called a plane $\boxed{\textbf{curve}}$. The variable t is called a $\boxed{\textbf{parameter}}$.

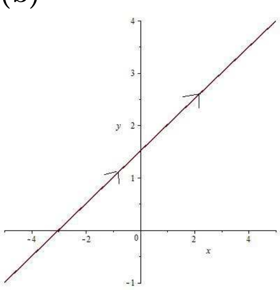
3. The parametric equations $x(t) = a \sin t, y(t) = a \cos t$ define a $\boxed{\textbf{(d) circle}}$.

5. $\boxed{\textbf{False}}$. A curve can be defined parametrically in an infinite number of ways.

Skill Building

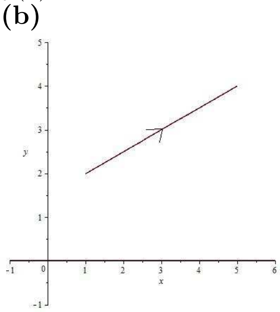
7. (a) Solving for t in the $x(t)$ equation, we have $t = \frac{1}{2}(x-1)$. Plug this into $y(t)$: $y = \frac{1}{2}(x-1)+2$ or $\boxed{y = \frac{1}{2}x + \frac{3}{2}}$. Since t varies from $-\infty$ to $+\infty$, the rectangular equation is defined for all x from $-\infty$ to ∞.

(b)

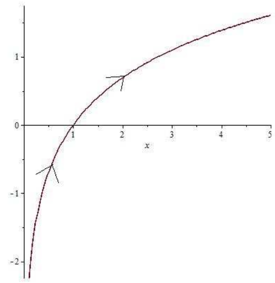


9. (a) Solving for t in the $y(t)$ equation, we have $t = y - 2$. Plug this into $x(t)$: $x = 2(y-2)+1$ or $\boxed{x = 2y - 3}$. Since t varies from 0 to 2, we have $1 \le x \le 5$ and y varies from $y(0) = 2$ to $y(2) = 4$.

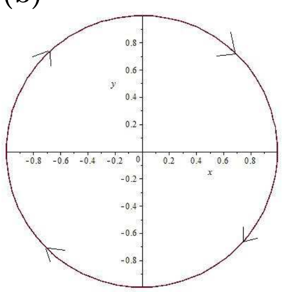
(b)



11. (a) Plug $y = t$ into $x(t)$: $\boxed{x = e^y}$. This can be rewritten as $y = \ln x$. Since $-\infty < t < \infty$, we see that $0 < x(t) < \infty$.
(b)



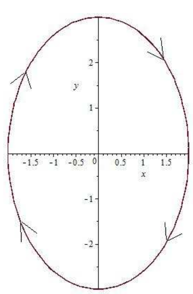
13. (a) Notice that $[x(t)]^2 + [y(t)]^2 = \sin^2 t + \cos^2 t = 1$, which means our parametrization is that of a circle of radius 1 centered at $(0, 0)$, namely $\boxed{x^2 + y^2 = 1}$. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. We find, respectively, the points $(0, 1), (1, 0), (0, -1), (-1, 0)$, and $(0, 1)$. Plotted in this order, we trace a circle starting at the point $(0, 1)$ and moving clockwise until we arrive back at $(0, 1)$.
(b)



15. (a) We eliminate the parameter t using the Pythagorean Theorem $\sin^2 t + \cos^2 t = 1$. Since $\sin t = \frac{x}{2}$ and $\cos t = \frac{y}{3}$,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t = 1$$
$$\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$$

which is an ellipse centered at the origin. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. We find, respectively, the points $(0, 3), (2, 0), (0, -3), (-2, 0)$, and $(0, 3)$. Plotted in this order, we trace an ellipse oriented clockwise.
(b)

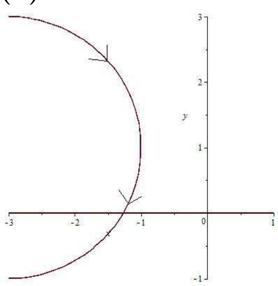


17. (a) Notice that $2\sin t = x+3$ and $2\cos t = y-1$. If we square and add both equations, then we have

$$(x+3)^2 + (y-1)^2 = 4\sin^2 t + 4\cos^2 t = 4$$
$$\boxed{\frac{(x+3)^2}{4} + \frac{(y-1)^2}{4} = 1},$$

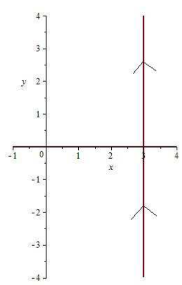
which is a circle centered at $(-3, 1)$ with radius 2. To find its starting point, ending point, and orientation, plot the points $t = 0, \dfrac{\pi}{2}$, and π. We find, respectively, the points $(-3, 3), (-1, 1)$, and $(-3, -1)$. Plotted in this order, we trace the right half of the circle centered at $(-3, 1)$ with radius 2 oriented clockwise.

(b)



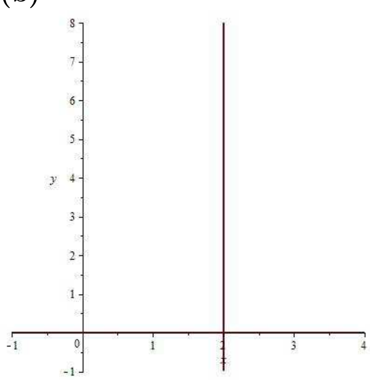
19. (a) Since $x(t) = 3$, every point on our plane curve has x-coordinate equal to 3. These points describe the line $\boxed{x = 3}$. Since t goes from $-\infty$ to ∞, so does y so the orientation of the curve is upward.

(b)



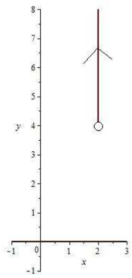
21. (a) Since $x(t) = 2$, the rectangular equation is $\boxed{x = 2}$.

(b)



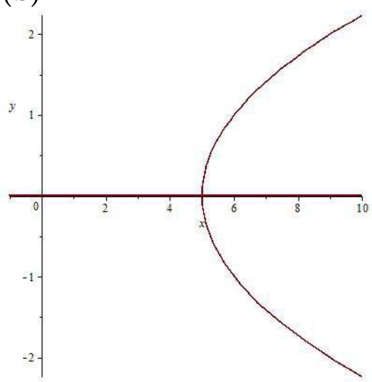
(c) Since $t > 0$, we know that $\boxed{y > 4}$.

(d)



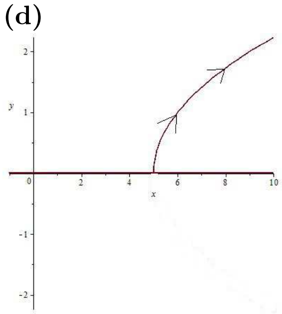
23. (a) Solving for t in the $y(t)$ equation, we have $t = y^2$. Plug this into the $x(t)$ equation: $\boxed{x = y^2 + 5}$.

(b)



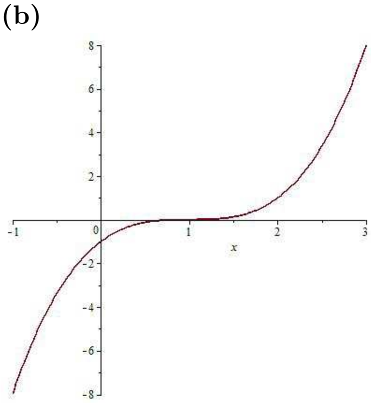
(c) Since $t \geq 0$, we see that the parametric curve is defined only for $\boxed{x \geq 5}$ and $\boxed{y \geq 0}$.

(d)



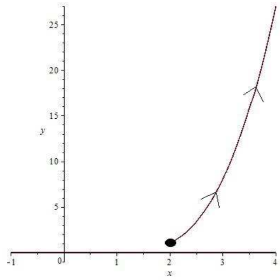
25. (a) Solving for t in the $y(t)$ equation: $t = y^{2/3}$. Plug this into the $x(t)$ equation: $x = (y^{2/3})^{1/2} + 1$ which is the same as $x = y^{1/3} + 1$, or, equivalently, $\boxed{y = (x-1)^3}$.

(b)



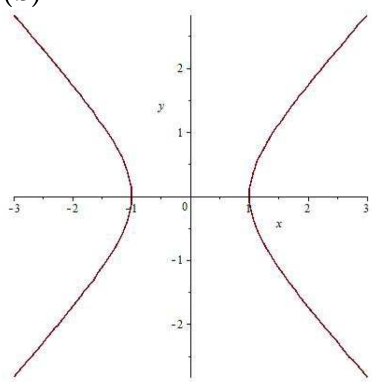
(c) Since $t \geq 1$, we see that $\boxed{x \geq 2}$ and $\boxed{y \geq 1}$.

(d)



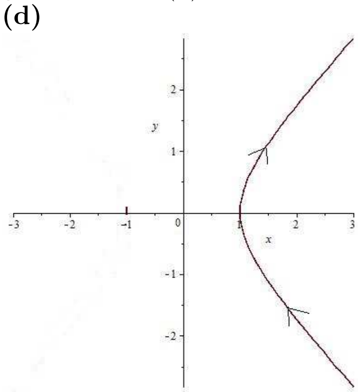
27. (a) Solving for t we have y equation, we have $t = \tan^{-1} y$, so that $\boxed{x = \sec\left(\tan^{-1} y\right)}$.

(b)



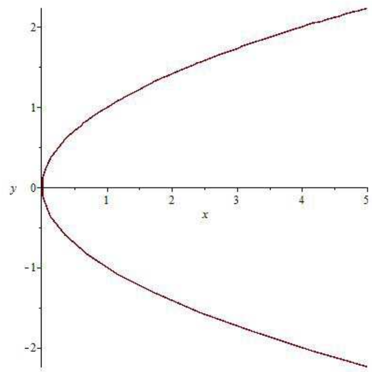
(c) Since $\sec t \geq 1$ for all $t \geq 1$, we have $\boxed{x \geq 1}$, so we only have the right-hand portion of the graph in part (b).

(d)



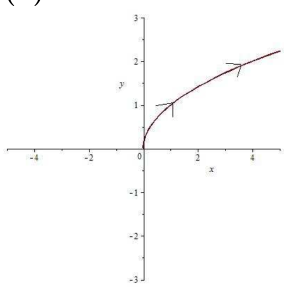
29. (a) Since $x = t^4$ and $y^2 = (t^2)^2 = t^4$, we see that $\boxed{x = y^2}$.

(b)



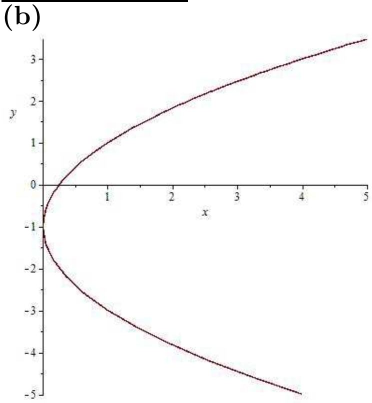
(c) There are no restrictions on t, so we assume $-\infty < t < \infty$. Since $y(t)$ is a even power of t, we have that $\boxed{y \geq 0}$ and also $\boxed{x \geq 0}$.

(d)



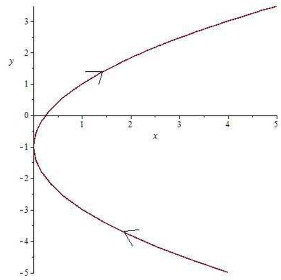
31. (a) Solving for t in the $y(t)$ equation: $t = \dfrac{y+1}{2}$. Plug this into the $x(t)$ equation: $\boxed{x = \left(\dfrac{y+1}{2}\right)^2}$.

(b)



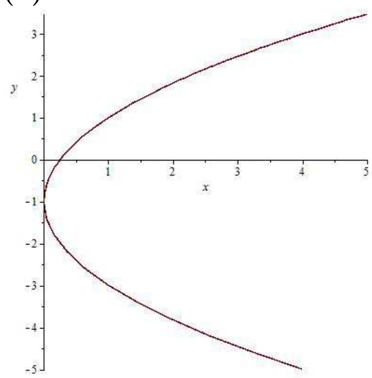
(c) Since $x \geq 0$, there are $\boxed{\text{no restrictions on the plane curve}}$.

(d)



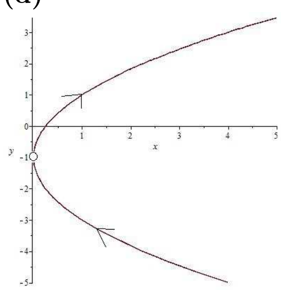
33. (a) Notice that this parametrization is the same as in Problem 31 replacing t with $\frac{1}{t}$. This means the plane curve is the same: $\boxed{x = \left(\frac{y+1}{2}\right)^2}$.

(b)



(c) Since $t \neq 0$, $\boxed{x > 0}$ and $\boxed{y \neq -1}$.

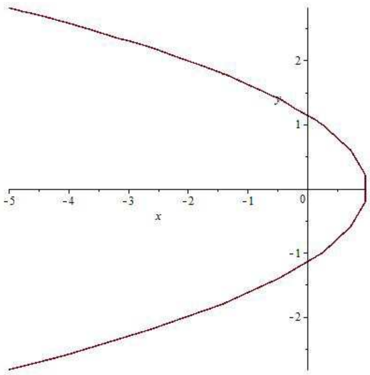
(d)



35. (a) Solving for $\sin^2 t$ in $x(t)$ we have $\sin^2 t = \frac{x+2}{3}$. Since $\cos^2 t = \left(\frac{y}{2}\right)^2$ we see that

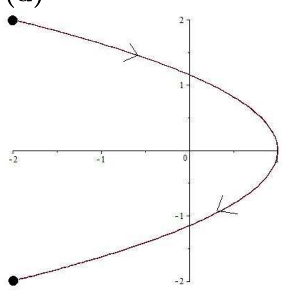
$$\boxed{\frac{x+2}{3} + \frac{y^2}{4} = 1}.$$

(b)



(c) For $0 \leq t \leq \pi$, we have $0 \leq \sin t \leq 1$ and $-1 \leq \cos t \leq 1$ so we see that $\boxed{-2 \leq x \leq 1}$ and $\boxed{-2 \leq y \leq 2}$.

(d)



37. Solving for t^2, we have $t^2 = 1/x$. Now plug this into y:

$$\boxed{y = \frac{2}{1/x + 1} = \frac{2x}{1+x}}.$$

As t increases, x approaches 0 and so does y. Also $\boxed{x \geq 0}$ and $\boxed{0 < y \leq 2}$.

39. Solving for t^2 in the $x(t)$ equation, we have

$$\begin{aligned} x^2 &= \frac{16}{4-t^2} \\ 4-t^2 &= \frac{16}{x^2} \\ t^2 &= 4 - \frac{16}{x^2}. \end{aligned}$$

Taking the square root to solve for t and plugging both t and t^2 into the $y(t)$ equation, we have

$$y = \frac{4\left(\sqrt{4 - \frac{16}{x^2}}\right)}{\sqrt{4 - \left(4 - \frac{16}{x^2}\right)}}$$

$$y = \frac{4\sqrt{4 - \frac{16}{x^2}}}{\sqrt{\frac{16}{x^2}}}$$

$$y = \frac{4\sqrt{4 - \frac{16}{x^2}}}{\frac{4}{x}}$$

$$y = x\sqrt{4 - \frac{16}{x^2}}$$

$$\boxed{y = 2\sqrt{x^2 - 4}}.$$

As t approaches 2, the denominators of $x(t)$ and $y(t)$ are getting close to 0, so $x(t)$ and $y(t)$ are approaching ∞. The points are going upward from the point $(2,0)$ along the curve $y = 2\sqrt{x^2-4}$. Also $\boxed{x \geq 2}$ and $\boxed{y \geq 0}$.

41. Solving for $\sin t$ and $\cos t$ in $x(t)$ and $y(t)$, respectively, and then squaring, we have

$$\sin t = x + 2$$
$$\cos t = \frac{y-4}{-2}$$
$$\boxed{(x+2)^2 + \frac{(y-4)^2}{4} = 1},$$

which is an ellipse centered at $(-2,4)$. At $t = 0$, the object is at the point $(-2,2)$; at $t = \frac{\pi}{2}$, the object is at the point $(-1,4)$. This means the object is tracing this ellipse in a counterclockwise manner. Also $\boxed{-3 \leq x \leq -1}$ and $\boxed{2‘y \leq 6}$.

43. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t,\ y(t) = 4t-2,\ -\infty < t < \infty}$$

$$\boxed{x(t) = t^3,\ y(t) = 4t^3-2,\ -\infty < t < \infty}$$

45. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t,\ y(t) = -2t^2+1,\ -\infty < t < \infty}$$

$$\boxed{x(t) = t+1,\ y(t) = -2(t+1)^2+1,\ -\infty < t < \infty}$$

47. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t,\ y(t) = 4t^3,\ -\infty < t < \infty}$$

$$\boxed{x(t) = t^3,\ y(t) = 4t,\ -\infty < t < \infty}$$

49. Answers will vary. Here are two parameterizations.

$$\boxed{y(t) = t,\ x(t) = \frac{1}{3}\sqrt{t}-3,\ t \geq 0}$$

$$\boxed{y(t) = t^3,\ x(t) = \frac{1}{3}t^{1/6}-3,\ t \geq 0}$$

51. Answers will vary. One such answer would be the following: The line through the given segment is $y = x-2$ so one pair of parametric equations would be $\boxed{x(t) = t,\ y(t) = t-2;\ 2 \leq t \leq 7}$.

53. Answers will vary. One such answer would be the following: The ellipse can be parametrized by $\boxed{x(t) = 3\cos t, y(t) = 2\sin t}$. To start at the left end of the ellipse, we can begin with $t = \pi$. At $t = \frac{3\pi}{2}$, we are at the bottom of the ellipse (which agrees with the counterclockwise orientation), so an appropriate range on t would be $\pi \leq t \leq \frac{5\pi}{2}$.

55. For counterclockwise orientation, we choose $x(t) = 3\cos(\omega t), y(t) = 2\sin(\omega t)$. If 1 revolution takes 3 seconds, the period is $\dfrac{2\pi}{\omega} = 3$, so $\omega = \dfrac{2\pi}{3}$. The parametrization is

$$\boxed{x(t) = 3\cos\left(\frac{2\pi}{3}t\right),\ y(t) = 2\sin\left(\frac{2\pi}{3}t\right)}.$$

To begin at $(3, 0)$ the interval on t would be $\boxed{0 \le t \le 3}$.

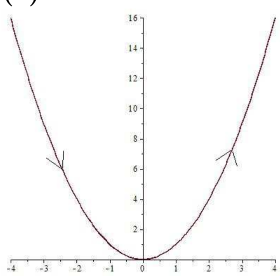
57. For clockwise orientation, we choose $x(t) = 3\sin(\omega t), y(t) = 2\cos(\omega t)$. If 1 revolution takes 2 seconds, the period is $\dfrac{2\pi}{\omega} = 2$, so $\omega = \pi$. The parametrization is

$$\boxed{x(t) = 3\sin(\pi t),\ y(t) = 2\cos(\pi t)}.$$

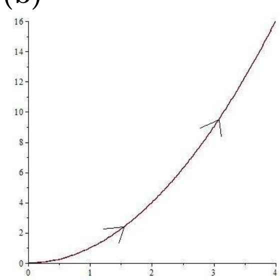
To begin at $(0, 2)$ the interval on t would be $\boxed{0 \le t \le 2}$.

59. Notice that the parametrization in (a) is the plane curve $\boxed{y = x^2}$ for $-4 \le x \le 4$. Each of the parameterizations in (b), (c), and (d) are some variation of the parametrization in (a). Part (b) is the same plane curve as (a) if we replace t with $\sqrt{t}$, but now we can only consider non-negative x-values since in (b) $x(t) = \sqrt{t} \ge 0$; there are no restrictions on x in (a). Part (c) is the same plane curve as (a) if we replace t with e^t, but again there are included restrictions since in (b) $x > 0$. Part (d) is the same plane curve as (a) if we replace t with $\cos t$ (with no included restrictions) except that the orientation is opposite of that in (a) and $-1 \le x \le 1$ whereas in (a) $-4 \le x \le 4$.

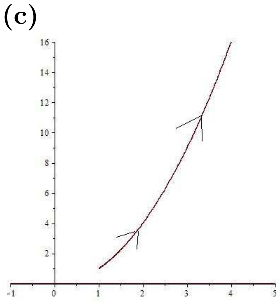
(a)



(b)



(c)



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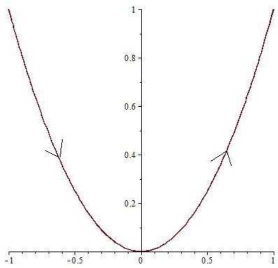


Applications and Extensions

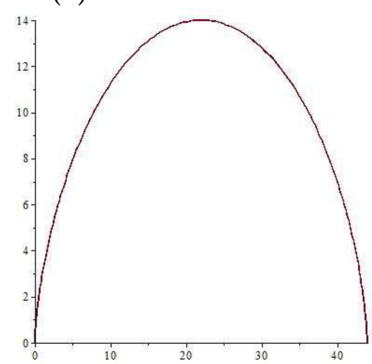
61. $\boxed{\text{I} \to \text{(d) counterclockwise, II} \to \text{(a) counterclockwise,}}$

$\boxed{\text{III} \to \text{(b) counterclockwise, IV} \to \text{(c) counterclockwise}}$

63. $\boxed{\text{I} \to \text{(c) from } (1,0) \text{ to } (-1,0), \text{ II} \to \text{(b) from } (-1,0) \text{ to } (1,0), \text{ III} \to \text{(a) clockwise,}}$

$\boxed{\text{IV} \to \text{(d) from } \left(-\frac{\sqrt{2}}{2}, 1\right) \text{ to } (1,0)}$

65. (a)



(b) $\boxed{(x(2.1), y(2.1)) \approx (8.66, 10.53)}$

67. Answers will vary.

69. (a) Using the given formulas $\boxed{x(t) = (125\cos 40^\circ)t \approx 95.8t}$ and

$\boxed{y(t) = -\frac{1}{2} \cdot 32t^2 + (125\sin 40^\circ)t + 3 \approx -16t^2 + 80.3t + 3}$.

(b) The height of the ball after 2 seconds is $y(2) = -\frac{1}{2} \cdot 32(2)^2 + (125\sin 40^\circ) \cdot 2 + 3 \boxed{\approx 99.6 \text{ feet}}$.

(c) The horizontal distance the ball has traveled after 2 seconds is $x(2) - x(0) = (125\cos 40^\circ) \cdot 2 - 0 \boxed{\approx 191.6 \text{ feet}}$.

(d) To find how long it takes, solve $x(t) = 300$ for t.

$$\begin{aligned} (125\cos 40^\circ)t &= 300 \\ t &= \frac{300}{125\cos 40^\circ} \\ &\boxed{t \approx 3.13 \text{ seconds}}. \end{aligned}$$

(e) The height at $t \approx 3.13$ seconds is $\boxed{y(3.1) \approx 97.6 \text{ feet}}$.

(f) To find the time when the ball hits the ground, we need to solve $y(t) = 0$ (i.e. the height of the is 0).

$$-\frac{1}{2} \cdot 32t^2 + (125 \sin 40°)t + 3 \quad = \quad 0.$$

Using the quadratic formula, we find the two solutions $t \approx -0.04$ seconds and $t \approx 5.1$ seconds. Since $t \geq 0$, it must be $\boxed{\approx 5.1}$ seconds that the ball is in the air.

(g) The ball has traveled $\boxed{x(5.1) \approx 488.6}$ feet horizontally before it hits the ground.

71. (a) Using the given formulas $\boxed{x(t) = (80 \cos 35°)t \approx 65.5t}$ and $\boxed{y(t) = -\frac{1}{2} \cdot 32t^2 + (80 \sin 35°)t + 6 \approx -16t^2 + 45.9t + 6}$.

(b) The height of the football after 1 second is $y(1) = -\frac{1}{2} \cdot 32 + (80 \sin 35°) + 6 \boxed{\approx 35.9 \text{ feet}}$.

(c) The horizontal distance the ball has traveled after 1 second is $x(1) - x(0) = (80 \cos 35°) - 0 \boxed{\approx 65.5 \text{ feet}}$.

(d) To find how long it takes, solve $x(t) = 120$ for t.

$$\begin{aligned} (80 \cos 35°)t &= 120 \\ t &= \frac{120}{80 \cos 35°} \\ &\boxed{t \approx 1.8 \text{ seconds}}. \end{aligned}$$

(e) The height of the ball at $t \approx 1.8$ seconds is $\boxed{y(1.8) \approx 36.8 \text{ feet}}$.

73. The parametrization $(2 \cos \theta, 2 \sin \theta)$, $0 \leq \theta \leq 2\pi$, traces a circle of radius 2 centered at the origin starting at $(2, 0)$ in the counterclockwise direction while the parametrization $(2 \sin \theta, 2 \cos \theta)$, $0 \leq \theta \leq 2\pi$, traces a circle of radius 2 centered at the origin starting at $(0, 2)$ in the clockwise direction.

Challenge Problems

75. The line through $(-R, 0)$ with slope m has equation $y = m(x + R)$. We want to find the point $P = (x, y)$ on the line $y = m(x + R)$ that intersects the circle $x^2 + y^2 = R^2$. By plugging $y = m(x + R) = mx + mR$ into the circle equation to solve for x, we have

$$\begin{aligned} x^2 + (mx + mR)^2 &= R^2 \\ x^2 + m^2x^2 + 2m^2Rx + m^2R^2 &= R^2 \\ (m^2 + 1)x^2 + (2m^2R)x + (m^2 - 1)R^2 &= 0. \end{aligned}$$

Using the quadratic equation to solve for x, the solutions are

$$\begin{aligned} x &= \frac{-(2m^2R) \pm \sqrt{(2m^2R)^2 - 4(m^2 + 1)(m^2 - 1)R^2}}{2(m^2 + 1)} \\ &= \frac{-2m^2R \pm \sqrt{4m^4R^2 - 4(m^4 - 1)R^2}}{2(m^2 + 1)} \\ &= \frac{-2m^2R \pm \sqrt{4R^2}}{2(m^2 + 1)} \\ &= \frac{-2m^2R \pm 2R}{2(m^2 + 1)} \\ &= \frac{-m^2 \pm 1}{m^2 + 1} \cdot R. \end{aligned}$$

Writing the two solutions separately, they are

$$x = \frac{-m^2+1}{m^2+1} \cdot R, \quad x = \frac{-m^2-1}{m^2+1} \cdot R = -R.$$

The second solution $x = -R$ is the original point we wrote the equation of the line through, so the new point $P = (x, y)$ has as its x-coordinate

$$x = \frac{-m^2+1}{m^2+1} \cdot R.$$

To find the y-coordinate, we know that P lies on the line $y = m(x+R)$. Plugging in x-coordinate, we have

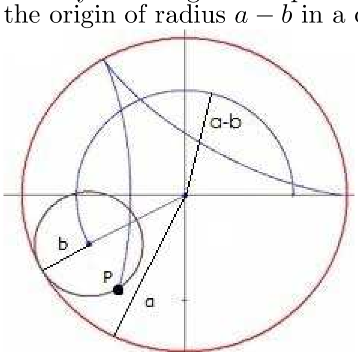
$$y = m\left(\frac{-m^2+1}{m^2+1} \cdot R + R\right) = \frac{2mR}{m^2+1}.$$

This means we can parametrize a circle using the slope m through the point $(-R, 0)$ using the equations

$$\boxed{x(m) = \frac{R(1-m^2)}{1+m^2} \qquad y(m) = \frac{2Rm}{1+m^2}}$$

for $-\infty < m < \infty$.

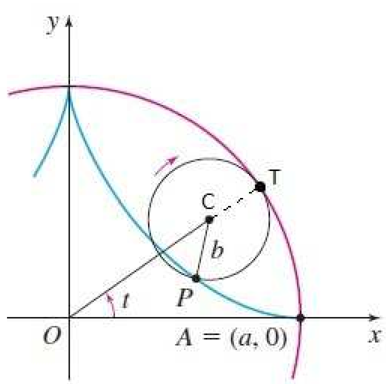
77. By referring to the picture, the center of the smaller circle traces out a circle centered at the origin of radius $a - b$ in a counterclockwise manner.



The center of the smaller circle $(x_c(t), y_c(t))$ is parametrized by

$$\begin{aligned} x_c(t) &= (a-b)\cos t \\ y_c(t) &= (a-b)\sin t, \end{aligned}$$

for $0 \le t \le 2\pi$. Now refer to the following picture.



Note that arc AT is equal to arc TP since the inner circle is rolling along the outer circle. Recall that in a circle of radius R, the measure of an arc with central angle θ is $(2\pi R) \cdot \frac{\theta}{2\pi} = R \cdot \theta$. With this in hand, we know that arc AT is $a \cdot t$. This means that arc TP is also $a \cdot t$ so the central angle PCT has measure satisfying the equation

$$b \cdot m\angle PCT = a \cdot t.$$

This means

$$m\angle PCT = \frac{a}{b}t.$$

Now picture the inner circle rotated so that CT is parallel to the x-axis. [Keep in mind that this amounts to a clockwise rotation of t radians.] Then since the center of the inner circle is located at (x_c, y_c) with radius b and rotates clockwise, this circle could be parametrized by $(x_c + b\cos t, y_c - b\sin t)$. The point P would be located on the circle after we moved through an angle with measure $m\angle PCT = \frac{a}{b}t$, so it would be located at

$$\left(x_c + b\cos\left(\frac{a}{b}t\right), y_c - b\sin\left(\frac{a}{b}t\right)\right).$$

But to find where P is *actually* located (remember we rotated the inner circle t radians clockwise), we need to rotate the inner circle back t radians (i.e. subtract t radians inside the sine and cosine). The parametrization of the point P, as it travels around the hypocycloid, is given by

$$\boxed{x(t) = x_c + b\cos\left(\frac{a}{b}t - t\right) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)}$$

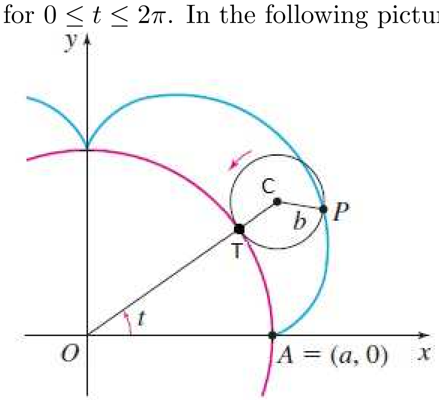
$$\boxed{y(t) = y_c - b\sin\left(\frac{a}{b}t - t\right) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)},$$

for $0 \le t \le 2\pi$.

79. It will be helpful to look through the solution to Problem 77 before reading this solution. The smaller circle's center traces out a circle centered at $(0, 0)$ with radius $a + b$ in a counterclockwise manner. The center of the smaller circle $(x_c(t), y_c(t))$, oriented counterclockwise, is parametrized by

$$\begin{aligned} x_c(t) &= (a+b)\cos t \\ y_c(t) &= (a+b)\sin t, \end{aligned}$$

for $0 \le t \le 2\pi$. In the following picture,



we have that arc AT is equal to arc TP. Reasoning as in Problem 77, we find that $m\angle PCT = \frac{a}{b}t$. If we rotate the smaller circle about its center so that TC is parallel to the x-axis with T to the right of C (which is a clockwise rotation of $\pi + t$ radians), then the smaller circle (also oriented counterclockwise) can be parametrized by $(x_c + b\cos t, y_c + b\sin t)$. The point P would be located on this circle after we moved through an angle with measure $m\angle PCT = \frac{a}{b}t$, so it would be located at

$$\left(x_c + b\cos\left(\frac{a}{b}t\right), y_c + b\sin\left(\frac{a}{b}t\right)\right).$$

If we undo the original clockwise rotation of $\pi + t$ radians (this would amount to adding $\pi + t$ radians inside the sine and cosine), we find that the parametrization of the point P on the epicycloid is given by

$$\boxed{x(t) = x_c + b\cos\left(\frac{a}{b}t + (\pi + t)\right) = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)}$$

$$\boxed{y(t) = y_c + b\sin\left(\frac{a}{b}t + (\pi + t)\right) = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)},$$

for $0 \leq t \leq 2\pi$ where we used that $\cos(\theta + \pi) = -\cos\theta$ and $\sin(\theta + \pi) = -\sin\theta$ for any angle θ.

AP® Practice Problems

1. For the pair of parametric equations in (C), eliminate the parameter t using a Pythagorean Identity.

$$\begin{aligned} \cos^2(2t) + \sin^2(2t) &= 1 \\ \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1 \\ x^2 + y^2 &= 3^2 \end{aligned}$$

The rectangular equation represents a circle with radius t and centered at the origin. In the parametric equations, $0 \leq t \leq \pi$, so the curve begins when $t = 0$ at the point $(3, 0)$ and ends when $t = \pi$ at the point $(3, 0)$.

The answer is C.

3. Eliminate the parameter t using a Pythagorean Identity.

$$\begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\frac{x}{4}\right)^2 + \left(\frac{y}{1/2}\right)^2 &= 1 \\ \frac{x^2}{16} + 4y^2 &= 1 \end{aligned}$$

The answer is D.

9.2 Tangent Lines

Concepts and Vocabulary

1. Let C denote a curve represented by the parametric equations $x = x(t), y = y(t), a \leq t \leq b$, where each function $x(t)$ and $y(t)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous and never simultaneously 0 on (a, b), then C is called a **(a) smooth** curve.

3. If in the formula for the slope of a tangent line, $\frac{dy}{dt} = 0$ $\left(\text{but } \frac{dx}{dt} \neq 0\right)$, then the curve has a **horizontal** tangent line at the point $(x(t), y(t))$. If $\frac{dx}{dt} = 0$ $\left(\text{but } \frac{dy}{dt} \neq 0\right)$, then the curve has a **vertical** tangent line at the point $(x(t), y(t))$.

Skill Building

For problems 5-12, we will use the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

5. $\frac{dx}{dt} = e^t \cos t - e^t \sin t$ and $\frac{dy}{dt} = e^t \sin t + e^t \cos t$, so

$$\frac{dy}{dx} = \frac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t} = \boxed{\frac{\sin t + \cos t}{\cos t - \sin t}}.$$

7. $\frac{dx}{dt} = 1 - \frac{1}{t^2}$ and $\frac{dy}{dt} = 1$, so

$$\frac{dy}{dx} = \boxed{\frac{1}{1 - \frac{1}{t^2}}}.$$

9. $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$ and $\frac{dy}{dt} = \cos t - \cos t + t \sin t = t\sin t$, so

$$\frac{dy}{dx} = \frac{t \sin t}{t \cos t} = \boxed{\tan t}.$$

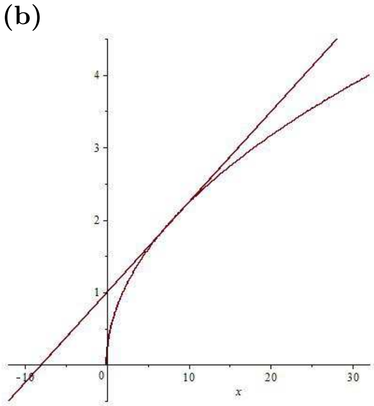
11. $\frac{dx}{dt} = -2\cot t \csc^2 t$ and $\frac{dy}{dt} = -\csc^2 t$, so

$$\frac{dy}{dx} = \frac{-\csc^2 t}{-2\cot t \csc^2 t} = \boxed{\frac{1}{2\cot t}}.$$

13. (a) The slope of the tangent line is $\left[\frac{1}{4t}\right]_{t=2} = \frac{1}{8}$. The equation of the tangent line at $(x(2), y(2)) = (8, 2)$ is

$$y - 2 = \frac{1}{8}(x - 8)$$
$$\boxed{y = \frac{1}{8}x + 1}.$$

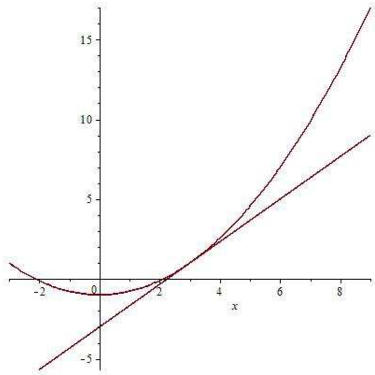
(b)



15. (a) The slope of the tangent line is $\left[\frac{4t}{3}\right]_{t=1} = \frac{4}{3}$. The equation of the tangent line at $(x(1), y(1)) = (3, 1)$ is

$$y - 1 = \frac{4}{3}(x - 3)$$
$$\boxed{y = \frac{4}{3}x - 3}.$$

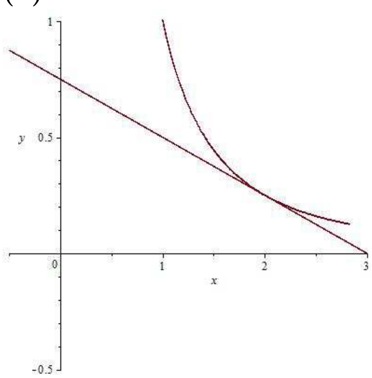
(b)



17. (a) The slope of the tangent line is $\left[\dfrac{-t^{-2}}{\frac{1}{2}t^{-1/2}}\right]_{t=4} = \left[\dfrac{-2}{t^{3/2}}\right]_{t=4} = -\dfrac{1}{4}$. The equation of the tangent line at $(x(4), y(4)) = (2, 1/4)$ is

$$y - \frac{1}{4} = -\frac{1}{4}(x-2)$$
$$\boxed{y = -\frac{1}{4}x + \frac{3}{4}}.$$

(b)



19. (a) The slope of the tangent line is $\left[\dfrac{-4(t+2)^{-2}}{2(t+2)^{-2}}\right]_{t=0} = -2$. The equation of the tangent line at $(x(0), y(0)) = (0, 2)$ is

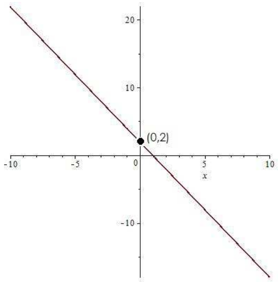
$$y - 2 = -2(x - 0)$$
$$\boxed{y = -2x + 2}.$$

If we add $x(t)$ and $y(t)$ together, then

$$\begin{aligned} x(t) + y(t) &= \frac{t}{t+2} + \frac{4}{t+2} = \frac{t+4}{t+2} = \frac{t+2+2}{t+2} \\ &= \frac{t+2}{t+2} + \frac{2}{t+2} = 1 + \frac{1}{2}y(t). \end{aligned}$$

So $x + y = 1 + \frac{1}{2}y$, or $y = -2x + 2$, is the equation of the plane curve. This means that the tangent line is the plane curve itself.

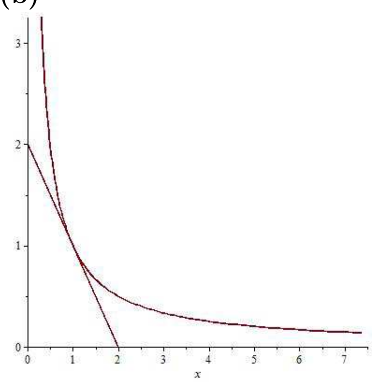
(b)



21. (a) The slope of the tangent line is $\left[\dfrac{-e^{-t}}{e^{t}}\right]_{t=0} = -1$. The equation of the tangent line at $(x(0), y(0)) = (1, 1)$ is

$$y - 1 = -(x - 1)$$
$$\boxed{y = -x + 2}.$$

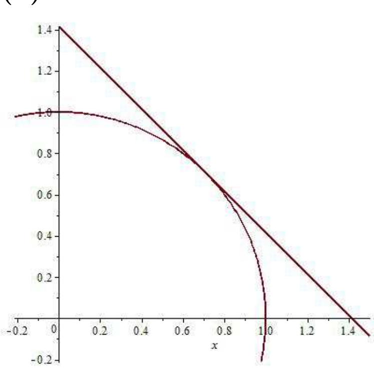
(b)



23. (a) The slope of the tangent line is $\left[\dfrac{-\sin t}{\cos t}\right]_{t=\pi/4} = \dfrac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$. The equation of the tangent line at $(x(\pi/4), y(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$ is

$$y - \frac{\sqrt{2}}{2} = -\left(x - \frac{\sqrt{2}}{2}\right)$$
$$\boxed{y = -x + \sqrt{2}}.$$

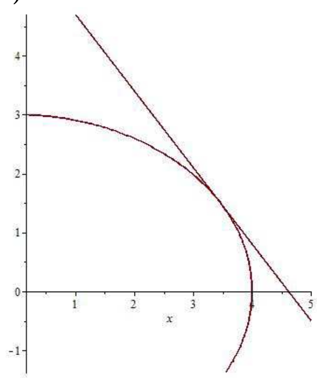
(b)



25. (a) The slope of the tangent line is $\left[\dfrac{-3\sin t}{4\cos t}\right]_{t=\pi/3} = \dfrac{-3\sqrt{3}/2}{4\cdot 1/2} = -\dfrac{3\sqrt{3}}{4}$. The equation of the tangent line at $(x(\pi/3), y(\pi/3)) = (2\sqrt{3}, 3/2)$ is

$$y - \frac{3}{2} = -\frac{3\sqrt{3}}{4}\left(x - 2\sqrt{3}\right)$$
$$\boxed{y = -\frac{3\sqrt{3}}{4}x + 6}.$$

(b)



27.

$$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 3t^2 - 4\end{aligned}$$

$\dfrac{dy}{dt} = 0$ when $t = \pm\dfrac{2}{\sqrt{3}} = \pm\dfrac{2\sqrt{3}}{3}$. At both values, $\dfrac{dx}{dt} \neq 0$, so we have horizontal tangent lines at $t = \pm\dfrac{2\sqrt{3}}{3}$ which correspond to the points $\boxed{\left(\dfrac{4}{3}, -\dfrac{16\sqrt{3}}{9}\right)}$ and $\boxed{\left(\dfrac{4}{3}, \dfrac{16\sqrt{3}}{9}\right)}$. Next, $\dfrac{dx}{dt} = 0$ when $t = 0$. At $t = 0$, $\dfrac{dy}{dt} \neq 0$, so we have a vertical tangent line at $t = 0$ which corresponds to the point $\boxed{(0,0)}$.

29.

$$\begin{aligned}\frac{dx}{dt} &= \sin t\\ \frac{dy}{dt} &= -\cos t\end{aligned}$$

On the interval $0 \leq t \leq 2\pi$, $\dfrac{dy}{dt} = 0$ when $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$. At both values, $\dfrac{dx}{dt} \neq 0$, so we have horizontal tangent lines at $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ which correspond to the points $\boxed{(1,0)}$ and $\boxed{(1,2)}$. Next, on the interval $0 \leq t \leq 2\pi$, $\dfrac{dx}{dt} = 0$ when $t = 0, \pi, 2\pi$. At all three values, $\dfrac{dy}{dt} \neq 0$, so we have vertical tangent lines at $t = 0, \pi, 2\pi$ which correspond to the points $\boxed{(0,1)}$ and $\boxed{(2,1)}$.

Applications and Extensions

31. (a) We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2 - 4$$

The vertical tangent lines occur when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$. Notice that this is when $t = 0$, which corresponds to the point $\boxed{(2, 0)}$. The horizontal tangent lines occur when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$. This is when

$$\begin{aligned} 3t^2 - 4 &= 0 \\ t^2 &= \frac{4}{3} \\ t &= \pm\frac{2}{\sqrt{3}}. \end{aligned}$$

For both of these values, $\dfrac{dx}{dt} \neq 0$, so the horizontal tangent lines occur at the points

$$(x(2/\sqrt{3}), y(2/\sqrt{3})) = \boxed{\left(\frac{10}{3}, -\frac{16\sqrt{3}}{9}\right)} \text{ and } (x(-2/\sqrt{3}), y(-2/\sqrt{3})) = \boxed{\left(\frac{10}{3}, \frac{16\sqrt{3}}{9}\right)}.$$

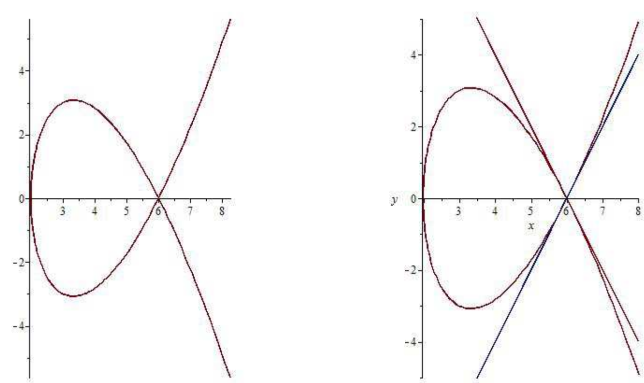
(b) The point $(6, 0)$ corresponds to the values $t = \pm 2$. Each of these values produces a slope of a tangent line at $(6, 0)$, namely

$$\frac{dy}{dx} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=2} = \frac{3(2)^2 - 4}{2(2)} = 2 \text{ and } \frac{dy}{dx} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=-2} = \frac{3(-2)^2 - 4}{2(-2)} = -2.$$

(c) The equations of the two tangent lines at $(6, 0)$ are

$$\boxed{y = 2x - 12}$$
$$\boxed{y = -2x + 12}.$$

(d)



Challenge Problems

33. If $\dfrac{dx}{dt}$ is never equal to 0, then $x(t)$ is always increasing or always decreasing, which means that $x(t)$ passes the horizontal line test. Since $x(t)$ passes the horizontal line test, we know an inverse of $x = f(t)$ exists, namely $t = f^{-1}(x)$. If we substitute $t = f^{-1}(x)$ into $y = g(t)$, then

$$y = g\left(f^{-1}(x)\right).$$

By defining the function $F = g \circ f^{-1}$, we see that $y = F(x)$. Furthermore, we know that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and since the curve C is smooth (and $\dfrac{dx}{dt} \neq 0$), $\dfrac{dy}{dx}$ exists, making F differentiable.

35. Using the formula developed in Problem 34, we start by finding the first and second derivatives of x and y with respect to θ.

$$\begin{aligned}
\frac{dx}{d\theta} &= -3a\cos^2\theta\sin\theta \\
\frac{dy}{d\theta} &= 3a\sin^2\theta\cos\theta \\
\frac{d^2x}{d\theta^2} &= 6a\sin^2\theta\cos\theta - 3a\cos^3\theta \\
\frac{d^2y}{d\theta^2} &= 6a\cos^2\theta\sin\theta - 3a\sin^3\theta.
\end{aligned}$$

So

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(6a\cos^2\theta\sin\theta - 3a\sin^3\theta)(-3a\cos^2\theta\sin\theta) - (3a\sin^2\theta\cos\theta)(6a\sin^2\theta\cos\theta - 3a\cos^3\theta)}{(-3a\cos^2\theta\sin\theta)^3} \\
&= \frac{-18a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta - 18a^2\sin^4\theta\cos^2\theta + 9a^2\sin^2\theta\cos^4\theta}{-27a^3\cos^6\theta\sin^3\theta} \\
&= \frac{-18a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta) + 9a^2\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)}{-27a^3\cos^6\theta\sin^3\theta} \\
&= \frac{-9a^2\sin^2\theta\cos^2\theta}{-27a^3\cos^6\theta\sin^3\theta} \\
&= \boxed{\frac{1}{3a\cos^4\theta\sin\theta}}.
\end{aligned}$$

AP® Practice Problems

1. $x(t) = \tan t$ $\qquad$ $y(t) = t^2 - 3t + 8$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$\dfrac{dx}{dt} = \sec^2 t$ $\qquad$ $\dfrac{dy}{dt} = 2t - 3$

Find numbers t that correspond to the point $(0, 8)$.

$$\begin{aligned}
\tan t &= 0 & t^2 - 3t + 8 &= 8 \\
t &= n\pi \text{ for integer } n & t^2 - 3t &= 0 \\
& & t(t-3) &= 0 \\
& & t = 0 \quad \text{and} \quad t &= 3
\end{aligned}$$

Since $-\dfrac{\pi}{4} \leq t < \dfrac{\pi}{4}$, $t = 0$.

The slope of the tangent lines is given by $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2t-3}{\sec^2 t}$.

At $t = 0$, $\dfrac{dy}{dx} = \dfrac{2(0) - 3}{\sec^2 0} = -3$.

The answer is A.

3. $x(t) = t^3 - 12t$ $\qquad y(t) = 4t^2 + t$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$\dfrac{dx}{dt} = 3t^2 - 12 \qquad \dfrac{dy}{dt} = 8t + 1$

The curve has a horizontal tangent when $\dfrac{dy}{dt} = 8t + 1 = 0$, but $\dfrac{dx}{dt} \neq 0$.

Note that $8t + 1 = 0$ when $t = -\dfrac{1}{8}$, and that $\left.\dfrac{dx}{dt}\right|_{t=-\frac{1}{8}} = 3\left(-\dfrac{1}{8}\right) - 12 = -\dfrac{765}{64}$.

The curve has a vertical tangent when $\dfrac{dx}{dt} = 3t^2 - 12 = 0$.

Note that $3t^2 - 12 = 3(t+2)(t-2) = 0$ when $t = -2, 2$.

The answer is C.

9.3 Arc Length; Surface Area of a Solid of Revolution

Concepts and Vocabulary

1. **False**. The formula should be

$$S = 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

3. **False**. The formula should be

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

which means under the square root we should square the first derivatives of x and y, not take the second derivative of x and y.

Skill Building

5. We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 3t^2 \quad \text{and} \quad \frac{dy}{dt} = 2t$$

The curve is smooth for $0 \le t \le 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt \\ &= \int_0^2 \sqrt{9t^4 + 4t^2}\, dt \\ &= \int_0^2 t\sqrt{9t^2 + 4}\, dt. \end{aligned}$$

We use the substitution $u = 9t^2 + 4$. Then $du = 18t\ dt$, or equivalently, $t\ dt = \dfrac{du}{18}$. Changing the limits of integration, we find that when $t = 0$, then $u = 4$, and when $t = 2$, then $u = 40$. The arc length s is

$$\begin{aligned} s &= \int_0^2 t\sqrt{9t^2+4}\ dt = \int_4^{40} \sqrt{u}\frac{du}{18} = \frac{1}{18}\int_4^{40} u^{1/2}\ du = \frac{1}{18}\left[\frac{u^{3/2}}{3/2}\right]_4^{40} \\ &= \frac{1}{27}\left[40^{3/2} - 4^{3/2}\right] = \boxed{\frac{1}{27}[80\sqrt{10} - 8]}. \end{aligned}$$

7. We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = t$$

The curve is smooth for $0 \le t \le 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(1)^2 + (t)^2}\ dt \\ &= \int_0^2 \sqrt{1 + t^2}\ dt. \end{aligned}$$

To compute this integral, we use the Table of Integrals 47 with $a = 1$. Then

$$\begin{aligned} s &= \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left|t + \sqrt{1+t^2}\right|\right]_0^2 \\ &= \sqrt{5} + \frac{1}{2}\ln|2 + \sqrt{5}| - 0 \\ &= \boxed{\sqrt{5} + \frac{1}{2}\ln(2 + \sqrt{5})}. \end{aligned}$$

9. We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 4\cos t \quad \text{and} \quad \frac{dy}{dt} = -4\sin t$$

The curve is smooth for $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_{-\pi/2}^{\pi/2} \sqrt{(4\cos t)^2 + (-4\sin t)^2}\ dt \\ &= \int_{-\pi/2}^{\pi/2} 4\ dt \\ &= [4t]_{-\pi/2}^{\pi/2} \\ &= 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) \\ &= \boxed{4\pi}. \end{aligned}$$

11. We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 2\cos t \quad \text{and} \quad \frac{dy}{dt} = -2\sin t$$

The curve is smooth for $0 \le t \le 2\pi$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(2\cos t)^2 + (-2\sin t)^2}\, dt \\ &= \int_0^{2\pi} 2\, dt \\ &= [2t]_0^{2\pi} \\ &= 2\,(2\pi) - 2\,(0) \\ &= \boxed{4\pi}. \end{aligned}$$

13. **(a)** We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

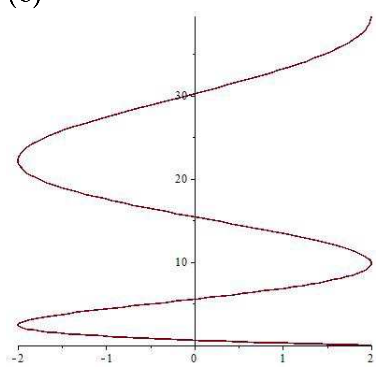
$$\frac{dx}{dt} = -4\sin(2t) \quad \text{and} \quad \frac{dy}{dt} = 2t$$

The curve is smooth for $0 \le t \le 2\pi$. Using the arc length formula, we have

$$\boxed{s = \int_0^{2\pi} \sqrt{(-4\sin(2t))^2 + (2t)^2}\, dt}.$$

(b) Using a Computer Algebra system to compute this integral, we find that $\boxed{s \approx 44.527}$.

(c)



15. **(a)** We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{2}(t+2)^{-1/2}$$

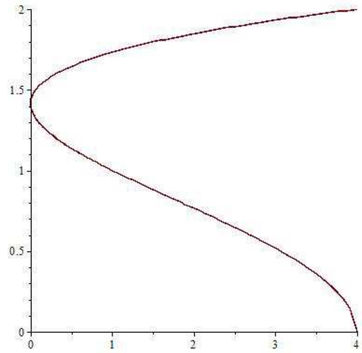
The curve is smooth for $-2 \le t \le 2$. Using the arc length formula, we have

$$\boxed{s = \int_{-2}^{2} \sqrt{(2t)^2 + \left(\frac{1}{2\sqrt{t+2}}\right)^2}\, dt}.$$

Note that this is an improper integral because of the discontinuity at $t = -2$.

(b) Using a Computer Algebra system to compute this integral, we find that $\boxed{s \approx 8.429}$.

(c)



17. **(a)** We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

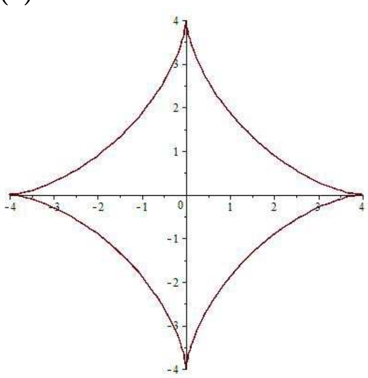
$$\frac{dx}{dt} = -3\sin t - 3\sin(3t) \quad \text{and} \quad \frac{dy}{dt} = 3\cos t - 3\cos(3t)$$

By using a Computer Algebra system, we find that $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are simultaneously 0 on $0 < t < 2\pi$ when $t = \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ so the curve is not smooth on $[0, 2\pi]$. To find the arc length, we will exploit symmetry and find the arc length on the interval $\left[0, \dfrac{\pi}{2}\right]$ (where the curve is smooth) and then quadruple it. Using the arc length formula, the full arc length is

$$\boxed{s = 4 \cdot \int_0^{\pi/2} \sqrt{(-3\sin t - 3\sin(3t))^2 + (3\cos t - 3\cos(3t))^2}\; dt}.$$

(b) Using a Computer Algebra system to compute this integral, we find that $\boxed{s = 24}$.

(c)



19. $x(t) = 3t^2$ $\qquad$ $y(t) = 6t$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 6$$

Use the formula for the surface area of the solid of revolution generated by revolving the curve C about the x-axis.

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= 2\pi \int_0^1 6t\sqrt{(6t)^2 + 6^2}\,dt \\ &= 72\pi \int_0^1 t\sqrt{t^2+1}\,dt \end{aligned}$$

Let $u = t^2 + 1$. Then $du = 2t\,dt$ or $t\,dt = \dfrac{du}{2}$. The lower limit of integration becomes $u = 0^2 + 1 = 1$ and the upper limit of integration becomes $u = 1^2 + 1 = 2$. Therefore,

$$\begin{aligned} S &= 72\pi \int_0^1 t\sqrt{t^2+1}\,dt = 72\pi \int_0^1 \sqrt{t^2+1}\,t\,dt \\ &= 72\pi \int_1^2 \sqrt{u}\frac{du}{2} = 36\pi \int_1^2 \sqrt{u}\,du = 36\pi \left[\frac{u^{3/2}}{3/2}\right]_1^2 = \boxed{24\pi\left(2\sqrt{2}-1\right)}. \end{aligned}$$

21. $x(t) = \cos^3 t$ $\qquad$ $y(t) = \sin^3 t$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = -3\sin t\cos^2 t \qquad \frac{dy}{dt} = 3\cos t\sin^2 t$$

Use the formula for the surface area of the solid of revolution generated by revolving the curve C about the x-axis.

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= 2\pi \int_0^{\pi/2} \sin^3 t\sqrt{\left(-3\sin t\cos^2 t\right)^2 + \left(3\cos t\sin^2 t\right)^2}\,dt \\ &= 2\pi \int_0^{\pi/2} \sin^3 t\sqrt{9\sin^2 t\cos^4 t + 9\cos^2 t\sin^4 t}\,dt \\ &= 2\pi \int_0^{\pi/2} 3\sin^3 t\sin t\cos t\sqrt{\cos^2 t + \sin^2 t}\,dt \\ &= 6\pi \int_0^{\pi/2} \sin^4 t\cos t\,dt \quad \text{since} \quad \cos^2 t + \sin^2 t = 1 \end{aligned}$$

Let $u = \sin t$. Then $du = \cos t\,dt$. The lower limit of integration becomes $u = \sin 0 = 0$ and the upper limit of integration becomes $u = \sin\dfrac{\pi}{2} = 1$.
Therefore,

$$\begin{aligned} S &= 6\pi \int_0^{\pi/2} \sin^4 t\cos t\,dt \\ &= 6\pi \int_0^1 u^4\,du \\ &= 6\pi \left[\frac{1}{5}u^5\right]_0^1 \\ &= \frac{6\pi}{5}(1-0) \\ &= \boxed{\frac{6\pi}{5}}. \end{aligned}$$

23. $x(t) = 3t^2 \qquad y(t) = 2t^3$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 6t^2$$

Use the formula for the surface area of the solid of revolution generated by revolving the curve C about the y-axis.

$$\begin{aligned}
S &= 2\pi \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\pi \int_0^1 3t^2\sqrt{(6t)^2 + (6t^2)^2}\,dt \\
&= 2\pi \int_0^1 3t^2\sqrt{36t^2 + 36t^4}\,dt \\
&= 2\pi \int_0^1 \left(3t^2\right)(6t)\sqrt{1+t^2}\,dt \\
&= 36\pi \int_0^1 t^3\sqrt{1+t^2}\,dt
\end{aligned}$$

Let $u = \sqrt{1+t^2}$. Then $u^2 = 1 + t^2$, $2u\,du = 2t\,dt$, and $u\,du = t\,dt$. The lower limit of integration becomes $u = \sqrt{1+0^2} = 1$ and the upper limit of integration becomes $u = \sqrt{1+1^2} = \sqrt{2}$. Therefore,

$$\begin{aligned}
S &= 36\pi \int_0^1 t^3\sqrt{1+t^2}\,dt = 36\pi \int_0^1 t^2\sqrt{1+t^2}\,t\,dt \\
&= 36\pi \int_1^{\sqrt{2}} \left(u^2 - 1\right)\cdot u \cdot u\,du \\
&= 36\pi \int_1^{\sqrt{2}} \left(u^4 - u^2\right) du \\
&= 36\pi \left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\sqrt{2}} \\
&= 36\pi \left[\left(\frac{1}{5}\cdot 4\sqrt{2} - \frac{1}{3}\cdot 2\sqrt{2}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right] \\
&= 36\pi \left(\frac{2}{15}\sqrt{2} + \frac{2}{15}\right) \\
&= \boxed{\frac{24}{5}\pi\left(\sqrt{2}+1\right)}
\end{aligned}$$

25. $x(t) = 2\sin t \qquad y(t) = 2\cos t$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 2\cos t \qquad \frac{dy}{dt} = -2\sin t$$

Use the formula for the surface area of the solid of revolution generated by revolving the curve C about the y-axis.

$$\begin{aligned}
S &= 2\pi \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\pi \int_0^{\pi/2} 2\sin t\sqrt{(2\cos t)^2 + (-2\sin t)^2}\,dt \\
&= 2\pi \int_0^{\pi/2} 4\sin t\sqrt{\cos^2 t + \sin^2 t}\,dt \\
&= 8\pi \int_0^{\pi/2} \sin t\,dt \quad \text{since} \quad \cos^2 t + \sin^2 t = 1 \\
&= 8\pi[-\cos t]_0^{\pi/2} \\
&= -8\pi\left(\cos\frac{\pi}{2} - \cos 0\right) \\
&= -8\pi(0-1) = \boxed{8\pi}
\end{aligned}$$

Applications and Extensions

27. (a) We begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 3b\sin^2 t\cos t \quad \text{and} \quad \frac{dy}{dt} = -3b\cos^2 t\sin t$$

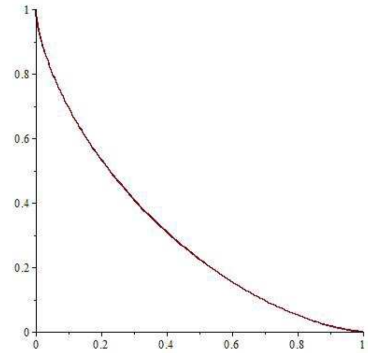
The curve is smooth for $0 \le t \le \dfrac{\pi}{2}$. Using the arc length formula, we have

$$\begin{aligned}
s &= \int_0^{\pi/2} \sqrt{(3b\sin^2 t\cos t)^2 + (-3b\cos^2 t\sin t)^2}\ dt \\
&= \int_0^{\pi/2} \sqrt{9b^2\sin^4\cos^2 t + 9b^2\cos^4 t\sin^2 t}\ dt \\
&= \int_0^{\pi/2} \sqrt{9b^2\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}\ dt \\
&= \int_0^{\pi/2} 3b\sin t\cos t\ dt.
\end{aligned}$$

Using the substitution $u = \sin t$, we have $du = \cos t\ dt$ and the new limits of integration are $u = 0$ to $u = 1$. So the arc length is

$$\begin{aligned}
s &= 3b\int_0^1 u\ du \\
&= \left[\frac{3b}{2}u^2\right]_0^1 = \boxed{\frac{3b}{2}}.
\end{aligned}$$

(b) The following graph shows the case $b = 1$.



29. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dy}{dt} = 2t$$

The curve is smooth for $0 \le t \le 2$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$s = \int_0^2 \sqrt{(3)^2 + (2t)^2}\, dt = \int_0^2 \sqrt{4\left(\frac{9}{4} + t^2\right)}\, dt = 2\int_0^2 \sqrt{\frac{9}{4} + t^2}\, dt.$$

Using the Table of Integrals 47 with $a = \frac{3}{2}$, we find that

$$\begin{aligned} s &= 2\left[\frac{t}{2}\sqrt{\frac{9}{4} + t^2} + \frac{9}{8}\ln\left|t + \sqrt{\frac{9}{4} + t^2}\right|\right]_0^2 \\ &= 2\left[\frac{5}{2} + \frac{9}{8}\ln\left|\frac{9}{2}\right|\right] - 2\left[\frac{9}{8}\ln\left|\frac{3}{2}\right|\right] \\ &= \boxed{5 + \frac{9}{4}\ln(3)}. \end{aligned}$$

31. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = t \quad \text{and} \quad \frac{dy}{dt} = \sqrt{2t+3}$$

The curve is smooth for $0 \le t \le 2$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(t)^2 + \left(\sqrt{2t+3}\right)^2}\, dt \\ &= \int_0^2 \sqrt{t^2 + 2t + 3}\, dt \\ &= \int_0^2 \sqrt{(t+1)^2 + 2}\, dt, \end{aligned}$$

where in the last step we completed the square. If we make a substitution $u = t+1$, then $du = dt$ and changing the limits of integration

$$s = \int_1^3 \sqrt{u^2 + 2}\, du.$$

Using the Table of Integrals 47 with $a = \sqrt{2}$, we find that

$$\begin{aligned} s &= \left[\frac{u}{2}\sqrt{u^2+2} + \ln\left|u + \sqrt{u^2+2}\right|\right]_1^3 \\ &= \left[\frac{3}{2}\sqrt{11} + \ln\left|3+\sqrt{11}\right|\right] - \left[\frac{1}{2}\sqrt{3} + \ln\left|1+\sqrt{3}\right|\right] \\ &= \boxed{\frac{3\sqrt{11}-\sqrt{3}}{2} + \ln\left(\frac{3+\sqrt{11}}{1+\sqrt{3}}\right)}. \end{aligned}$$

33. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = -2\sin(2t) \quad \text{and} \quad \frac{dy}{dt} = 2\sin t\cos t$$

The curve is smooth for $0 \le t \le \frac{\pi}{2}$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{(-2\sin(2t))^2 + (2\sin t\cos t)^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{4\sin^2(2t) + (2\sin t\cos t)^2}\, dt. \end{aligned}$$

Using the double angle formula $\sin(2t) = 2\sin t\cos t$, we have

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{4\sin^2(2t) + \sin(2t)^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{5}\sin(2t)\, dt = \left[-\frac{\sqrt{5}}{2}\cos(2t)\right]_0^{\pi/2} \\ &= -\frac{\sqrt{5}}{2}(-1) + \frac{\sqrt{5}}{2}(1) \\ &= \boxed{\sqrt{5}}. \end{aligned}$$

35. $x(t) = t$ $\qquad$ $y(t) = \cosh t$

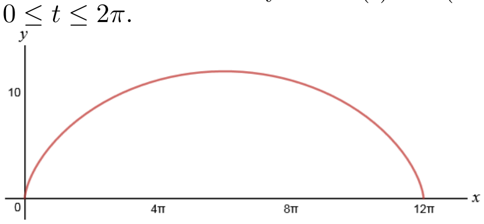
Begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 1 \qquad\qquad \frac{dy}{dt} = \sinh t$$

Use the formula for the surface area of the solid of revolution generated by revolving the curve C about the x-axis.

$$\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\pi \int_a^b \cosh t\sqrt{1^2 + (\sinh t)^2}dt \\
&= 2\pi \int_a^b \cosh t\sqrt{1 + \sinh^2 t}\,dt \\
&= 2\pi \int_a^b \cosh t\sqrt{\cosh^2 t}\,dt \quad \text{since} \quad \cosh^2 t = 1 + \sinh^2 t \\
&= 2\pi \int_a^b \cosh^2 t\,dt \\
&= 2\pi \int_a^b \frac{1}{2}[1 + \cosh(2t)]\,dt \quad \text{since} \quad \cosh^2 t = \frac{1}{2}[\cosh(2t) + 1] \\
&= \pi\left[\frac{1}{2}\sinh(2t) + t\right]_a^b \\
&= \pi\left\{\left[\frac{1}{2}\sinh(2b) + b\right] - \left[\frac{1}{2}\sinh(2a) + a\right]\right\} \\
&= \boxed{\frac{\pi}{2}[\sinh(2b) - \sinh(2a)] - \pi(b - a)}
\end{aligned}$$

37. One arch of the cycloid $x(t) = 6(t - \sin t), y(t) = 6(1 - \cos t)$ is generated on the interval $0 \le t \le 2\pi$.



Begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = 6(1 - \cos t)$, $\frac{dy}{dt} = 6\sin t$.

Use the formula for the surface area of the solid of revolution generated by revolving the curve C about the x-axis.

$$\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\pi \int_0^{2\pi} 6(1 - \cos t)\sqrt{[6(1 - \cos t)]^2 + (6\sin t)^2}\,dt \\
&= 2\pi \int_0^{2\pi} 36(1 - \cos t)\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt \\
&= 72\pi \int_0^{2\pi} (1 - \cos t)\sqrt{2 - 2\cos t}\,dt \quad \text{since} \quad \cos^2 t + \sin^2 t = 1 \\
&= 72\sqrt{2}\,\pi \int_0^{2\pi} (1 - \cos t)^{3/2}\,dt \\
&= 72\sqrt{2}\,\pi \int_0^{2\pi} \left[2\sin^2\frac{t}{2}\right]^{3/2} dt \quad \text{since} \quad 1 - \cos t = 2\sin^2\frac{t}{2} \\
&= 288\pi \int_0^{2\pi} \sin^3\frac{t}{2}\,dt
\end{aligned}$$

Now use the substitution $u = \dfrac{t}{2}$ and $du = \dfrac{1}{2}\,dt$. Then $dt = 2\,du$. The lower limit of integration becomes $u = 0$ and the upper limit of integration becomes $u = \dfrac{2\pi}{2} = \pi$. The integral becomes $S = 288\pi \displaystyle\int_0^{2\pi} \sin^3 \frac{t}{2}\,dt = 288\pi \int_0^{\pi} \sin^3 u \cdot 2\,du = 576\pi \int_0^{\pi} \sin^3 u\,du.$
The exponent of $\sin u$ is 3, a positive, odd integer. Factor $\sin u$ from $\sin^3 u$ and write the rest of the integrand in terms of cosines.

$$S = 576\pi \int_0^{\pi} \sin^3 u\,du = 576\pi \int_0^{\pi} \sin^2 u \sin u\,du = 576\pi \int_0^{\pi} \left(1 - \cos^2 u\right) \sin u\,du$$

Now use the substitution $v = \cos u$ and $dv = -\sin u\,du$. Then $\sin u\,du = -dv$. The lower limit of integration becomes $v = \cos 0 = 1$ and the upper limit of integration becomes $v = \cos \pi = -1$. Therefore,

$$\begin{aligned} S &= 576\pi \int_0^{\pi} \left(1 - \cos^2 u\right) \sin u\,du \\ &= 576\pi \int_1^{-1} \left(1 - v^2\right)(-dv) \\ &= 576\pi \int_{-1}^{1} \left(1 - v^2\right) dv \\ &= 576\pi \left[v - \frac{1}{3}v^3\right]_{-1}^{1} \\ &= 576\pi \left\{\left(1 - \frac{1}{3}\right) - \left[-1 - \left(-\frac{1}{3}\right)\right]\right\} \\ &= \boxed{768\pi} \end{aligned}$$

39. For a function $y = f(x)$ on $a \le x \le b$ representing a smooth curve C, we can parametrize C by choosing $x(t) = t$ and $y(t) = f(t)$ on the interval $a \le t \le b$. It is given that C is smooth on $a \le t \le b$, so we compute the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t)$$

The arc length s of C on $a \le t \le b$ using the arc length formula is

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{1 + [f'(t)]^2}\,dt = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

For Problems 40-43, we note that since ds approximates s for nearby points, we will compute ds by the formula $ds = \sqrt{(dx)^2 + (dy)^2}$ where dx and dy represent that changes in x and y, respectively, between the nearby points.
41. We start by computing dx and dy.

$$\begin{aligned} dx = x(1.2) - x(1) &= \sqrt{1.2} - \sqrt{1} = \sqrt{1.2} - 1 \\ dy = y(1.2) - y(1) &= 1.2^3 - 1^3 = 0.728. \end{aligned}$$

The arc length s is approximately

$$s \approx ds = \sqrt{(\sqrt{1.2} - 1)^2 + (0.728)^2}\boxed{\approx 0.7342}.$$

To see how this estimate compares to the actual value of

$$s = \int_1^{1.2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

using a CAS, we find that $s \approx 0.7343$.

43. We start by computing dx and dy.

$$\begin{aligned} dx = x(0.2) - x(0) &= e^{0.2a} - e^{0a} = e^{0.2a} - 1 \\ dy = y(0.2) - y(0) &= e^{0.2b} - e^{0b} = e^{0.2b} - 1. \end{aligned}$$

The arc length s is approximately

$$s \approx ds = \sqrt{(e^{0.2a}-1)^2 + (e^{0.2b}-1)^2} = \boxed{\sqrt{e^{0.4a} - 2e^{0.2a} + e^{0.4b} - 2e^{0.2b} + 2}},$$

for given values of a and b.

AP® Practice Problems

1. $x(t) = 2\sin t$ $\qquad y(t) = 1 + e^t$

Begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$\frac{dx}{dt} = 2\cos t$ $\qquad \frac{dy}{dt} = e^t$

Using the arc length formula for parametric equations,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-\pi}^{2} \sqrt{(2\cos t)^2 + (e^t)^2}\, dt = \int_{-\pi}^{2} \sqrt{4\cos^2 t + e^{2t}}\, dt.$$

The answer is C.

3. $x(t) = \cos(2t)$ $\qquad y(t) = \cos^2 t$

Begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$\frac{dx}{dt} = -2\sin(2t)$ $\qquad \frac{dy}{dt} = -2\cos t \sin t$

Using the arc length formula for parametric equations,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{\pi/4}^{\pi/3} \sqrt{[-2\sin(2t)]^2 + (-2\cos t \sin t)^2}\, dt.$$

Using the identity $2\cos t \sin t = \sin(2t)$,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_{\pi/4}^{\pi/3} \sqrt{[-2\sin(2t)]^2 + [-\sin(2t)]^2}\, dt = \sqrt{5}\int_{\pi/4}^{\pi/3} \sin(2t)\, dt.$$

Let $u = 2t$. Then $du = 2\,dt$ or $dt = \frac{du}{2}$. The lower limit of integration becomes $u = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$ and the upper limit of integration becomes $u = 2 \cdot \frac{\pi}{3} = \frac{2\pi}{3}$. Therefore,

$$s = \sqrt{5}\int_{\pi/2}^{2\pi/3} \sin(u)\frac{du}{2} = \frac{\sqrt{5}}{2}[-\cos u]_{\pi/2}^{2\pi/3} = \frac{\sqrt{5}}{2}\left[-\left(-\frac{1}{2}\right) - 0\right] = \frac{\sqrt{5}}{4}.$$

The answer is B.

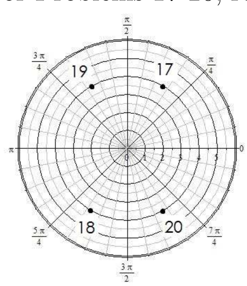
9.4 Polar Coordinates

Concepts and Vocabulary

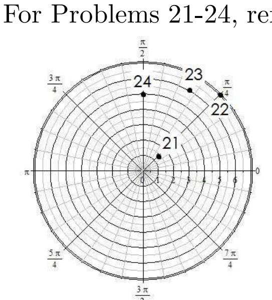
1. In a polar coordinate system, the origin is called the $\boxed{\textbf{pole}}$, and the $\boxed{\textbf{polar axis}}$ coincides with the positive x-axis of the rectangular coordinate system.
3. $\boxed{\textbf{False}}$. A point in polar coordinates has infinitely many representations in polar coordinates by adding multiples of 2π to the angle to obtain a new, equivalent point.
5. $\boxed{\textbf{True}}$. If $r < 0$, then we move r units from the origin in the opposite direction of θ to plot the point.
7. To convert the point (r, θ) in polar coordinates to a point (x, y) in rectangular coordinates, use the formulas $x = \boxed{r\cos\theta}$ and $y = \boxed{r\sin\theta}$.

Skill Building

9. A
11. C
13. B
15. A
For Problems 17-20, refer to the following graph.



For Problems 21-24, refer to the following graph.



For Problems 25-32, we use the facts that $(r, \theta) = (-r, \theta \pm \pi)$ and $(r, \theta) = (r, \theta \pm 2\pi k)$ for all non-negative integers k.
25.
(a) $\left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} - 2\pi\right) = \boxed{\left(5, -\frac{4\pi}{3}\right)}$

(b) $\left(5, \frac{2\pi}{3}\right) = \left(-5, \frac{2\pi}{3} + \pi\right) = \boxed{\left(-5, \frac{5\pi}{3}\right)}$

(c) $\left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} + 2\pi\right) = \boxed{\left(5, \frac{8\pi}{3}\right)}$

27.
(a) $(-2, 3\pi) = (2, 3\pi - \pi) = (2, 2\pi) = (2, 2\pi - 4\pi) = \boxed{(2, -2\pi)}$

(b) $(-2, 3\pi) = (-2, 3\pi - 2\pi) = \boxed{(-2, \pi)}$

(c) $(-2, 3\pi) = (2, 3\pi - \pi) = \boxed{(2, 2\pi)}$

29.

(a) $\left(1, \frac{\pi}{2}\right) = \left(1, \frac{\pi}{2} - 2\pi\right) = \boxed{\left(1, -\frac{3\pi}{2}\right)}$

(b) $\left(1, \frac{\pi}{2}\right) = \left(-1, \frac{\pi}{2} + \pi\right) = \boxed{\left(-1, \frac{3\pi}{2}\right)}$

(c) $\left(1, \frac{\pi}{2}\right) = \left(1, \frac{\pi}{2} + 2\pi\right) = \boxed{\left(1, \frac{5\pi}{2}\right)}$

31.

(a) $\left(-3, -\frac{\pi}{4}\right) = \left(3, -\frac{\pi}{4} - \pi\right) = \boxed{\left(3, -\frac{5\pi}{4}\right)}$

(b) $\left(-3, -\frac{\pi}{4}\right) = \left(-3, -\frac{\pi}{4} + 2\pi\right) = \boxed{\left(-3, \frac{7\pi}{4}\right)}$

(c) $\left(-3, -\frac{\pi}{4}\right) = \left(3, -\frac{\pi}{4} + \pi\right) = \left(3, \frac{3\pi}{4}\right) = \left(3, \frac{3\pi}{4} + 2\pi\right) = \boxed{\left(3, \frac{11\pi}{4}\right)}$

33.

$$\begin{aligned} x &= r\cos\theta = 6\cos\left(\frac{\pi}{6}\right) = 3\sqrt{3} \\ y &= r\sin\theta = 6\sin\left(\frac{\pi}{6}\right) = 3 \end{aligned}$$

The rectangular coordinates of the point are $\boxed{\left(3\sqrt{3}, 3\right)}$.

35.

$$\begin{aligned} x &= r\cos\theta = -6\cos\left(-\frac{\pi}{6}\right) = -3\sqrt{3} \\ y &= r\sin\theta = -6\sin\left(-\frac{\pi}{6}\right) = 3 \end{aligned}$$

The rectangular coordinates of the point are $\boxed{\left(-3\sqrt{3}, 3\right)}$.

37.

$$\begin{aligned} x &= r\cos\theta = 5\cos\left(\frac{\pi}{2}\right) = 0 \\ y &= r\sin\theta = 5\sin\left(\frac{\pi}{2}\right) = 5 \end{aligned}$$

The rectangular coordinates of the point are $\boxed{(0, 5)}$.

39.

$$\begin{aligned} x &= r\cos\theta = 2\sqrt{2}\cos\left(-\frac{\pi}{4}\right) = 2 \\ y &= r\sin\theta = 2\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = -2 \end{aligned}$$

The rectangular coordinates of the point are $\boxed{(2, -2)}$.

41. The point $(5, 0)$ is located on the polar axis $\theta = 0$, 5 units from the origin, so the polar coordinates of the point are $\boxed{(5, 0)}$.

43. The point $(-2, 2)$ is in the second quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}$ from the origin. The angle θ satisfies $\frac{\pi}{2} \leq \theta \leq \pi$ where

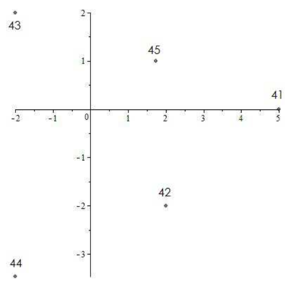
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{2}{-2}\right) + \pi = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$$

The polar coordinates of the point are $\boxed{\left(2\sqrt{2}, \frac{3\pi}{4}\right)}$.

45. The point $\left(\sqrt{3}, 1\right)$ is in the first quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ from the origin. The angle θ satisfies $0 \le \theta \le \frac{\pi}{2}$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

The polar coordinates of the point are $\boxed{\left(2, \frac{\pi}{6}\right)}$.



For Problems 46-50, the graph of these points follows Problem 50.

47. The point $\left(-\sqrt{3}, 1\right)$ is in the second quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} =$ 2 from the origin. The angle θ satisfies $\frac{\pi}{2} \le \theta \le \pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}.$$

The polar coordinates of the point are $\boxed{\left(2, \frac{5\pi}{6}\right)}$.

49. The point $(3, 2)$ is in the first quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$ from the origin. The angle θ satisfies $0 \le \theta \le \frac{\pi}{2}$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{3}\right).$$

Using a calculator $\tan^{-1}\left(\frac{2}{3}\right) \approx 0.588$ radians. The polar coordinates of the point are $\boxed{\left(\sqrt{13}, \tan^{-1}\left(\frac{2}{3}\right)\right) \approx \left(\sqrt{13}, 0.588\right)}$.

51. If r is fixed at 2, then the set of points are all a distance of 2 units from the origin, forming a circle of radius 2 centered at the origin, which matches graph $\boxed{(E)}$.

53. If we multiply both sides by r, the equation becomes $r^2 = 2r\cos\theta$. Now use the formulas $r^2 = x^2 + y^2$ and $x = r\cos\theta$ to convert the equation to rectangular coordinates.

$$\begin{aligned} r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

This is a circle of radius 1 centered at $(1, 0)$, which matches graph $\boxed{(F)}$.

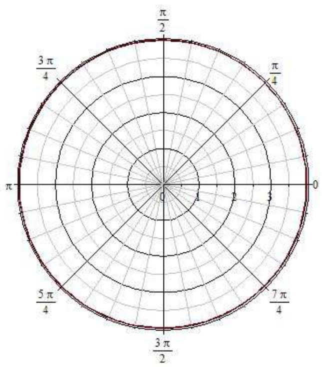
55. If we multiply both sides by r, the equation becomes $r^2 = -2r\cos\theta$. Now use the formulas $r^2 = x^2 + y^2$ and $x = r\cos\theta$ to convert the equation to rectangular coordinates.

$$\begin{aligned} r^2 &= -2r\cos\theta \\ x^2 + y^2 &= -2x \\ x^2 + 2x + y^2 &= 0 \\ (x+1)^2 + y^2 &= 1 \end{aligned}$$

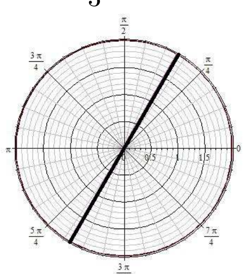
This is a circle of radius 1 centered at $(-1, 0)$, which matches graph $\boxed{(H)}$.

57. If θ is fixed at $\dfrac{3\pi}{4}$ and r is allowed to vary, the result is a line containing the pole, making an angle of $\dfrac{3\pi}{4}$ with the polar axis. Such a line has slope $\tan\dfrac{3\pi}{4} = -1$, which has rectangular equation $y = -x$ and matches graph $\boxed{(D)}$.

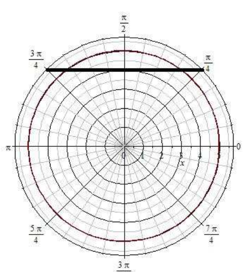
59. $r = 4$ is a circle centered at $(0, 0)$ of radius 4.



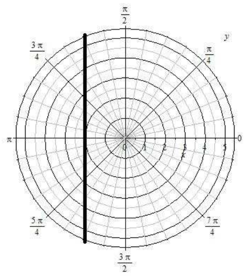
61. $\theta = \dfrac{\pi}{3}$ is a line through the origin making an angle of $\dfrac{\pi}{3}$ with the positive x-axis.



63. $r\sin\theta = 4$ is the equivalent of $y = 4$.



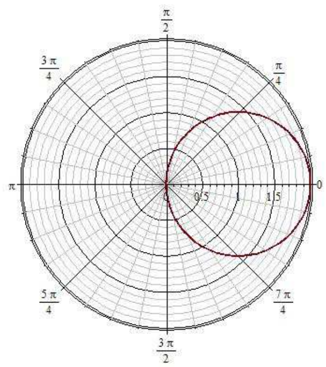
65. $r\cos\theta = -2$ is the equivalent of $x = -2$.



67. If we multiply both sides of the equation by r, then we have $r^2 = 2r\cos\theta$. Since $r^2 = x^2+y^2$ and $x = r\cos\theta$, the equation becomes $x^2+y^2 = 2x$. By moving the $2x$ term to the left side and completing the square, we have

$$(x-1)^2+y^2 = 1,$$

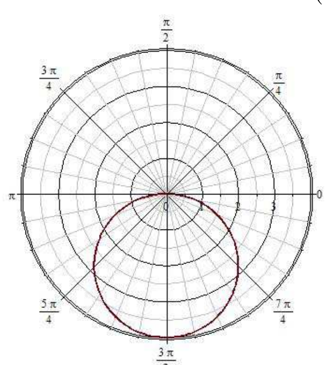
which is a circle centered at $(1,0)$ with radius 1.



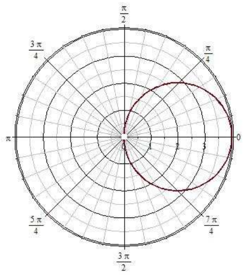
69. If we multiply both sides of the equation by r, then we have $r^2 = -4r\sin\theta$. Since $r^2 = x^2+y^2$ and $y = r\sin\theta$, the equation becomes $x^2+y^2 = -4y$. By moving the $-4y$ term to the left side and completing the square, we have

$$x^2+(y+2)^2 = 4,$$

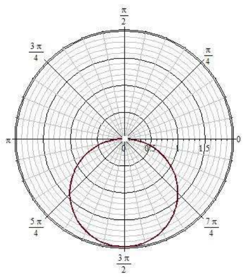
which is a circle centered at $(0,-2)$ with radius 2.



71. The polar equation $r\sec\theta = 4$ can be rewritten as $r = 4\cos\theta$ when $\cos\theta \neq 0$. The θ values where $\cos\theta = 0$ are $\theta = \dfrac{\pi}{2}+k\pi$ for some integer k. This is a circle centered at $(2,0)$ with radius 2, excluding the points where $\theta = \dfrac{\pi}{2}+k\pi$ which is the pole.



73. The polar equation $r\csc\theta = -2$ can be rewritten as $r = -2\sin\theta$ when $\sin\theta \neq 0$. The θ values where $\sin\theta = 0$ are $\theta = k\pi$ for some integer k. This is a circle centered at $(0, -1)$ with radius 1, excluding the points where $\theta = k\pi$ which is the pole.



75. Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation becomes

$$\frac{(r\cos\theta)^2}{4} + \frac{(r\sin\theta)^2}{9} = 1$$
$$9r^2\cos^2\theta + 4r^2\sin^2\theta = 36$$
$$r^2 = \frac{36}{9\cos^2\theta + 4\sin^2\theta}$$
$$\boxed{r = 6\frac{\sqrt{9\cos^2\theta + 4\sin^2\theta}}{9\cos^2\theta + 4\sin^2\theta}}.$$

77. Substituting $r^2 = x^2 + y^2$ and $x = r\cos\theta$, the equation becomes

$$r^2 - 4r\cos\theta = 0$$
$$\boxed{r = 4\cos\theta}.$$

79. Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation becomes

$$(r\cos\theta)^2 = 1 - 4r\sin\theta$$
$$\boxed{r^2\cos^2\theta + 4\cos\theta - 1 = 0}.$$

81. Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation becomes

$$(r\cos\theta)(r\sin\theta) = 1$$
$$r^2 = \frac{1}{\cos\theta\sin\theta}$$
$$\boxed{r = \frac{\sqrt{\cos\theta\sin\theta}}{\cos\theta\sin\theta}}.$$

83. Multiplying both sides by r, we have $r^2 = r\cos\theta$. Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$, the equation becomes

$$x^2 + y^2 = x$$
$$\boxed{\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}},$$

where we completed the square in the final step.

85. Multiply both sides by r and use the equations $r = \sqrt{x^2+y^2}$ and $y = r\sin\theta$.

$$\begin{aligned} r^3 &= r\sin\theta \\ \left(\sqrt{x^2+y^2}\right)^3 &= y \\ \boxed{(x^2+y^2)^{3/2} = y}. \end{aligned}$$

87. Multiply both sides by $1-\cos\theta$ and use the equations $r = \sqrt{x^2+y^2}$ and $x = r\cos\theta$.

$$\begin{aligned} r - r\cos\theta &= 4 \\ \boxed{\sqrt{x^2+y^2} - x = 4}. \end{aligned}$$

89. Substituting $r^2 = x^2+y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ the equation becomes

$$\begin{aligned} x^2+y^2 &= \tan^{-1}\left(\frac{y}{x}\right) \\ \tan(x^2+y^2) &= \frac{y}{x} \\ \boxed{y = x\tan\left(x^2+y^2\right)}. \end{aligned}$$

91. Substituting $r = \sqrt{x^2+y^2}$ the equation becomes

$$\begin{aligned} \sqrt{x^2+y^2} &= 2 \\ \boxed{y\sqrt{4-x^2}}. \end{aligned}$$

93. Substituting $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ the equation becomes

$$\begin{aligned} \tan\left(\tan^{-1}\left(\frac{y}{x}\right)\right) &= 4 \\ \frac{y}{x} &= 4 \\ \boxed{y = 4x}. \end{aligned}$$

Applications and Extensions

95. (a) Wrigley Field resides at the point $\boxed{(-10, 36)}$ in rectangular coordinates.

(b) The point $(-10, 36)$ lies in the second quadrant so the angle θ satisfies $\frac{\pi}{2} \le \theta \le \pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{36}{-10}\right) + \pi \approx 1.842 \text{ radians.}$$

Wrigley Field is $r = \sqrt{(-10)^2 + (36)^2} = \sqrt{1396} = 2\sqrt{349} \approx 37.363$ blocks from the intersection of Madison and State Streets, so in polar coordinates, Wrigley Field is located approximately at $\boxed{(37.363, 1.842)}$.

(c) U.S. Cellular Field resides at the point $\boxed{(-3, -35)}$ in rectangular coordinates.

(d) The point $(-3, -35)$ lies in the third quadrant so the angle θ satisfies $\pi \le \theta \le \frac{3\pi}{2}$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{-35}{-3}\right) + \pi \approx 4.627 \text{ radians.}$$

U.S. Cellular Field is $r = \sqrt{(-3)^2 + (-35)^2} = \sqrt{1234} \approx 35.128$ blocks from the intersection of Madison and State Streets, so in polar coordinates, U.S. Cellular Field is located approximately at $\boxed{(35.128, 4.627)}$.

97. Since $y = r\sin\theta$, the polar equation $r\sin\theta = a$ converts to $y = a$ in rectangular coordinates, which is a horizontal line a units above the origin (i.e. above the pole) if $a > 0$ and $|a|$ units below the origin (i.e. below the pole) if $a < 0$.

99. If we multiply the equation on both sides by r, then we have $r^2 = 2ar\sin\theta$. If we use the equations $r^2 = x^2 + y^2$ and $y = r\sin\theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= 2ay \\ x^2 + (y-a)^2 &= a^2 \end{aligned}$$

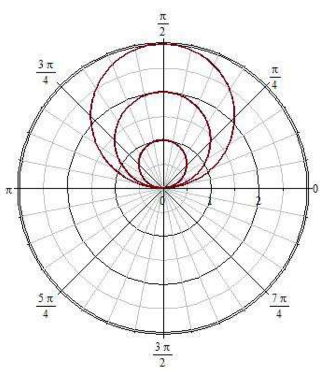
in rectangular coordinates. Since $a > 0$, then this is the equation of a circle centered at $(0, a)$ with radius a.

101. If we multiply the equation on both sides by r, then we have $r^2 = 2ar\cos\theta$. If we use the equations $r^2 = x^2 + y^2$ and $x = r\cos\theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= 2ax \\ (x-a)^2 + y^2 &= a^2 \end{aligned}$$

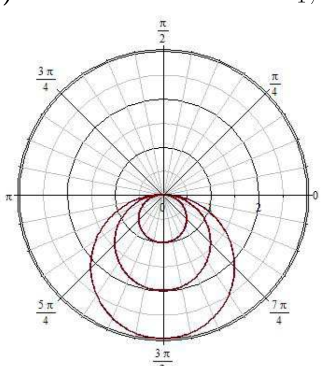
in rectangular coordinates. Since $a > 0$, then this is the equation of a circle centered at $(a, 0)$ with radius a.

103. (a) The smallest circle is r_1, the middle circle is r_2, and the largest circle is r_3.



(b) Answers will vary.

(c) The smallest circle is r_1, the middle circle is r_2, and the largest circle is r_3.



(d) Answers will vary.

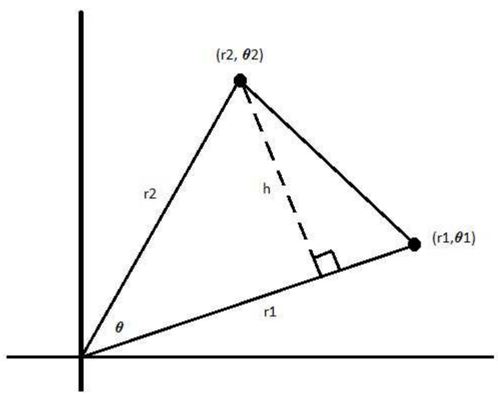
Challenge Problems

105. If we multiply both sides of the equation by r, then the equation becomes $r^2 = ar\sin\theta + br\cos\theta$. Now if we use the equations $r^2 = x^2 + y^2$, $x = r\cos\theta$, nd $y = r\sin\theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= ay + bx \\ x^2 - bx + y^2 - ay &= 0 \\ x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4} + y^2 - ay + \frac{a^2}{4} - \frac{a^2}{4} &= 0 \\ \left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 &= \frac{a^2 + b^2}{4}. \end{aligned}$$

This is the equation of $\boxed{\text{a circle centered at } \left(\frac{b}{2}, \frac{a}{2}\right) \text{ of radius } \frac{\sqrt{a^2+b^2}}{2}}$.

107. Consider the following picture.

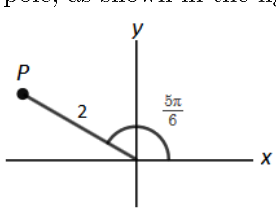


From the point (r_2, θ_2), drop a perpendicular to the segment connecting the pole and the point (r_1, θ_1). If we denote the length of the perpendicular by h, then the area of the triangle is $\frac{1}{2}r_1 h$. The angle $\theta = \theta_2 - \theta_1$ and then we have the equation $\sin\theta = \frac{h}{r_2}$ or $h = r_2 \sin\theta$. Using these equations, the area of the triangle is

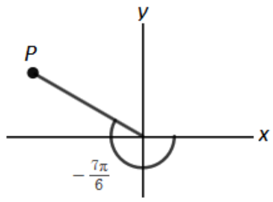
$$\frac{1}{2}r_1 r_2 \sin(\theta_2 - \theta_1).$$

AP® Practice Problems

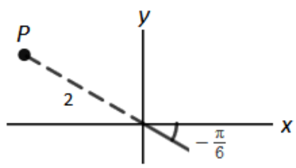
1. The point $P = \left(2, \frac{5}{6}\pi\right)$ is located by first drawing the angle $\frac{5}{6}\pi$. Then P is on the extension of the terminal side of $\theta = \frac{5}{6}\pi$ through the pole at a distance 2 units from the pole, as shown in the figure below.



The point P is also on the extension of the terminal side of $\theta = -\frac{7}{6}\pi$ through the pole at a distance 2 units from the pole, as shown in the figure below. So, $P = \left(2, -\frac{7}{6}\pi\right)$.



The point P is also on the extension of the terminal side of $\theta = -\frac{1}{6}\pi$ through the pole at a distance 2 units in the opposite direction from the pole, as shown in the figure below. So, $P = \left(-2, -\frac{1}{6}\pi\right)$.



The answer is B.

3. We use the equations $x = r\cos\theta$ and $y = r\sin\theta$ with $r = -1$ and $\theta = \frac{\pi}{2}$.

$$x = -1\cos\frac{\pi}{2} = -1(0) = 0$$
$$y = -1\sin\frac{\pi}{2} = -1(1) = -1$$

The rectangular coordinates are $(0, -1)$.

The answer is D.

5. To convert the equation $r = -2\cos\theta$ to rectangular coordinates, we multiply the equation by r to obtain $r^2 = -2r\cos\theta$. Since $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we have

$$\begin{aligned} r^2 &= -2r\cos\theta \\ x^2 + y^2 &= -2x \\ x^2 + 2x + y^2 &= 0 \\ x^2 + 2x + 1 + y^2 &= 1 \\ (x+1)^2 + y^2 &= 1 \end{aligned}$$

This is the standard form of the equation of a circle with its center at $(-1, 0)$ and radius 1 in rectangular coordinates.

The answer is A.

9.5 Polar Equations; Parametric Equations of Polar Equations; Arc Length of Polar Equations

Concepts and Vocabulary

1. **True**.

3. **True**. The rose has eight petals.

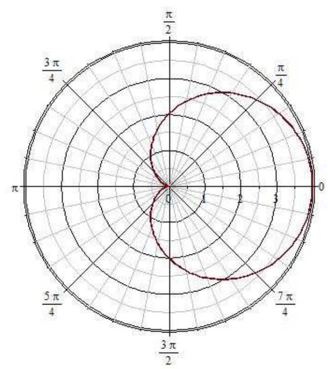
Skill Building

5. (a) The polar equation $r = 2+2\cos\theta$ contains $\cos\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π, plot the points $(r,\theta) = (2+2\cos\theta,\theta)$, and trace out the graph, beginning at the point $(4,0)$ and ending at $(4,2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(4,0)$	$\left(2+\sqrt{3},\frac{\pi}{6}\right)$	$\left(2+\sqrt{2},\frac{\pi}{4}\right)$	$\left(3,\frac{\pi}{3}\right)$	$\left(2,\frac{\pi}{2}\right)$	$\left(1,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(2-\sqrt{2},\frac{3\pi}{4}\right)$	$\left(2-\sqrt{3},\frac{5\pi}{6}\right)$	$(0,\pi)$	$\left(2-\sqrt{3},\frac{7\pi}{6}\right)$	$\left(2-\sqrt{2},\frac{5\pi}{4}\right)$	$\left(1,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(2,\frac{3\pi}{2}\right)$	$\left(3,\frac{5\pi}{3}\right)$	$\left(2+\sqrt{2},\frac{7\pi}{4}\right)$	$\left(2+\sqrt{3},\frac{11\pi}{6}\right)$	$(4,2\pi)$



(b) Parametric equations for $r = 2+2\cos\theta$:

$$x = r\cos\theta = 2(1+\cos\theta)\cos\theta \qquad y = r\sin\theta = 2(1+\cos\theta)\sin\theta$$

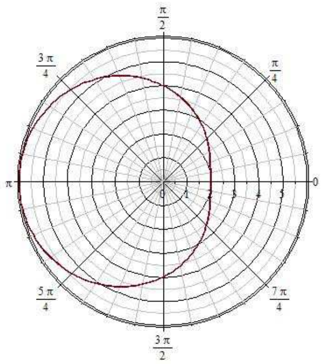
where θ is the parameter, and if $0 \le \theta \le 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

7. (a) The polar equation $r = 4-2\cos\theta$ contains $\cos\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π, plot the points $(r,\theta) = (4-2\cos\theta,\theta)$, and trace out the graph, beginning at the point $(2,0)$ and ending at $(2,2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(2,0)$	$\left(4-\sqrt{3},\frac{\pi}{6}\right)$	$\left(4-\sqrt{2},\frac{\pi}{4}\right)$	$\left(3,\frac{\pi}{3}\right)$	$\left(4,\frac{\pi}{2}\right)$	$\left(5,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(4+\sqrt{2},\frac{3\pi}{4}\right)$	$\left(4+\sqrt{3},\frac{5\pi}{6}\right)$	$(6,\pi)$	$\left(4+\sqrt{3},\frac{7\pi}{6}\right)$	$\left(4+\sqrt{2},\frac{5\pi}{4}\right)$	$\left(5,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(4,\frac{3\pi}{2}\right)$	$\left(3,\frac{5\pi}{3}\right)$	$\left(4-\sqrt{2},\frac{7\pi}{4}\right)$	$\left(4-\sqrt{3},\frac{11\pi}{6}\right)$	$(2,2\pi)$



(b) Parametric equations for $r = 4 - 2\cos\theta$:

$$\boxed{x = r\cos\theta = (4 - 2\cos\theta)\cos\theta \qquad y = r\sin\theta = (4 - 2\cos\theta)\sin\theta}$$

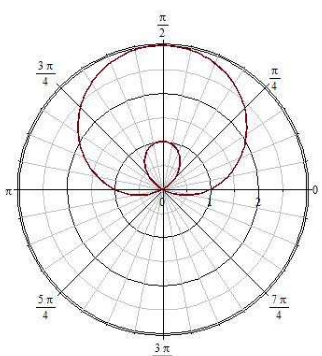
where θ is the parameter, and if $0 \le \theta \le 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

9. (a) The polar equation $r = 1 + 2\sin\theta$ contains $\sin\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π, plot the points $(r,\theta) = (1 + 2\sin\theta, \theta)$, and trace out the graph, beginning at the point $(1, 0)$ and ending at $(1, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(1,0)$	$\left(2, \frac{\pi}{6}\right)$	$\left(1+\sqrt{2}, \frac{\pi}{4}\right)$	$\left(1+\sqrt{3}, \frac{\pi}{3}\right)$	$\left(3, \frac{\pi}{2}\right)$	$\left(1+\sqrt{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(1+\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(2, \frac{5\pi}{6}\right)$	$(1,\pi)$	$\left(0, \frac{7\pi}{6}\right)$	$\left(1-\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(1-\sqrt{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(-1, \frac{3\pi}{2}\right)$	$\left(1-\sqrt{3}, \frac{5\pi}{3}\right)$	$\left(1-\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(0, \frac{11\pi}{6}\right)$	$(1, 2\pi)$



(b) Parametric equations for $r = 1 + 2\sin\theta$:

$$\boxed{x = r\cos\theta = (1 + 2\sin\theta)\cos\theta \qquad y = r\sin\theta = (1 + 2\sin\theta)\sin\theta}$$

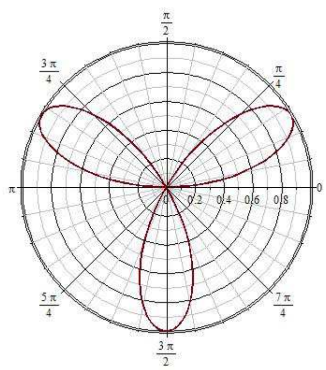
where θ is the parameter, and if $0 \le \theta \le 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

11. (a) The polar equation $r = \sin(3\theta)$ contains $\sin(3\theta)$, which has period $\frac{2\pi}{3}$. So we construct a table of common values for θ that range from $0 \le \theta \le 2\pi$, noting that the values $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$ and $\frac{4\pi}{3} \le \theta \le 2\pi$ repeat the values for $0 \le \theta \le \frac{2\pi}{3}$. Then we plot the points and trace out the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(0,0)$	$\left(1,\frac{\pi}{6}\right)$	$\left(\frac{\sqrt{2}}{2},\frac{\pi}{4}\right)$	$\left(0,\frac{\pi}{3}\right)$	$\left(-1,\frac{\pi}{2}\right)$	$\left(0,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(\frac{\sqrt{2}}{2},\frac{3\pi}{4}\right)$	$\left(1,\frac{5\pi}{6}\right)$	$(0,\pi)$	$\left(-1,\frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{2}}{2},\frac{5\pi}{4}\right)$	$\left(0,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(1,\frac{3\pi}{2}\right)$	$\left(0,\frac{5\pi}{3}\right)$	$\left(-\frac{\sqrt{2}}{2},\frac{7\pi}{4}\right)$	$\left(-1,\frac{11\pi}{6}\right)$	$(0,2\pi)$



(b) Parametric equations for $r = \sin(3\theta)$:

$$\boxed{x = r\cos\theta = \sin(3\theta)\cos\theta \qquad y = r\sin\theta = \sin(3\theta)\sin\theta}$$

where θ is the parameter, and if $0 \le \theta \le 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

13. To find the points of intersection, where $0 \le \theta \le 2\pi$, set both polar equations equal.

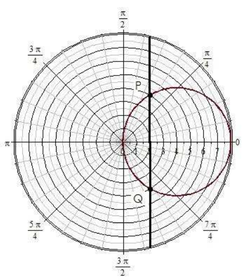
$$\begin{aligned} 8\cos\theta &= 2\sec\theta \\ 4\cos^2\theta &= 1 \\ \cos\theta &= \pm\frac{1}{2} \\ \theta &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}. \end{aligned}$$

The points of intersection are

$$\begin{aligned}
\left(8\cos\left(\frac{\pi}{3}\right), \frac{\pi}{3}\right) &= \left(4, \frac{\pi}{3}\right) \\
\left(8\cos\left(\frac{2\pi}{3}\right), \frac{2\pi}{3}\right) &= \left(-4, \frac{2\pi}{3}\right) \\
\left(8\cos\left(\frac{4\pi}{3}\right), \frac{4\pi}{3}\right) &= \left(-4, \frac{4\pi}{3}\right) \\
\left(8\cos\left(\frac{5\pi}{3}\right), \frac{5\pi}{3}\right) &= \left(4, \frac{5\pi}{3}\right).
\end{aligned}$$

Notice that the first and third points are the same, and the second and fourth points are the same, so the two unique points of intersection are

$$\boxed{P = \left(4, \frac{\pi}{3}\right) \text{ and } Q = \left(4, \frac{5\pi}{3}\right)}.$$

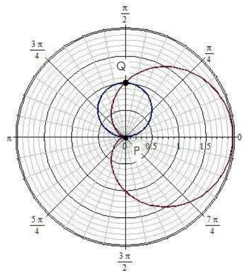


15. To find the points of intersection, where $0 \le \theta \le 2\pi$, set both polar equations equal.

$$\begin{aligned}
\sin\theta &= 1 + \cos\theta \\
\sin\theta - \cos\theta &= 1 \\
(\sin\theta - \cos\theta)^2 &= 1^2 \\
\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta &= 1 \\
\sin\theta\cos\theta &= 0 \\
\sin\theta = 0 \quad &\text{or} \quad \cos\theta = 0 \\
\theta &= 0,\ \pi,\ 2\pi,\ \frac{\pi}{2},\ \frac{3\pi}{2}.
\end{aligned}$$

By squaring both sides in the third step, we may have introduced extraneous solutions, so we need to check all of them. The values $\theta = 0,\ 2\pi,\ \frac{3\pi}{2}$ do not make $\sin\theta = 1 + \cos\theta$, so our only values of intersection are when $\theta = \pi,\ \frac{\pi}{2}$. The points of intersection are

$$\begin{aligned}
P = (\sin(\pi), \pi) &= \boxed{(0, \pi)} \\
Q = \left(\sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}\right) &= \boxed{\left(1, \frac{\pi}{2}\right)}.
\end{aligned}$$

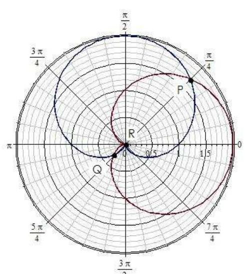


17. To find the points of intersection, where $0 \le \theta \le 2\pi$, set both polar equations equal.

$$\begin{aligned} 1+\sin\theta &= 1+\cos\theta \\ \sin\theta &= \cos\theta \\ \tan\theta &= 1 \\ \theta &= \frac{\pi}{4},\ \frac{5\pi}{4}. \end{aligned}$$

We also note that both polar curves go through the pole, so the pole is another point of intersection. The points of intersection are the pole, R, and

$$\begin{aligned} P &= \left(1+\sin\left(\frac{\pi}{4}\right), \frac{\pi}{4}\right) = \boxed{\left(1+\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)} \\ Q &= \left(1+\sin\left(\frac{5\pi}{4}\right), \frac{5\pi}{4}\right) = \boxed{\left(1-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)}. \end{aligned}$$



19. We use the arc length formula $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ with $r = e^{\theta/2}$. Then $\frac{dr}{d\theta} = \frac{1}{2}e^{\theta/2}$ and

$$\begin{aligned} s &= \int_0^2 \sqrt{(e^{\theta/2})^2 + \left(\frac{1}{2}e^{\theta/2}\right)^2}\, d\theta = \int_0^2 \sqrt{\frac{5}{4}e^\theta}\, d\theta = \frac{\sqrt{5}}{2}\int_0^2 e^{\theta/2}\, d\theta \\ &= \frac{\sqrt{5}}{2}\left[2e^{\theta/2}\right]_0^2 = \boxed{\sqrt{5}\,(e-1)}. \end{aligned}$$

21. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ with $r = \cos^2 \frac{\theta}{2}$. Then $\frac{dr}{d\theta} = -\cos\frac{\theta}{2}\sin\frac{\theta}{2}$ and

$$\begin{aligned} s &= \int_0^{\pi} \sqrt{\left(\cos^2\frac{\theta}{2}\right)^2 + \left(-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{\cos^4\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\, d\theta \\ &= \int_0^{\pi} \sqrt{\cos^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}\, d\theta = \int_0^{\pi} \cos\frac{\theta}{2}\, d\theta \\ &= \left[2\sin\frac{\theta}{2}\right]_0^{\pi} = \boxed{2}, \end{aligned}$$

where in the second line we are using that $\cos\frac{\theta}{2} \geq 0$ on $[0, \pi]$ when taking the square root.

Applications and Extensions

23. $\boxed{r = 3 + 3\cos\theta}$

25. $\boxed{r = 4 + \sin\theta}$

27. Since $r = f(\theta) = 2\cos(3\theta)$, we have the parametrization $x(\theta) = 2\cos(3\theta)\cos\theta$ and $y(\theta) = 2\cos(3\theta)\sin\theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned} \frac{dx}{d\theta} &= -6\sin(3\theta)\cos\theta - 2\cos(3\theta)\sin\theta \\ \frac{dy}{d\theta} &= -6\sin(3\theta)\sin\theta + 2\cos(3\theta)\cos\theta. \end{aligned}$$

The slope of the tangent line is then

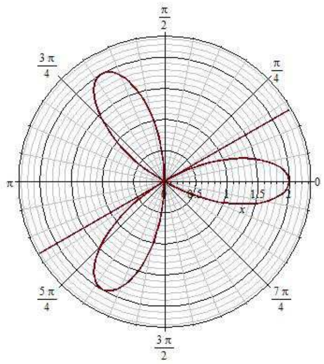
$$\left[\frac{dy}{dx}\right]_{\theta=\pi/6} = \left[\frac{-6\sin(3\theta)\sin\theta + 2\cos(3\theta)\cos\theta}{-6\sin(3\theta)\cos\theta - 2\cos(3\theta)\sin\theta}\right]_{\theta=\pi/6} = \frac{\sqrt{3}}{3}.$$

The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned} x\left(\frac{\pi}{6}\right) &= 0 \\ y\left(\frac{\pi}{6}\right) &= 0. \end{aligned}$$

The equation of the tangent line is

$$y - 0 = \frac{\sqrt{3}}{3}(x - 0)$$
$$\boxed{y = \frac{\sqrt{3}}{3}x}.$$



29. Since $r = f(\theta) = 2 + \cos\theta$, we have the parametrization $x(\theta) = (2 + \cos\theta)\cos\theta$ and $y(\theta) = (2 + \cos\theta)\sin\theta$. Now we find $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= -\sin\theta\cos\theta - (2 + \cos\theta)\sin\theta \\ \frac{dy}{d\theta} &= -\sin\theta\sin\theta + (2 + \cos\theta)\cos\theta.\end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx}\right]_{\theta=\pi/4} = \left[\frac{-\sin\theta\sin\theta + (2 + \cos\theta)\cos\theta}{-\sin\theta\cos\theta - (2 + \cos\theta)\sin\theta}\right]_{\theta=\pi/4} = \sqrt{2} - 2.$$

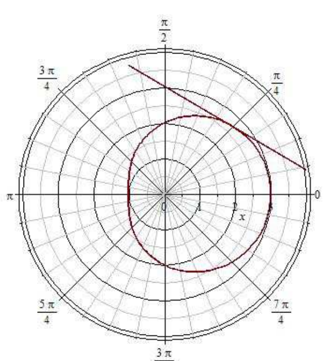
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= \sqrt{2} + \frac{1}{2} \\ y\left(\frac{\pi}{4}\right) &= \sqrt{2} + \frac{1}{2}.\end{aligned}$$

The equation of the tangent line is

$$y - \left(\sqrt{2} + \frac{1}{2}\right) = (\sqrt{2} - 2)\left(x - \left(\sqrt{2} + \frac{1}{2}\right)\right)$$

$$\boxed{y = (\sqrt{2} - 2)x + \frac{5\sqrt{2}}{2} - \frac{1}{2}}.$$



31. Since $r = f(\theta) = 4 + 5\sin\theta$, we have the parametrization $x(\theta) = (4 + 5\sin\theta)\cos\theta$ and $y(\theta) = (4 + 5\sin\theta)\sin\theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= 5\cos\theta\cos\theta - (4 + 5\sin\theta)\sin\theta \\ \frac{dy}{d\theta} &= 5\cos\theta\sin\theta + (4 + 5\sin\theta)\cos\theta.\end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx}\right]_{\theta=\pi/4} = \left[\frac{5\cos\theta\sin\theta + (4 + 5\sin\theta)\cos\theta}{5\cos\theta\cos\theta - (4 + 5\sin\theta)\sin\theta}\right]_{\theta=\pi/4} = -\frac{5\sqrt{2}}{4} - 1.$$

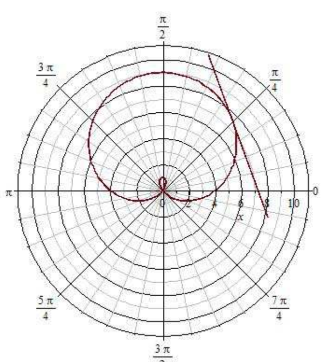
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= 2\sqrt{2} + \frac{5}{2} \\ y\left(\frac{\pi}{4}\right) &= 2\sqrt{2} + \frac{5}{2}.\end{aligned}$$

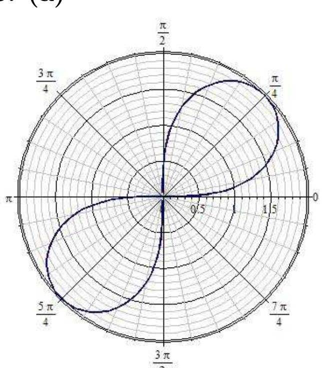
The equation of the tangent line is

$$y - \left(2\sqrt{2} + \frac{5}{2}\right) = \left(-\frac{5\sqrt{2}}{4} - 1\right)\left(x - \left(2\sqrt{2} + \frac{5}{2}\right)\right)$$

$$\boxed{y = -\left(1 + \frac{5\sqrt{2}}{4}\right)x + \frac{57\sqrt{2}}{8} + 10}.$$



33. (a)

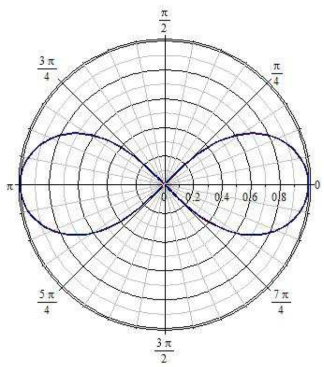


(b) The equation $r^2 = 4\sin(2\theta)$ can be rewritten as $r = \pm 2\sqrt{\sin(2\theta)}$. Besides orientation, the $\pm$ does not affect the graph of the lemniscate, so one set of parametric equations would be:

$$\boxed{x = r\cos\theta = 2\sqrt{\sin(2\theta)}\cos\theta \qquad y = r\sin\theta = 2\sqrt{\sin(2\theta)}\sin\theta}$$

where θ is the parameter.

35. (a)



(b) The equation $r^2 = \cos(2\theta)$ can be rewritten as $r = \pm\sqrt{\cos(2\theta)}$. Besides orientation, the $\pm$ does not affect the graph of the lemniscate, so one set of parametric equations would be:

$$\boxed{x = r\cos\theta = \sqrt{\cos(2\theta)}\cos\theta \qquad y = r\sin\theta = \sqrt{\cos(2\theta)}\sin\theta}$$

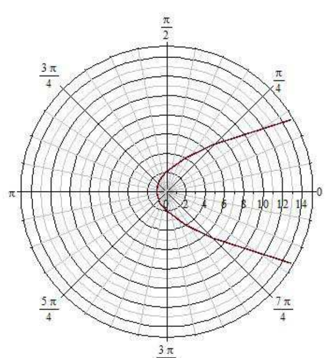
where θ is the parameter.

37. (a) The polar equation $r = \dfrac{2}{1-\cos\theta}$ contains $\cos\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π (excluding these values since r is not defined there) and plot the points (r, θ).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	undefined	$\left(\frac{4}{2-\sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\frac{4}{2-\sqrt{2}}, \frac{\pi}{4}\right)$	$\left(4, \frac{\pi}{3}\right)$	$\left(2, \frac{\pi}{2}\right)$	$\left(\frac{4}{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(\frac{4}{\sqrt{2}+2}, \frac{3\pi}{4}\right)$	$\left(\frac{4}{\sqrt{3}+2}, \frac{5\pi}{6}\right)$	$(1, \pi)$	$\left(\frac{4}{\sqrt{3}+2}, \frac{7\pi}{6}\right)$	$\left(\frac{4}{\sqrt{2}+2}, \frac{5\pi}{4}\right)$	$\left(\frac{4}{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(2, \frac{3\pi}{2}\right)$	$\left(4, \frac{5\pi}{3}\right)$	$\left(\frac{4}{2-\sqrt{2}}, \frac{7\pi}{4}\right)$	$\left(\frac{4}{2-\sqrt{3}}, \frac{11\pi}{6}\right)$	undefined



(b) Parametric equations for the polar equation $r = \dfrac{2}{1-\cos\theta}$ are

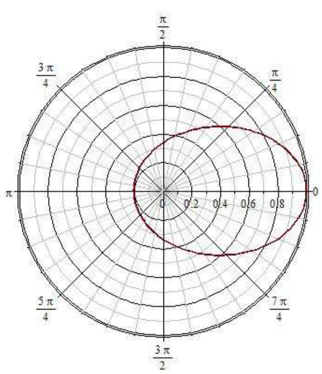
$$\boxed{x = \frac{2\cos\theta}{1-\cos\theta}}$$
$$\boxed{y = \frac{2\sin\theta}{1-\cos\theta}}.$$

39. **(a)** The polar equation $r = \dfrac{1}{3-2\cos\theta}$ contains $\cos\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π (excluding the values of θ such that $3-2\cos\theta = 0$, however there are none) and plot the points (r,θ).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(1,0)$	$\left(\frac{1}{3-\sqrt{3}},\frac{\pi}{6}\right)$	$\left(\frac{1}{3-\sqrt{2}},\frac{\pi}{4}\right)$	$\left(\frac{1}{2},\frac{\pi}{3}\right)$	$\left(\frac{1}{3},\frac{\pi}{2}\right)$	$\left(\frac{1}{4},\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(\frac{1}{3+\sqrt{2}},\frac{3\pi}{4}\right)$	$\left(\frac{1}{3+\sqrt{3}},\frac{5\pi}{6}\right)$	$\left(\frac{1}{5},\pi\right)$	$\left(\frac{1}{3+\sqrt{3}},\frac{7\pi}{6}\right)$	$\left(\frac{1}{3+\sqrt{2}},\frac{5\pi}{4}\right)$	$\left(\frac{1}{4},\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(\frac{1}{3},\frac{3\pi}{2}\right)$	$\left(\frac{1}{2},\frac{5\pi}{3}\right)$	$\left(\frac{1}{3-\sqrt{2}},\frac{7\pi}{4}\right)$	$\left(\frac{1}{3-\sqrt{3}},\frac{11\pi}{6}\right)$	$(1,2\pi)$



(b) Parametric equations for the polar equation $r = \dfrac{1}{3-2\cos\theta}$ are

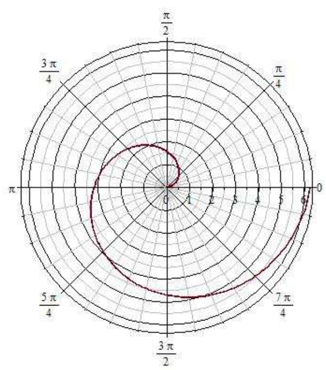
$$\boxed{x = \frac{\cos\theta}{3-2\cos\theta}}$$
$$\boxed{y = \frac{\sin\theta}{3-2\cos\theta}}.$$

41. (a) Constructing a table of common values for $\theta > 0$, we find some points on the polar graph $r = \theta$.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(\frac{3\pi}{4}, \frac{3\pi}{4}\right)$	$\left(\frac{5\pi}{6}, \frac{5\pi}{6}\right)$	(π, π)	$\left(\frac{7\pi}{6}, \frac{7\pi}{6}\right)$	$\left(\frac{5\pi}{4}, \frac{5\pi}{4}\right)$	$\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$	$\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$	$\left(\frac{7\pi}{4}, \frac{7\pi}{4}\right)$	$\left(\frac{11\pi}{6}, \frac{11\pi}{6}\right)$	$(2\pi, 2\pi)$



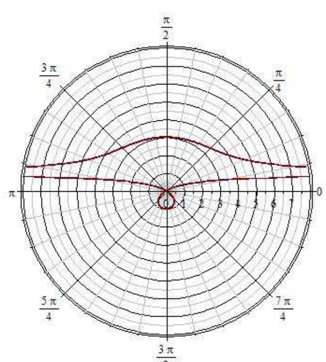
(b) Parametric equations for the polar equation $r = \theta$ are

$$\boxed{x = \theta \cos\theta}$$
$$\boxed{y = \theta \sin\theta}.$$

43. (a) Constructing a table of common values for θ between 0 and 2π (excluding 0, π, and 2π which make r undefined), we find some points on the polar graph $r = \csc\theta - 2$.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
(r,θ)	$\left(0, \frac{\pi}{6}\right)$	$\left(-2+\sqrt{2}, \frac{\pi}{4}\right)$	$\left(\frac{2\sqrt{3}}{3} - 2, \frac{\pi}{3}\right)$	$\left(-1, \frac{\pi}{2}\right)$	$\left(\frac{2\sqrt{3}}{3} - 2, \frac{2\pi}{3}\right)$	$\left(-2+\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(0, \frac{5\pi}{6}\right)$

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
(r,θ)	$\left(-4, \frac{7\pi}{6}\right)$	$\left(-2-\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(-\frac{2\sqrt{3}}{3} - 2, \frac{4\pi}{3}\right)$	$\left(-3, \frac{3\pi}{2}\right)$	$\left(-\frac{2\sqrt{3}}{3} - 2, \frac{5\pi}{3}\right)$	$\left(-2-\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(-4, \frac{11\pi}{6}\right)$



(b) Parametric equations for the polar equation $r = \csc\theta - 2$ are

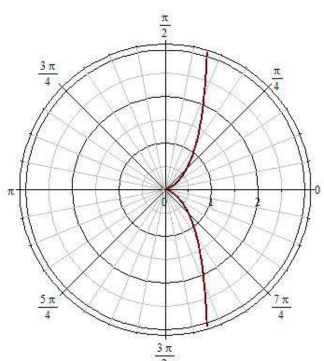
$$\boxed{x = (\csc\theta - 2)\cos\theta}$$
$$\boxed{y = (\csc\theta - 2)\sin\theta}.$$

45. (a) The polar equation $r = \sin\theta\tan\theta$ contains $\sin\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π (excluding the values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ since r is not defined there) and plot the points (r, θ).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(0,0)$	$\left(\frac{\sqrt{3}}{6}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$	$\left(\frac{3}{2}, \frac{\pi}{3}\right)$	undefined	$\left(-\frac{3}{2}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$	$\left(-\frac{\sqrt{3}}{6}, \frac{5\pi}{6}\right)$	$(0,\pi)$	$\left(-\frac{\sqrt{3}}{6}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$	$\left(-\frac{3}{2}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	undefined	$\left(\frac{3}{2}, \frac{5\pi}{3}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$	$\left(\frac{\sqrt{3}}{6}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$



(b) Parametric equations for the polar equation $r = \sin\theta\tan\theta$ are

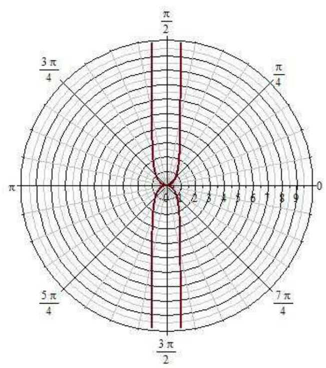
$$\boxed{x = \sin\theta\tan\theta\cos\theta = \sin^2\theta}$$
$$\boxed{y = \sin\theta\tan\theta\sin\theta = \sin^2\theta\tan\theta}.$$

47. (a) The polar equation $r = \tan\theta$ contains $\tan\theta$, which has the period π. We construct a table of common values of θ that range from 0 to 2π (excluding $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ since r is not defined there), noting that $\pi \le \theta \le 2\pi$ duplicates the values taken on $0 \le \theta \le \pi$, and plot the points (r, θ).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(0,0)$	$\left(\frac{\sqrt{3}}{3},\frac{\pi}{6}\right)$	$\left(1,\frac{\pi}{4}\right)$	$\left(\sqrt{3},\frac{\pi}{3}\right)$	undefined	$\left(-\sqrt{3},\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(-1,\frac{3\pi}{4}\right)$	$\left(-\frac{\sqrt{3}}{3},\frac{5\pi}{6}\right)$	$(0,\pi)$	$\left(\frac{\sqrt{3}}{3},\frac{7\pi}{6}\right)$	$\left(1,\frac{5\pi}{4}\right)$	$\left(\sqrt{3},\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	undefined	$\left(-\sqrt{3},\frac{5\pi}{3}\right)$	$\left(-1,\frac{7\pi}{4}\right)$	$\left(-\frac{\sqrt{3}}{3},\frac{11\pi}{6}\right)$	$(0,2\pi)$



(b) Parametric equations for the polar equation $r = \tan\theta$ are

$$\boxed{x = \tan\theta\cos\theta = \sin\theta}$$
$$\boxed{y = \sin\theta\tan\theta}.$$

49. First we recall that for a polar point (r,θ), by adding π to the angle θ and changing the sign of r, we arrive at the same point. In other words, $(r,\theta) = (-r,\theta+\pi)$. To show that $r_1 = 4(\cos\theta + 1)$ has the same graph as $r_2 = 4(\cos\theta - 1)$, we will show that every point on r_1 is also on r_2, and vice versa.
Recalling the angle addition formula for cosine, we have

$$\cos(\theta+\pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = -\cos\theta.$$

If (r,θ) is a point on the graph of r_1, then $r_1(\theta) = r$ and

$$r_2(\theta+\pi) = 4(\cos(\theta+\pi) - 1) = 4(-\cos\theta - 1) = -4(\cos\theta + 1) = -r_1(\theta) = -r$$

and so $(-r,\theta+\pi) = (r,\theta)$ is also a point on the graph of r_2.
If (r,θ) is a point on the graph of r_2, then $r_2(\theta) = r$ and

$$r_1(\theta+\pi) = 4(\cos(\theta+\pi) + 1) = 4(-\cos\theta + 1) = -4(\cos\theta - 1) = -r_2(\theta) = -r$$

and so $(-r,\theta+\pi) = (r,\theta)$ is also a point on the graph of r_1.

51. We use the arc length formula $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ with $r = \theta$. Then $\frac{dr}{d\theta} = 1$ and

$$s = \int_0^{2\pi} \sqrt{(\theta)^2 + (1)^2}\,d\theta = \left[\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left|\theta + \sqrt{\theta^2+1}\right|\right]_0^{2\pi},$$

where we have used the Table of Integrals 49 with $a = 1$. Then

$$s = \boxed{\pi\sqrt{4\pi^2+1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)}.$$

53. The perimeter of the cardioid is found by calculating the arc length of the cardioid over the interval $[-\pi, \pi]$. We will exploit symmetry and find the length s of the top half of the cardioid using the arc length formula $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ with $r = 1 - \cos\theta$ and then double it. With the function $r = 1 - \cos\theta$, we have $\frac{dr}{d\theta} = \sin\theta$ and

$$s = \int_0^\pi \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta = \int_0^\pi \sqrt{2 - 2\cos\theta}\, d\theta.$$

Let $u = 2 - 2\cos\theta$ and then $du = 2\sin\theta\, d\theta$. Solving for $\cos\theta$ in the first equation, we have

$$\cos\theta = \frac{2-u}{2}.$$

Using the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$, on the interval $0 \le \theta \le \pi$ we find that

$$\sin\theta = \frac{\sqrt{4u - u^2}}{2} = \frac{\sqrt{u}\sqrt{4-u}}{2},$$

and so

$$\begin{aligned} du &= 2\sin\theta\, d\theta \\ \frac{1}{\sqrt{u}\sqrt{4-u}}\, du &= d\theta. \end{aligned}$$

Changing the limits of integration to $u = 2 - 2\cos(0) = 0$ to $u = 2 - 2\cos\pi = 4$, the integral for s becomes

$$s = \int_0^4 \sqrt{u} \cdot \frac{1}{\sqrt{u}\sqrt{4-u}}\, du = \int_0^4 \frac{1}{\sqrt{4-u}}\, du.$$

Another substitution, $w = 4 - u$ with $dw = -du$, the limits of integration change to $w = 4 - (0) = 4$ to $w = 4 - (4) = 0$, and we have

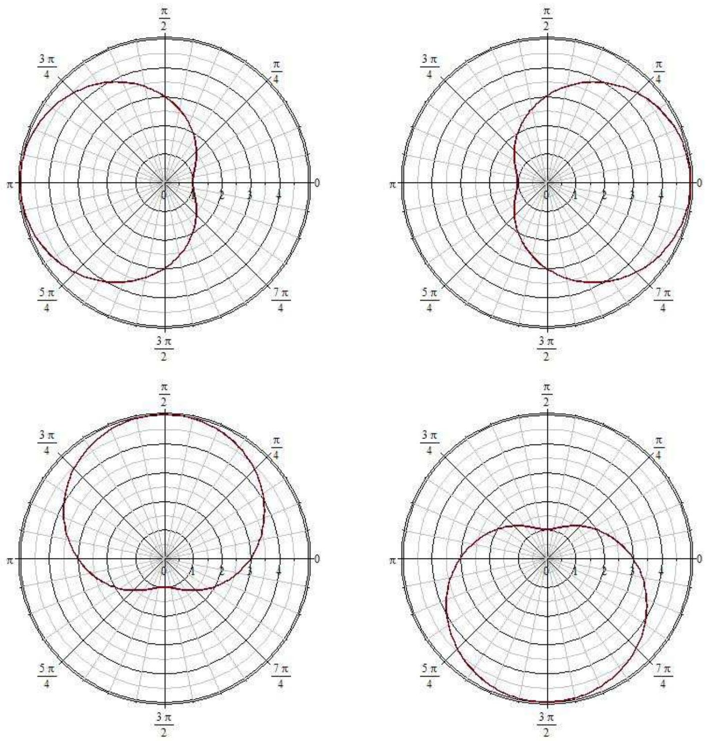
$$s = \int_4^0 -\frac{1}{\sqrt{w}}\, dw = \int_0^4 \frac{1}{\sqrt{w}}\, dw.$$

Notice that this is an improper integral. So

$$\begin{aligned} s &= \lim_{b\to 0^+} \int_b^4 \frac{1}{\sqrt{w}}\, dw \\ &= \lim_{b\to 0^+} \left[2\sqrt{w}\right]_b^4 = \lim_{b\to 0^+} \left[4 - \sqrt{b}\right] \\ &= 4. \end{aligned}$$

Since the top half of the cardioid has length 4, the full cardioid has perimeter equal to $\boxed{8}$.

55. The graphs below, from left to right and top to bottom, are r_1, r_2, r_3, r_4.



For Problems 56-61, using the parameterizations $x(\theta) = r\cos\theta = f(\theta)\cos\theta$ and $y(\theta) = r\sin\theta = f(\theta)\sin\theta$ we can find $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. Then we have a horizontal tangent line if $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$, and we have a vertical tangent line if $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$.

57. Parametric equations for $r = 3 + 3\cos\theta$ are

$$x = (3 + 3\cos\theta)\cos\theta \qquad y = (3 + 3\cos\theta)\sin\theta$$

and

$$\begin{aligned}
\frac{dy}{d\theta} &= (-3\sin\theta)\sin\theta + (3 + 3\cos\theta)\cos\theta = -3\sin^2\theta + 3\cos\theta + 3\cos^2\theta = 6\cos^2\theta + 3\cos\theta - 3 \\
&= 3(\cos\theta + 1)(2\cos\theta - 1) \\
\frac{dx}{d\theta} &= (-3\sin\theta)\cos\theta - (3 + 3\cos\theta)\sin\theta = -3\sin\theta\cos\theta - 3\sin\theta - 3\cos\theta\sin\theta \\
&= -\sin\theta(6\cos\theta + 3).
\end{aligned}$$

Horizontal Tangent Lines on $0 \leq \theta \leq 2\pi$:

$$\begin{aligned}
\frac{dy}{d\theta} &= 0 \\
3(\cos\theta + 1)(2\cos\theta - 1) &= 0 \\
\cos\theta + 1 = 0 \quad &\text{or} \quad 2\cos\theta - 1 = 0 \\
\theta &= \pi,\ \frac{\pi}{3},\ \frac{5\pi}{3}.
\end{aligned}$$

Notice that $\theta = \pi$ makes $\frac{dx}{d\theta} = 0$ so we must exclude it but the others do not make $\frac{dx}{d\theta}$ zero. The y-coordinates of the points corresponding to $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$ are $\frac{9\sqrt{3}}{4}$ and $-\frac{9\sqrt{3}}{4}$, respectively. The horizontal tangent lines are then $\boxed{y = \frac{9\sqrt{3}}{4}}$ and $\boxed{y = -\frac{9\sqrt{3}}{4}}$.

Vertical Tangent Lines on $0 \le \theta \le 2\pi$:

$$\begin{aligned} \frac{dx}{d\theta} &= 0 \\ -\sin\theta(6\cos\theta + 3) &= 0 \\ -\sin\theta = 0 \quad &\text{or} \quad 6\cos\theta + 3 = 0 \\ \theta &= 0,\ \pi,\ 2\pi,\ \frac{2\pi}{3},\ \frac{4\pi}{3}. \end{aligned}$$

Notice that $\theta = \pi$ makes $\frac{dy}{d\theta} = 0$ so we must exclude it but the others do not make $\frac{dy}{d\theta}$ zero. The x-coordinates of the points corresponding to $\theta = 0,\ 2\pi,\ \frac{2\pi}{3}$, and $\frac{4\pi}{3}$ are $6,\ 6,\ -\frac{3}{4}$, and $-\frac{3}{4}$, respectively. The vertical tangent lines are then $\boxed{x = 6}$ and $\boxed{x = -\frac{3}{4}}$.

59. Parametric equations for $r = 2\cos(2\theta)$ are

$$x = (2\cos(2\theta))\cos\theta \qquad y = (2\cos(2\theta))\sin\theta$$

and

$$\begin{aligned} \frac{dy}{d\theta} &= (-4\sin(2\theta))\sin\theta + (2\cos(2\theta))\cos\theta \\ \frac{dx}{d\theta} &= (-4\sin(2\theta))\cos\theta - (2\cos(2\theta))\sin\theta. \end{aligned}$$

Horizontal Tangent Lines: Using a CAS to find where $\frac{dy}{d\theta} = 0$, we find that $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right), -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right), -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right) + \pi, \tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right) - \pi$. None of these values make $\frac{dx}{d\theta} = 0$. The y-coordinates of the points corresponding to $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right), -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right), -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right) + \pi, \tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{30}}\right) - \pi$ are $-2, 2, \frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}, \frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}$, respectively. The horizontal tangent lines are then $\boxed{y = \pm 2}$ and $\boxed{y \pm \frac{4}{3\sqrt{6}}}$.

Vertical Tangent Lines: Using a CAS to find where $\frac{dx}{d\theta} = 0$, we find that $\theta = 0, \pi, 2\pi, \tan^{-1}\left(\sqrt{5}\right), -\tan^{-1}\left(\sqrt{5}\right) + \pi, -\tan^{-1}\left(\sqrt{5}\right), \tan^{-1}\left(\sqrt{5}\right) - \pi$. None of these values make $\frac{dy}{d\theta} = 0$. The x-coordinates of the points corresponding to $\theta = 0, \pi, 2\pi, \tan^{-1}\left(\sqrt{5}\right), -\tan^{-1}\left(\sqrt{5}\right) + \pi, -\tan^{-1}\left(\sqrt{5}\right), \tan^{-1}\left(\sqrt{5}\right) - \pi$ are $2, -2, 2, \frac{4}{3\sqrt{6}}, \frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}$, respectively. The vertical tangent lines are then $\boxed{x = \pm 2}$ and $\boxed{x = \pm\frac{4}{3\sqrt{6}}}$.

61. In Problem 33, we found parametric equations for $r^2 = 4\sin(2\theta)$.

$$x = r\cos\theta = 2\sqrt{\sin(2\theta)}\cos\theta \qquad y = r\sin\theta = 2\sqrt{\sin(2\theta)}\sin\theta, \quad \sin(2\theta) > 0.$$

Using a CAS to find the derivatives,

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{2\cos\theta\left(4\cos^2\theta - 3\right)}{\sqrt{\sin(2\theta)}}\\ \frac{dy}{d\theta} &= \frac{2\sin\theta\left(4\cos^2\theta - 1\right)}{\sqrt{\sin(2\theta)}}.\end{aligned}$$

Horizontal Tangent Lines: Using a CAS to find where $\frac{dy}{d\theta} = 0$ on $0 \le \theta \le 2\pi$ where $\sin(2\theta) > 0$, we find that $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. Neither of these values make $\frac{dx}{d\theta} = 0$. The y-coordinates of the points corresponding to $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$ are $3^{3/4}\frac{\sqrt{2}}{2}$ and $-3^{3/4}\frac{\sqrt{2}}{2}$, respectively. The horizontal tangent lines are then $\boxed{y = \pm\frac{3^{3/4}}{\sqrt{2}}}$.

Vertical Tangent Lines: Using a CAS to find where $\frac{dx}{d\theta} = 0$ on $0 \le \theta \le 2\pi$ where $\sin(2\theta) > 0$, we find that $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. Neither of these values make $\frac{dy}{d\theta} = 0$. The x-coordinates of the points corresponding to $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$ are $3^{3/4}\frac{\sqrt{2}}{2}$ and $-3^{3/4}\frac{\sqrt{2}}{2}$, respectively. The vertical tangent lines are then $\boxed{x = \pm\frac{3^{3/4}}{\sqrt{2}}}$.

63. (a) Answers will vary.
(b) Since $r^2 = (-r)^2$, we have $(-r)^2 = r^2 = 4\sin(2\theta)$ and so the lemniscate is symmetric with respect to the pole.

Challenge Problems

65. Symmetry with respect to the pole: Substituting $\theta + \pi$ for θ and using the angle addition formula for sine, we have

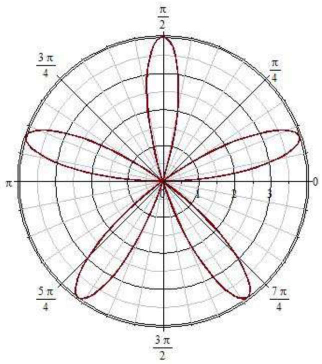
$$\begin{aligned}r &= \sin(2(\theta+\pi)) = \sin(2\theta + 2\pi)\\ r &= \sin(2\theta)\cos(2\pi) + \sin(2\pi)\cos(2\theta)\\ r &= \sin(2\theta),\end{aligned}$$

and an equivalent equation results implying that the graph is symmetric with respect to the pole.
Symmetry with respect to the polar axis: Substituting $-\theta$ for θ and recalling that $\sin\theta$ is an odd function, we have

$$\begin{aligned}r &= \sin(-2\theta)\\ r &= -\sin(2\theta),\end{aligned}$$

which is not an equivalent equation so the test fails.

67. (a)



(b) The graph intersects the pole when $r = 0$. Setting $4\sin(5\theta) = 0$ we see that $5\theta = \pi n$ for any integer n, or

$$\theta = \frac{n}{5}\pi.$$

We seek the smallest positive value of θ, which is when $n = 1$. So $\boxed{\alpha = \frac{1}{5}\pi}$.

(c) The arc length of the petal is given by

$$s = \int_0^{\pi/5} \sqrt{(4\sin(5\theta))^2 + (20\cos(5\theta))^2}\, d\theta.$$

Using a CAS, we find that $s \approx \boxed{8.404}$.

AP® Practice Problems

1. We obtain parametric equations for $r = 1 + \cos\theta$ by using the conversion formulas

$x = r\cos\theta$ and $y = r\sin\theta$.

$x = r\cos\theta = (1 + \cos\theta)\cos\theta$ and $y = r\sin\theta = (1 + \cos\theta)\sin\theta$.

Then

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[(1 + \cos\theta)\cos\theta] = (1 + \cos\theta)(-\sin\theta) + (\cos\theta)(-\sin\theta) = -\sin\theta - 2\cos\theta\sin\theta,$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[(1 + \cos\theta)\sin\theta] = (1 + \cos\theta)(\cos\theta) + (\sin\theta)(-\sin\theta) = \cos\theta + \cos^2\theta - \sin^2\theta,$$

and $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta}$.

At $\theta = \dfrac{\pi}{6}$,

$$\frac{dy}{dx} = \frac{\cos\frac{\pi}{6} + \left(\cos\frac{\pi}{6}\right)^2 - \left(\sin\frac{\pi}{6}\right)^2}{-\sin\frac{\pi}{6} - 2\cos\frac{\pi}{6}\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2}{-\frac{1}{2} - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)} = \frac{\sqrt{3} + 1}{-1 - \sqrt{3}} = -\frac{\sqrt{3} + 1}{1 + \sqrt{3}} = -1.$$

The answer is A.

3. Obtain parametric equations for $r = 5\theta$ by using the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$x = r\cos\theta = 5\theta\cos\theta$ and $y = r\sin\theta = 5\theta\sin\theta$.

The answer is A.

5. Use the arc length formula $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ with $r = 4e^{3\theta/2}$, $\frac{dr}{d\theta} = 6e^{3\theta/2}$, $\propto = 0$, and $\beta = \ln 9$.

$$\begin{aligned} s &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{\ln 9} \sqrt{\left(4e^{3\theta/2}\right)^2 + \left(6e^{3\theta/2}\right)^2}\, d\theta \\ &= \int_0^{\ln 9} \sqrt{16e^{3\theta} + 36e^{3\theta}}\, d\theta \\ &= \sqrt{52}\int_0^{\ln 9} e^{3\theta/2}\, d\theta \\ &= \frac{2}{3}\sqrt{52}\left[\left(e^{3\theta/2}\right)^2\right]_0^{\ln 9} \\ &= \frac{2}{3}\sqrt{52}\left(e^{3\ln 9/2} - 1\right) \\ &= \frac{4}{3}\sqrt{13}\left[9^{3/2} - 1\right] = \frac{104}{3}\sqrt{13} \end{aligned}$$

The answer is C.

9.6 Area in Polar Coordinates

Note: Throughout this assignment, we will be integrating $\sin^2(a\theta)$ or $\cos^2(a\theta)$ frequently. To integrate them we use the formulas $\sin^2(a\theta) = \frac{1-\cos(2a\theta)}{2}$ and $\cos^2(a\theta) = \frac{1+\cos(2a\theta)}{2}$. With these formulas, we have

$$\begin{aligned} \int \sin^2(a\theta)\ d\theta &= \frac{1}{2}\theta - \frac{1}{4a}\sin(2a\theta) = \frac{1}{2}\theta - \frac{1}{2a}\cos(a\theta)\sin(a\theta) \\ \int \cos^2(a\theta)\ d\theta &= \frac{1}{2}\theta + \frac{1}{4a}\sin(2a\theta) = \frac{1}{2}\theta + \frac{1}{2a}\cos(a\theta)\sin(a\theta) \end{aligned}$$

Each problem in which the above equations are used will be marked with an asterisk (*).

Concepts and Vocabulary

1. The area A of a sector of a circle of radius r and central angle θ is $A = \boxed{\frac{1}{2}r^2\theta}$.

3. $\boxed{\textbf{False}}$. The area is given by $A = \int_\alpha^\beta \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$.

Skill Building

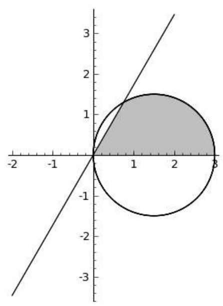
5*. The area of the shaded region in quadrant I equals one-fourth of the area A of the shaded region. The area of the shaded region in quadrant I is equal to the area of the non-shaded region inside the rose in quadrant I which is swept out starting at $\theta = 0$ and, by symmetry, ends at $\theta = \frac{\pi}{4}$. The area of the non-shaded region in quadrant I is given by $\int_0^{\pi/4} \frac{1}{2}r^2\ d\theta$, and the area A we seek is 4 times this area,

$$\begin{aligned} A &= 4\int_0^{\pi/4} \frac{1}{2}r^2\ d\theta = 2\int_0^{\pi/4} \cos^2(2\theta)\ d\theta \\ &= \left[\theta + \frac{1}{2}\cos(2\theta)\sin(2\theta)\right]_0^{\pi/4} = \boxed{\frac{\pi}{4}}. \end{aligned}$$

7*. The shaded region is swept out beginning with the ray $\theta = 0$ and ending with the ray $\theta = \pi$. The area A of the shaded region is then equal to $\int_0^{\pi} \frac{1}{2}r^2\ d\theta$.

$$\begin{aligned} A &= \int_0^{\pi} \frac{1}{2}r^2\ d\theta = \int_0^{\pi} \frac{1}{2}(2+2\sin\theta)^2\ d\theta = 2\int_0^{\pi} (1+2\sin\theta+\sin^2\theta)\ d\theta \\ &= [-4\cos\theta + 3\theta - \sin\theta\cos\theta]_0^{\pi} = \boxed{8+3\pi}. \end{aligned}$$

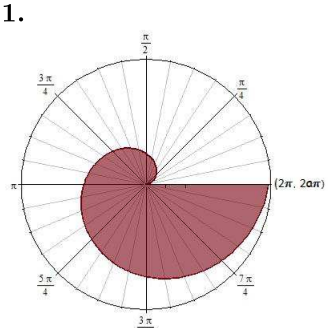
9*.



The area A of the shaded region is equal to $\int_0^{\pi/3} \frac{1}{2}r^2\ d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2}r^2\ d\theta = \int_0^{\pi/3} \frac{1}{2}(3\cos\theta)^2\ d\theta = \frac{9}{2}\int_0^{\pi/3} \cos^2\theta\ d\theta \\ &= \left[\frac{9}{4}\theta + \frac{9}{4}\cos\theta\sin\theta\right]_0^{\pi/3} = \frac{9\sqrt{3}}{16} + \frac{3\pi}{4} \\ &= \boxed{\frac{3}{16}\left(3\sqrt{3}+4\pi\right)}. \end{aligned}$$

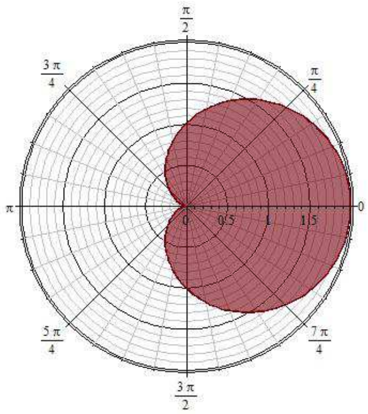
11.



[The graph shown is when $a = 1$.] The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2}r^2\ d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2}r^2\ d\theta = \int_0^{2\pi} \frac{1}{2}(a\theta)^2\ d\theta = \frac{a^2}{2}\int_0^{2\pi} \theta^2\ d\theta \\ &= \left[\frac{a^2}{6}\theta^3\right]_0^{2\pi} = \boxed{\frac{4\pi^3 a^2}{3}}. \end{aligned}$$

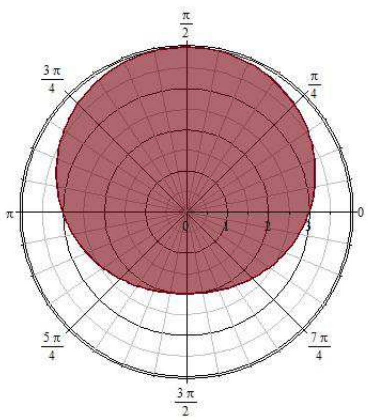
13*.



The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (1+\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta \\ &= \left[\frac{3}{4}\theta + \sin\theta + \frac{1}{4}\cos\theta\sin\theta\right]_0^{2\pi} = \boxed{\frac{3\pi}{2}}. \end{aligned}$$

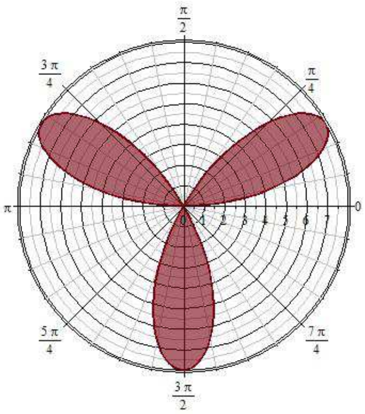
15*.

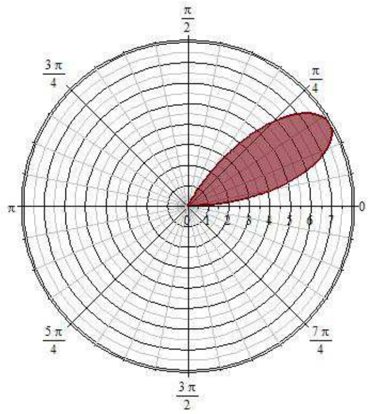


The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (3+\sin\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (9 + 6\sin\theta + \sin^2\theta) \, d\theta \\ &= \left[\frac{19}{4}\theta - 3\cos\theta - \frac{1}{4}\cos\theta\sin\theta\right]_0^{2\pi} = \boxed{\frac{19\pi}{2}}. \end{aligned}$$

17*.



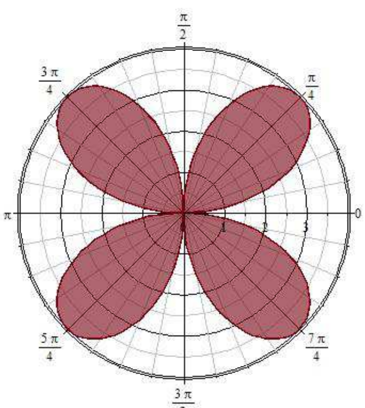


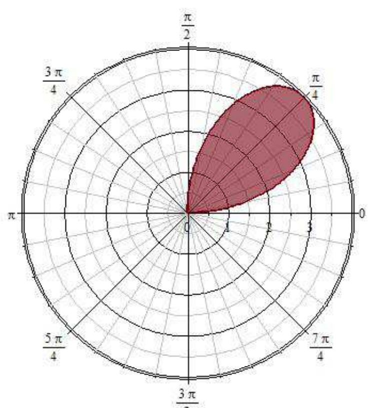
Exploiting symmetry, we will find the area A of the petal in quadrant I (see the second picture) which is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \dfrac{\pi}{3}$ and then triple it to find the full area of the rose. The area A of the petal in quadrant I is equal to $\displaystyle\int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/3} \frac{1}{2} \left(8\sin(3\theta)\right)^2 \, d\theta = 32 \int_0^{\pi/3} \sin^2(3\theta) \, d\theta \\ &= \left[16\theta - \frac{16}{3}\sin(3\theta)\cos(3\theta)\right]_0^{\pi/3} = \frac{16\pi}{3}. \end{aligned}$$

The total area of the full rose is then $3 \cdot \dfrac{16\pi}{3} = \boxed{16\pi}$.

19*.

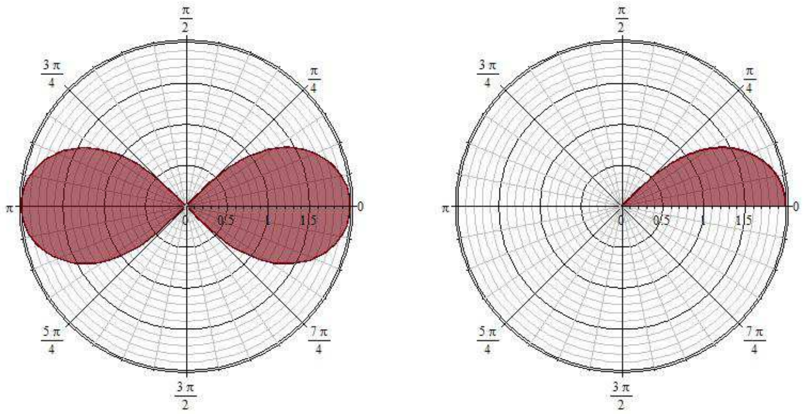




The area A of the petal in quadrant I (see the second picture) is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \dfrac{\pi}{2}$ so A is equal to $\displaystyle\int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2} \left(4\sin(2\theta)\right)^2 \, d\theta = 8 \int_0^{\pi/2} \sin^2(2\theta) \, d\theta \\ &= \left[4\theta - 2\sin(2\theta)\cos(2\theta)\right]_0^{\pi/2} = \boxed{2\pi}. \end{aligned}$$

21.

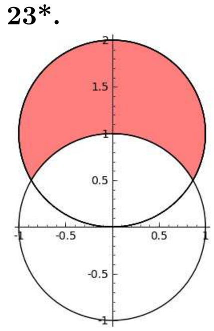


The area A of the half-petal in quadrant I (see the second picture) is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{4}$ so A is equal to $\int_0^{\pi/4} \frac{1}{2} r^2 \; d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} r^2 \; d\theta = \int_0^{\pi/4} \frac{1}{2} \left(4 \cos(2\theta)\right) \; d\theta = 2 \int_0^{\pi/4} \cos(2\theta) \; d\theta \\ &= \left[\sin(2\theta)\right]_0^{\pi/4} = 1. \end{aligned}$$

Then a full petal has area $2 \cdot 1 = \boxed{2}$.

23*.



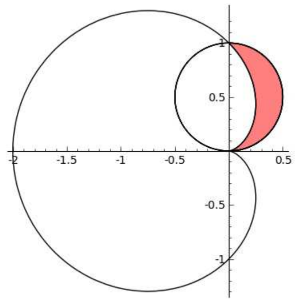
Referring to the picture we need to find the intersection points of the two circles, so we set them equal.

$$\begin{aligned} 2\sin\theta &= 1 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}. \end{aligned}$$

Then

$$\begin{aligned} A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2} (2\sin\theta)^2 \; d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2} (1)^2 \; d\theta \\ &= \left[-\frac{1}{2} \sin\theta\cos\theta + \frac{1}{2}\theta \right]_{\pi/6}^{5\pi/6} - \left[\frac{1}{2}\theta \right]_{\pi/6}^{5\pi/6} \\ &= \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\pi}{3} = \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}. \end{aligned}$$

25*.



Referring to the picture we can see that the shaded region is swept out beginning with $\theta = 0$ and ending at $\theta = \frac{\pi}{2}$. Then the area A of the shaded region is

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2}(\sin\theta)^2\ d\theta - \int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\ d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} (\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta)\ d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} (-\cos(2\theta) - 1 + 2\cos\theta)\ \ d\theta \\ &= \frac{1}{2}\left[-\frac{1}{2}\sin(2\theta) - \theta + 2\sin\theta\right]_0^{\pi/2} \\ &= \boxed{1 - \frac{\pi}{4}}. \end{aligned}$$

27*. Using formula (1) with $f(\theta) = \sin\theta$ and $f'(\theta) = \cos\theta$, the surface area S is

$$\begin{aligned} S &= 2\pi\int_0^{\pi/2} (\sin\theta)\sin\theta\sqrt{[\sin\theta]^2 + [\cos\theta]^2}\ d\theta \\ &= 2\pi\int_0^{\pi/2} \sin^2\theta\ d\theta \\ &= 2\pi\left[\frac{1}{2}\theta - \frac{1}{2}\cos\theta\sin\theta\right]_0^{\pi/2} \\ &= 2\pi\left(\frac{1}{4}\pi\right) = \boxed{\frac{\pi^2}{2}}. \end{aligned}$$

29. Using formula (1) with $f(\theta) = e^\theta$ and $f'(\theta) = e^\theta$, the surface area S is

$$\begin{aligned} S &= 2\pi\int_0^{\pi} (e^\theta)\sin\theta\sqrt{\left[e^\theta\right]^2 + \left[e^\theta\right]^2}\ d\theta = 2\sqrt{2}\pi\int_0^{\pi} e^{2\theta}\sin\theta\ d\theta \\ &= 2\sqrt{2}\pi\left[\frac{e^{2\theta}}{5}(2\sin\theta - \cos\theta)\right]_0^{\pi} = \boxed{\frac{2\sqrt{2}\pi}{5}\left(e^{2\pi}+1\right)}, \end{aligned}$$

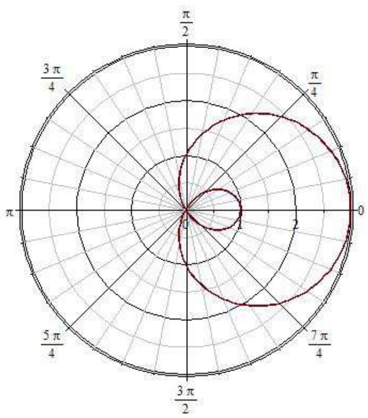
where we used Table of Integrals 122.

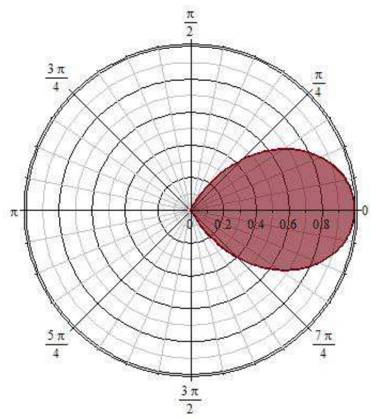
Applications and Extensions

31*. To find the area of the small loop, we need to find the values of θ that sweep out the inner loop. We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 1 + 2\cos\theta &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

The two graphs show the full limaçon and just the small loop (zoomed in).

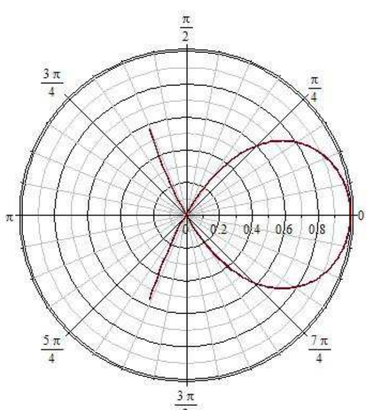


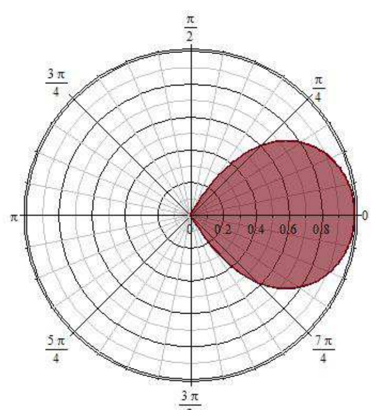


The area A of the small loop is

$$\begin{aligned} A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1 + 2\cos\theta)^2 \, d\theta = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta \\ &= \left[\frac{3}{2}\theta + 2\sin\theta + \cos\theta\sin\theta\right]_{2\pi/3}^{4\pi/3} = \boxed{\pi - \frac{3\sqrt{3}}{2}}. \end{aligned}$$

33. The two graphs show the full graph and the loop.





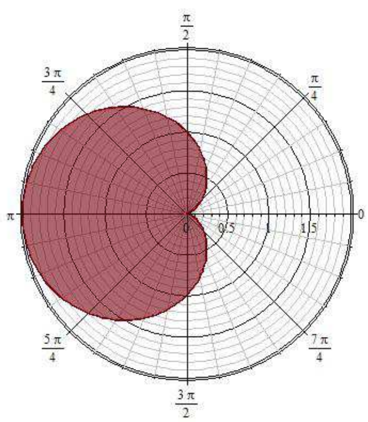
To find the area of loop, we need to find the values of θ that sweep out the loop. We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 2 - \sec\theta &= 0 \\ \sec\theta &= 2 \\ \theta &= \pm\frac{\pi}{3}. \end{aligned}$$

The area A of the small loop is

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2-\sec\theta)^2\ d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4-4\sec\theta+\sec^2\theta)\ d\theta \\ &= \left[2\theta - 2\ln|\sec\theta+\tan\theta| + \frac{1}{2}\tan\theta\right]_{-\pi/3}^{\pi/3} \\ &= \left(\frac{2\pi}{3} - 2\ln\left(2+\sqrt{3}\right) + \frac{\sqrt{3}}{2}\right) - \left(-\frac{2\pi}{3} - 2\ln\left(2-\sqrt{3}\right) - \frac{\sqrt{3}}{2}\right) \\ &= \frac{4\pi}{3} - 2\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) + \sqrt{3} = \frac{4\pi}{3} - 2\ln\left((2+\sqrt{3})^2\right) + \sqrt{3} \\ &= \boxed{\frac{4\pi}{3} - 4\ln(2+\sqrt{3}) + \sqrt{3}}. \end{aligned}$$

35.



Exploting symmetry, we will find the area A of the regions in the first and second quadrant, and then double that area. The area in the first two quadrants is swept out starting at the ray $\theta = 0$ and ending at $\theta = \pi$. Then

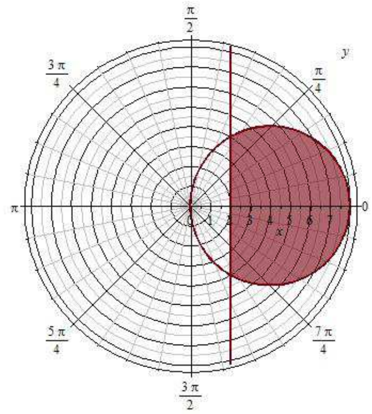
$$A = \int_0^{\pi} \frac{1}{2}\left[2\sin^2\frac{\theta}{2}\right]^2\ d\theta = 2\int_0^{\pi}\sin^4\frac{\theta}{2}\ d\theta.$$

Using the substitution $u = \dfrac{\theta}{2}$ and changing the limits of integration,

$$\begin{aligned} A &= 4\int_0^{\pi/2}\sin^4 u\ du \\ &= 4\left[-\frac{1}{4}\sin^3 u\cos u - \frac{3}{8}\cos u\sin u + \frac{3}{8}u\right]_0^{\pi/2} = \frac{3\pi}{4}, \end{aligned}$$

where to compute the integral we used Table of Integrals 82 & 88. Then the full shaded region has area equal to $2\cdot\dfrac{3\pi}{4} = \boxed{\dfrac{3\pi}{2}}$.

37*.



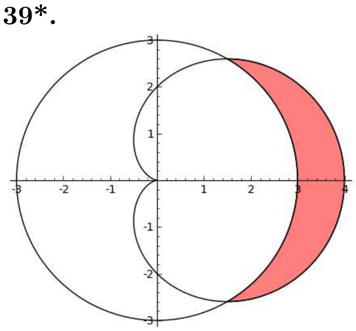
We need to find the angles that sweep out the shaded region so we set the equations equal.

$$\begin{aligned} 8\cos\theta &= 2\sec\theta \\ \cos^2\theta &= \frac{1}{4} \\ \cos\theta &= \pm\frac{1}{2} \\ \theta &= \pm\frac{\pi}{3},\ \pm\frac{2\pi}{3}. \end{aligned}$$

The shaded region is swept out beginning with $\theta = -\frac{\pi}{3}$ and ending at $\theta = \frac{\pi}{3}$. The area A of the shaded region is then

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(8\cos\theta)^2\ d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\sec\theta)^2\ d\theta = \int_{-\pi/3}^{\pi/3} (32\cos^2\theta - 2\sec^2\theta)\ d\theta \\ &= [16\cos\theta\sin\theta + 16\theta - 2\tan\theta]_{-\pi/3}^{\pi/3} = \boxed{4\sqrt{3} + \frac{32\pi}{3}}. \end{aligned}$$

39*.



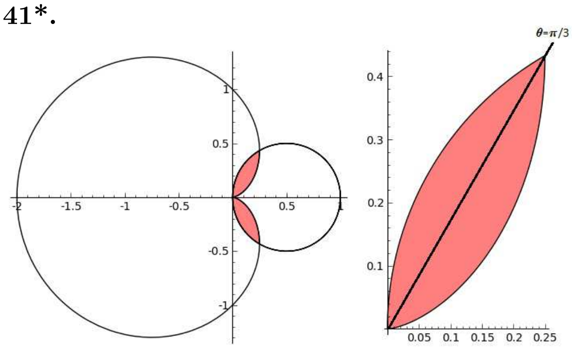
We need to find the angles that sweep out the shaded region so we set the equations equal.

$$\begin{aligned} 2 + 2\cos\theta &= 3 \\ \cos\theta &= \frac{1}{2} \\ \theta &= \pm\frac{\pi}{3}. \end{aligned}$$

The shaded region is swept out beginning with $\theta = -\dfrac{\pi}{3}$ and ending at $\theta = \dfrac{\pi}{3}$. The area A of the shaded region is then

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2+2\cos\theta)^2\ d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(3)^2\ d\theta = \int_{-\pi/3}^{\pi/3} \left(-\frac{5}{2} + 4\cos\theta + 2\cos^2\theta\right)\ d\theta \\ &= \left[-\frac{3}{2}\theta + 4\sin\theta + \cos\theta\sin\theta\right]_{-\pi/3}^{\pi/3} = \boxed{\frac{9\sqrt{3}}{2} - \pi}. \end{aligned}$$

41*.



Exploiting symmetry, we will find the area of the shaded region in quadrant I and then double it to find the full shaded area. Referring to the second picture, the shaded region in quadrant I can be divided by the ray from the pole through the intersection point of the two curves. The angle that represents the ray through the intersection point can be found by setting the equations equal.

$$\begin{aligned} \cos\theta &= 1 - \cos\theta \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3}. \end{aligned}$$

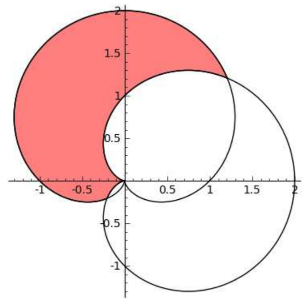
The shaded region in quadrant I can then be found by finding the area of the shaded region on $\left[0, \dfrac{\pi}{3}\right]$ of the cardioid and then adding the area of the shaded region on $\left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$ of the circle. This means the area A in quadrant I is

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2}(1-\cos\theta)^2\ d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\cos\theta)^2\ d\theta \\ &= \int_0^{\pi/3} \left(\frac{1}{2} - \cos\theta + \frac{1}{2}\cos^2\theta\right)\ d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}\cos^2\theta\ d\theta \\ &= \left[\frac{3}{4}\theta - \sin\theta + \frac{1}{4}\cos\theta\sin\theta\right]_0^{\pi/3} + \left[\frac{1}{4}\theta + \frac{1}{4}\cos\theta\sin\theta\right]_{\pi/3}^{\pi/2} \\ &= \left[\frac{\pi}{4} - \frac{7\sqrt{3}}{16}\right] + \left[\frac{\pi}{24} - \frac{\sqrt{3}}{16}\right] = \frac{7\pi}{24} - \frac{\sqrt{3}}{2}. \end{aligned}$$

Then the full shaded area is

$$2 \cdot \left(\frac{7\pi}{24} - \frac{\sqrt{3}}{2}\right) = \boxed{\frac{7\pi}{12} - \sqrt{3}}.$$

43*.



First we find the intersection points by setting the equations equal.

$$\begin{aligned} 1+\cos\theta &= 1+\sin\theta \\ \cos\theta &= \sin\theta \\ \theta &= \frac{\pi}{4}, \frac{5\pi}{4}. \end{aligned}$$

The area A of the shaded region is swept out beginning with the ray $\theta = \dfrac{\pi}{4}$ and ending with the ray $\theta = \dfrac{5\pi}{4}$. Then

$$\begin{aligned} A &= \int_{\pi/4}^{5\pi/4} \frac{1}{2}(1+\sin\theta)^2\ d\theta - \int_{\pi/4}^{5\pi/4} \frac{1}{2}(1+\cos\theta)^2\ d\theta = \int_{\pi/4}^{5\pi/4} \left(\sin\theta - \cos\theta + \frac{1}{2}\sin^2\theta - \frac{1}{2}\cos^2\theta\right)\ d\theta \\ &= \left[-\cos\theta - \sin\theta - \frac{1}{2}\sin\theta\cos\theta\right]_{\pi/4}^{5\pi/4} = \boxed{2\sqrt{2}}. \end{aligned}$$

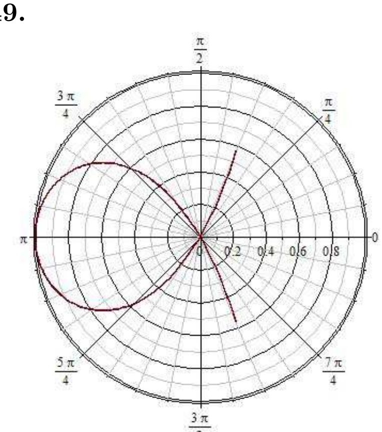
45. Using the area formula, the enclosed area A is

$$\begin{aligned} A &= \int_0^1 \frac{1}{2}(e^{-\theta})^2\ d\theta = \int_0^1 \frac{1}{2}e^{-2\theta}\ d\theta \\ &= \left[-\frac{1}{4}e^{-2\theta}\right]_0^1 = \boxed{\frac{1-e^{-2}}{4}}. \end{aligned}$$

47. Using the area formula, the area A enclosed is

$$\begin{aligned} A &= \int_1^{\pi} \frac{1}{2}\left(\frac{1}{\theta}\right)^2\ d\theta = \int_1^{\pi} \frac{1}{2}\theta^{-2}\ d\theta \\ &= \left[-\frac{1}{2}\theta^{-1}\right]_1^{\pi} = \boxed{\frac{\pi-1}{2\pi}}. \end{aligned}$$

49.



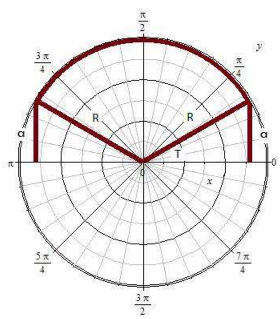
We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 2 + \sec\theta &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

The area A of the loop is

$$\begin{aligned} A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2+\sec\theta)^2 \, d\theta = \int_{2\pi/3}^{4\pi/3} \left(\frac{1}{2}\sec^2\theta + 2\sec\theta + 2\right) \, d\theta \\ &= \left[\frac{1}{2}\tan\theta + 2\ln|\sec\theta + \tan\theta| + 2\theta\right]_{2\pi/3}^{4\pi/3} = \sqrt{3} + \frac{4\pi}{3} + 2\ln\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right) \\ &= \boxed{\sqrt{3} + \frac{4\pi}{3} - 4\ln\left(2+\sqrt{3}\right)}. \end{aligned}$$

51. (a)



If we rotate the top half of the circle $r = R$ about the polar axis for $T \le \theta \le \pi - T$, then we will have the surface of the bead. Using formula (1) with $f(\theta) = R$, $f'(\theta) = 0$, and $T \le \theta \le \pi - T$, the surface area S of the bead is

$$\begin{aligned} S &= 2\pi \int_T^{\pi-T} R\sin\theta \sqrt{[R]^2 + [0]^2} \, d\theta = 2\pi R^2 \int_T^{\pi-T} \sin\theta \, d\theta \\ &= 2\pi R^2 \left[-\cos\theta\right]_T^{\pi-T} = 2\pi R^2 \left[-\cos(\pi - T) + \cos(T)\right]. \end{aligned}$$

From the picture, we see that $\cos(T) = \dfrac{\sqrt{R^2 - a^2}}{R}$ and so $-\cos(\pi - T) = \dfrac{\sqrt{R^2-a^2}}{R}$. Then

$$S = 2\pi R^2 \cdot \frac{2\sqrt{R^2 - a^2}}{R} = \boxed{4\pi R\sqrt{R^2 - a^2}}.$$

(b) Answers will vary.
53*. We first find where the curve intersects the pole by setting $r = 0$.

$$\begin{aligned} \sec\theta - 2\cos\theta &= 0 \\ 1 - 2\cos^2\theta &= 0 \\ \cos^2\theta &= \frac{1}{2} \\ \cos\theta &= \pm\frac{\sqrt{2}}{2} \\ \theta &= -\frac{\pi}{4}, \frac{\pi}{4}. \end{aligned}$$

From the picture we see that the shaded region is swept out beginning with $\theta = -\dfrac{\pi}{4}$ and ending with $\theta = \dfrac{\pi}{4}$. So the area A of the shaded region is

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(\sec\theta - 2\cos\theta\right)^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\sec^2\theta - 2 + 2\cos^2\theta\right) \, d\theta \\ &= \left[\frac{1}{2}\tan\theta - \theta + \sin\theta\cos\theta\right]_{-\pi/4}^{\pi/4} = \boxed{2 - \frac{\pi}{2}}. \end{aligned}$$

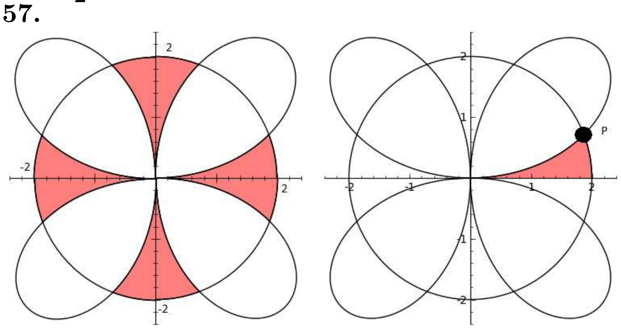
Challenge Problems

55. The area enclosed by the graph $r\theta = a$ and the rays $\theta = \theta_1$ and $\theta = \theta_2$ is

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \frac{1}{2}\left(\frac{a}{\theta}\right)^2 \, d\theta &= \frac{a^2}{2}\int_{\theta_1}^{\theta_2} \theta^{-2} \, d\theta \\ &= \frac{a^2}{2}\left[-\theta^{-1}\right]_{\theta_1}^{\theta_2} = \frac{a^2}{2}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right) = \frac{a}{2}\left(\frac{a}{\theta_1} - \frac{a}{\theta_2}\right) \\ &= \frac{a}{2}\left(r_1 - r_2\right). \end{aligned}$$

Since $\dfrac{a}{2}$ is a constant, we see that the area is a constant multiple of (i.e. proportional to) $r_1 - r_2$.

57.



To find the area of the shaded region in the first figure, we will exploit symmetry and find the area A of the shaded region in the second figure and multiply it by eight to find the area we seek. We must first find the intersection point P of the rose and circle.

$$\begin{aligned} 3\sin(2\theta) &= 2 \\ \sin(2\theta) &= \frac{2}{3} \\ 2\theta &= \sin^{-1}\left(\frac{2}{3}\right) \\ \theta &= \frac{1}{2}\sin^{-1}\left(\frac{2}{3}\right). \end{aligned}$$

Then the area A is swept out beginning at $\theta = 0$ and ending at $\theta = \dfrac{1}{2}\sin^{-1}\left(\dfrac{2}{3}\right)$, so

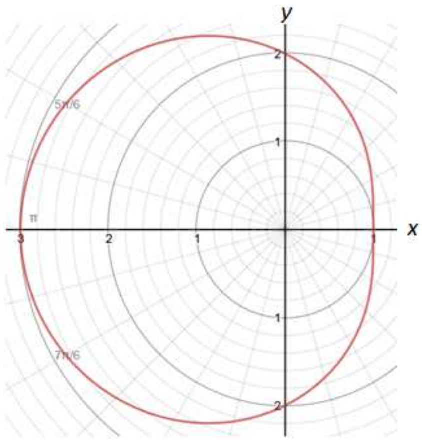
$$\begin{aligned} A &= \int_0^{1/2\sin^{-1}(2/3)} \frac{1}{2}(2)^2 \, d\theta - \int_0^{1/2\sin^{-1}(2/3)} \frac{1}{2}(3\sin(2\theta))^2 \, d\theta = \int_0^{1/2\sin^{-1}(2/3)} \left(2 - \frac{9}{2}\sin^2(2\theta) \, d\theta\right) \\ &= \left[-\frac{1}{4}\theta + \frac{9}{8}\sin(2\theta)\cos(2\theta)\right]_0^{1/2\sin^{-1}(2/3)} = \frac{\sqrt{5}}{4} - \frac{1}{8}\sin^{-1}\left(\frac{2}{3}\right). \end{aligned}$$

Multiplying this value by eight, we have the area we sought.

$$8\left(\frac{\sqrt{5}}{4}-\frac{1}{8}\sin^{-1}\left(\frac{2}{3}\right)\right)=\boxed{2\sqrt{5}-\sin^{-1}\left(\frac{2}{3}\right)}.$$

AP® Practice Problems

1. The graph of $r=2-\cos\theta$ is pictured below.



Note that for $\theta=\dfrac{\pi}{2}, r=2-\cos\dfrac{\pi}{2}=2$ and for $\theta=\pi, r=2-\cos\pi=3$.

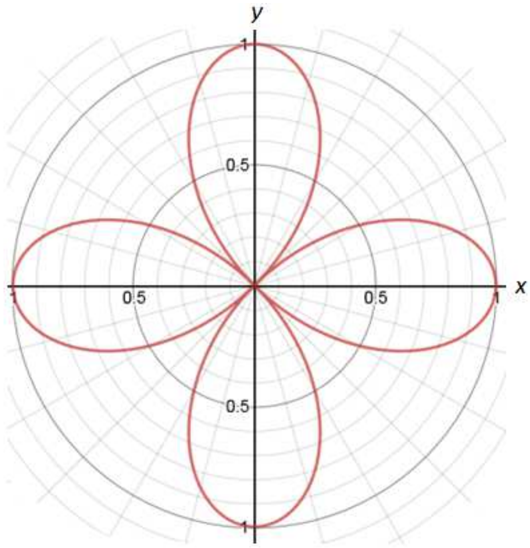
Therefore, the interval $\dfrac{\pi}{2}\le\theta\le\pi$ sweeps out the region in the second quadrant enclosed by the graph of $r=2-\cos\theta$.

Use the area formula $A=\displaystyle\int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta$ with $r=2-\cos\theta,\ \propto=\dfrac{\pi}{2}$, and $\beta=\pi$.

$$\begin{aligned}
A&=\int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta\\
&=\int_{\pi/2}^{\pi}\frac{1}{2}(2-\cos\theta)^2\,d\theta\\
&=\frac{1}{2}\int_{\pi/2}^{\pi}\left(4-4\cos\theta+\cos^2\theta\right)d\theta\\
&=\frac{1}{2}\left\{4\theta-4\sin\theta+\frac{1}{2}\left[\theta+\frac{1}{2}\sin(2\theta)\right]\right\}_{\pi/2}^{\pi}\\
&=\frac{1}{2}\left[\frac{9}{2}\theta-4\sin\theta+\frac{1}{4}\sin(2\theta)\right]_{\pi/2}^{\pi}\\
&=\frac{1}{2}\left[\left(\frac{9}{2}\pi-0+0\right)-\left(\frac{9}{4}\pi-4+0\right)\right]=\frac{9}{8}\pi+2
\end{aligned}$$

The answer is D.

3. The graph of $r = \cos(2\theta)$ is pictured below.



The interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ sweeps out one petal region of the rose $r = \cos(2\theta)$.

Use the area formula $A = \int_{\alpha}^{\beta} \frac{1}{2}r^2\,d\theta$ with $r = \cos(2\theta)$, $\alpha = -\frac{\pi}{4}$, and $\beta = \frac{\pi}{4}$.

$$\begin{aligned}
A &= \int_{\alpha}^{\beta} \frac{1}{2}r^2\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{2}[\cos(2\theta)]^2\,d\theta \\
&= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\,d\theta \\
&= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}[1+\cos(4\theta)]\,d\theta \\
&= \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{-\pi/4}^{\pi/4} \\
&= \frac{1}{4}\left\{\left[\frac{\pi}{4} + \frac{1}{4}\sin\pi\right] - \left[\left(-\frac{\pi}{4}\right) + \frac{1}{4}\sin(-\pi)\right]\right\} \\
&= \frac{\pi}{8}
\end{aligned}$$

The answer is B.

5. (a) Find the points of intersection of the two graphs by solving the equation.

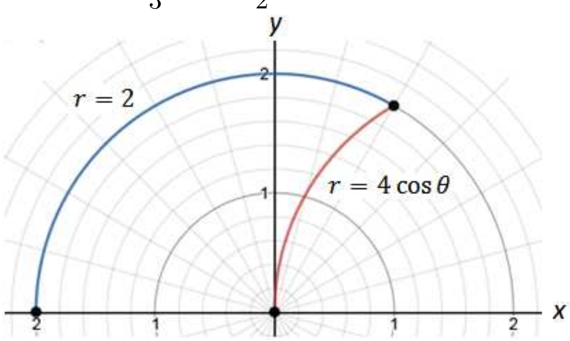
$$\begin{aligned}
4\cos\theta &= 2 \\
\cos\theta &= \frac{1}{2} \\
\theta &= -\frac{\pi}{3}, \frac{\pi}{3}
\end{aligned}$$

The two graphs intersect at $(r,\theta) = \boxed{(2, -\frac{\pi}{3})}$ and $\boxed{(2, \frac{\pi}{3})}$.

(b) The area of the region that lies outside the circle $r = 2$ and inside the circle $r = 4\cos\theta$ is

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (4\cos\theta)^2\, d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2^2\, d\theta$$
$$= \int_{-\pi/3}^{\pi/3} \left(8\cos^2\theta - 2\right) d\theta$$
$$= \int_{-\pi/3}^{\pi/3} \left\{8\cdot\frac{1}{2}[1+\cos(2\theta)] - 2\right\} d\theta$$
$$= \int_{-\pi/3}^{\pi/3} [2 + 4\cos(2\theta)]\, d\theta$$
$$= [2\theta + 2\sin(2\theta)]_{-\pi/3}^{\pi/3}$$
$$= \left(\frac{2\pi}{3} + 2\sin\frac{2\pi}{3}\right) - \left[-\frac{2\pi}{3} + 2\sin\left(-\frac{2\pi}{3}\right)\right]$$
$$= \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$

(c) The area of the region that lies inside the circle $r = 2$ and outside the circle $r = 4\cos\theta$ is equal to twice the area of the region swept out by the circle $r = 2$ on the interval $\frac{\pi}{3} \le \theta \le \pi$ minus twice the area of the region swept out by the circle $r = 4\cos\theta$ on the interval $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$.



The area of the region is

$$A = 2\cdot\frac{1}{2}\int_{\pi/3}^{\pi} 2^2\, d\theta - 2\cdot\frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\, d\theta$$
$$= \int_{\pi/3}^{\pi} 4\, d\theta - \int_{\pi/3}^{\pi/2} 16\cdot\frac{1}{2}[1+\cos(2\theta)]\, d\theta$$
$$= 4[\theta]_{\pi/3}^{\pi} - 8\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\pi/3}^{\pi/2}$$
$$= 4\left(\pi - \frac{\pi}{3}\right) - 8\left\{\left[\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right] - \left[\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right]\right\}$$
$$= 4\left(\frac{2\pi}{3}\right) - 8\left[\left(\frac{\pi}{2} + \frac{1}{2}\cdot 0\right) - \left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$$
$$= \frac{8\pi}{3} - 4\pi + \frac{8\pi}{3} + 2\sqrt{3}$$
$$= \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$

9.7 The Polar Equation of a Conic

Concepts and Vocabulary

1. A **(a) parabola** is the set of points P in the plane for which the distance from a fixed point called the focus P equals the distance from a fixed line called the directrix.
3. **In both cases, $e = 1$, so both graphs are parabolas. Answers will vary.**

Skill Building

5. Using Table 9,

$$r = \frac{1}{1+\cos\theta}$$

which means $\boxed{e=1}$ and $p = 1$ so we have a $\boxed{\text{parabola}}$ where the directrix is perpendicular to the polar axis 1 unit to the right of the pole.
7. Using Table 9,

$$r = \frac{4}{2-3\sin\theta} = \frac{2}{1-\frac{3}{2}\sin\theta}$$

which means $\boxed{e = \frac{3}{2}}$ and $ep = 2$ so $p = \frac{4}{3}$ which means we have a $\boxed{\text{hyperbola}}$ where the directrix is parallel to the polar axis $\frac{4}{3}$ units below the pole.
9. Using Table 9,

$$r = \frac{3}{4-2\cos\theta} = \frac{\frac{3}{4}}{1-\frac{1}{2}\cos\theta}$$

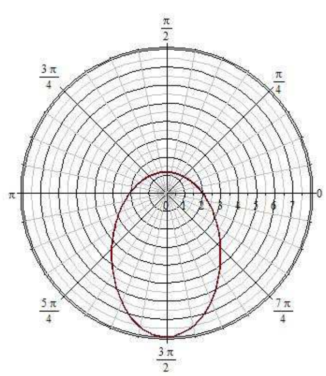
which means $\boxed{e = \frac{1}{2}}$ and $ep = \frac{3}{4}$ so $p = \frac{3}{2}$ which means we have an $\boxed{\text{ellipse}}$ where the directrix is perpendicular to the polar axis $\frac{3}{2}$ units to the left of the pole.
11. Using Table 9,

$$r = \frac{4}{3+3\sin\theta} = \frac{\frac{4}{3}}{1+\sin\theta}$$

which means $\boxed{e=1}$ and $ep = \frac{4}{3}$ so $p = \frac{4}{3}$ which means we have a $\boxed{\text{parabola}}$ where the directrix is parallel to the polar axis $\frac{4}{3}$ units above the pole.
13. **(a)** Using Table 9,

$$r = \frac{8}{4+3\sin\theta} = \frac{2}{1+\frac{3}{4}\sin\theta}$$

which means $e = \frac{3}{4}$ and $ep = 2$, so $p = \frac{8}{3}$. Since $e < 1$, this is an $\boxed{\text{ellipse}}$ with directrix parallel to the polar axis $\frac{8}{3}$ units above the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 2)$ and $(\pi, 2)$, respectively, which are vertices of the ellipse. The y-intercepts are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, which gives rise to the polar points $\left(\frac{\pi}{2}, \frac{8}{7}\right)$ and $\left(\frac{3\pi}{2}, 8\right)$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
r &= \frac{8}{4+3\sin\theta} \\
4r + 3r\sin\theta &= 8 \\
4r &= 8 - 3r\sin\theta \\
16r^2 &= (8-3r\sin\theta)^2 \\
16(x^2+y^2) &= (8-3y)^2 \\
16x^2+16y^2 &= 64 - 48y + 9y^2 \\
16x^2 + 7\left(y^2 + \frac{48}{7}y\right) &= 64 \\
16x^2 + 7\left(y + \frac{24}{7}\right)^2 &= 64 + 7\left(\frac{24}{7}\right)^2 \\
\boxed{16x^2 + 7\left(y + \frac{24}{7}\right)^2 = \frac{1024}{7}}&.
\end{aligned}$$

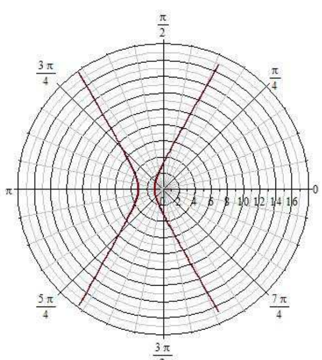
(c) Parametric equations are

$$\boxed{x = \frac{8\cos\theta}{4+3\sin\theta}, \quad y = \frac{8\sin\theta}{4+3\sin\theta}}.$$

15. (a) Using Table 9,

$$r = \frac{9}{3-6\cos\theta} = \frac{3}{1-2\cos\theta}$$

which means $e = 2$. Since $e > 1$, this is a $\boxed{\text{hyperbola}}$.



(b) To obtain a rectangular equation of the hyperbola, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
r &= \frac{9}{3-6\cos\theta} \\
3r - 6r\cos\theta &= 9 \\
3r &= 9 + 6r\cos\theta \\
9r^2 &= (9+6r\cos\theta)^2 \\
9(x^2+y^2) &= (9+6x)^2 \\
9x^2+9y^2 &= 81 + 108x + 36x^2 \\
27\left(x^2+4x\right) - 9y^2 &= -81 \\
27\left(x+2\right)^2 - 9y^2 &= -81 + 27(4) \\
\boxed{3\left(x+2\right)^2 - y^2 = 3}&.
\end{aligned}$$

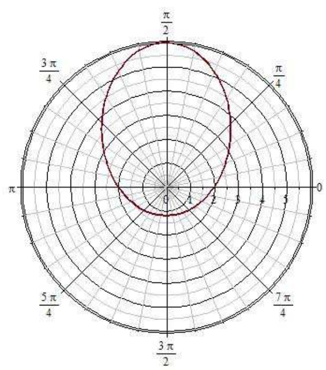
(c) Parametric equations are

$$\boxed{x = \frac{9\cos\theta}{3 - 6\cos\theta}, \quad y = \frac{9\sin\theta}{3 - 6\cos\theta}}.$$

17. (a) Using Table 9,

$$r = \frac{6}{3 - 2\sin\theta} = \frac{2}{1 - \frac{2}{3}\sin\theta}$$

which means $e = \dfrac{2}{3}$ and $ep = 2$, so $p = 3$. Since $e < 1$, this is an $\boxed{\text{ellipse}}$ with directrix parallel to the polar axis 3 units below the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 2)$ and $(\pi, 2)$, respectively, which are vertices of the ellipse. The y-intercepts are at $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$, which gives rise to the points $\left(\dfrac{\pi}{2}, 6\right)$ and $\left(\dfrac{3\pi}{2}, \dfrac{6}{5}\right)$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
3r - 2r\sin\theta &= 6 \\
9r^2 = (3r)^2 &= (6 + 2r\sin\theta)^2 \\
9(x^2 + y^2) &= (6 + 2y)^2 \\
9x^2 + 9y^2 &= 36 + 24y + 4y^2 \\
9x^2 + 5\left(y^2 - \frac{24}{5}y\right) &= 36 \\
9x^2 + 5\left(y - \frac{12}{5}\right)^2 &= 36 + 5\left(\frac{12}{5}\right)^2 \\
\boxed{9x^2 + 5\left(y - \frac{12}{5}\right)^2 = \frac{324}{5}}&.
\end{aligned}$$

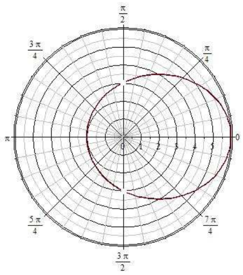
(c) Parametric equations are

$$\boxed{x = \frac{6\cos\theta}{3 - 2\sin\theta}, \quad y = \frac{6\sin\theta}{3 - 2\sin\theta}}.$$

19. (a) Using Table 9,

$$r = \frac{6\sec\theta}{2\sec\theta - 1} = \frac{6}{2 - \cos\theta} = \frac{3}{1 - \frac{1}{2}\cos\theta}, \quad \theta \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

which means $e = \dfrac{1}{2}$ and $ep = 3$, so $p = 6$. Since $e < 1$, this is an $\boxed{\text{ellipse}}$ with directrix perpendicular to the polar axis 6 units to the left of the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 6)$ and $(\pi, 2)$, respectively, which are vertices of the ellipse. The y-intercepts are undefined since the function is undefined at $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$.



(b) To obtain a rectangular equation of the hyperbola, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned} 2r - r\cos\theta &= 6 \\ 4r^2 &= (6 + r\cos\theta)^2 \\ 4(x^2+y^2) &= (6+x)^2 \\ 4x^2 + 4y^2 &= 36 + 12x + x^2 \\ 3(x^2 - 4x) + 4y^2 &= 36 \\ 3(x-2)^2 + 4y^2 &= 36 + 3(2)^2 \\ \boxed{3(x-2)^2 + 4y^2 = 48}&. \end{aligned}$$

(c) Parametric equations are

$$\boxed{x = \frac{6\cos\theta}{2-\cos\theta}, \quad y = \frac{6\sin\theta}{2-\cos\theta}}.$$

Applications and Extensions

For Problems 21-26, we will use the parametric form of each graph $x = r\cos\theta$ and $y = r\sin\theta$ along with the formula $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ to determine the slope at the given value of θ.

21. Parametric equations for $r = \dfrac{9}{4-\cos\theta}$ are

$$\begin{aligned} y &= \frac{9\sin\theta}{4-\cos\theta} \\ x &= \frac{9\cos\theta}{4-\cos\theta}, \end{aligned}$$

so

$$\begin{aligned} \left[\frac{dy}{d\theta}\right]_{\theta=0} &= \left[\frac{(4-\cos\theta)(9\cos\theta) - (9\sin\theta)(\sin\theta)}{(4-\cos\theta)^2}\right]_{\theta=0} = 3 \\ \left[\frac{dx}{d\theta}\right]_{\theta=0} &= \left[\frac{(4-\cos\theta)(-9\sin\theta) - (9\cos\theta)(\sin\theta)}{(4-\cos\theta)^2}\right]_{\theta=0} = 0. \end{aligned}$$

Since $\frac{dx}{d\theta} = 0$, this means $\frac{dy}{dx}$ is undefined which means the tangent line at $\theta = 0$ is vertical so the slope is $\boxed{\text{undefined}}$.

23. Parametric equations for $r = \dfrac{8}{4+\sin\theta}$ are

$$\begin{aligned} y &= \frac{8\sin\theta}{4+\sin\theta} \\ x &= \frac{8\cos\theta}{4+\sin\theta}, \end{aligned}$$

so

$$\left[\frac{dy}{d\theta}\right]_{\theta=\pi/2} = \left[\frac{(4+\sin\theta)(8\cos\theta)-(8\sin\theta)(\cos\theta)}{(4+\sin\theta)^2}\right]_{\theta=\pi/2} = 0$$

$$\left[\frac{dx}{d\theta}\right]_{\theta=\pi/2} = \left[\frac{(4+\sin\theta)(-8\sin\theta)-(8\cos\theta)(\cos\theta)}{(4+\sin\theta)^2}\right]_{\theta=\pi/2} = -\frac{40}{25}.$$

Then

$$\left[\frac{dy}{dx}\right]_{\theta=\pi/2} = \frac{0}{-40/25} = 0$$

which means the tangent line has slope $\boxed{0}$ at $\theta = \frac{\pi}{2}$.

25. Parametric equations for $r = \dfrac{4}{2+\cos\theta}$ are

$$y = \frac{4\sin\theta}{2+\cos\theta}$$

$$x = \frac{4\cos\theta}{2+\cos\theta},$$

so

$$\left[\frac{dy}{d\theta}\right]_{\theta=\pi} = \left[\frac{(2+\cos\theta)(4\cos\theta)-(4\sin\theta)(-\sin\theta)}{(2+\cos\theta)^2}\right]_{\theta=\pi} = -4$$

$$\left[\frac{dx}{d\theta}\right]_{\theta=\pi} = \left[\frac{(2+\cos\theta)(-4\sin\theta)-(4\cos\theta)(-\sin\theta)}{(2+\cos\theta)^2}\right]_{\theta=\pi} = 0.$$

Since $\dfrac{dx}{d\theta} = 0$, this means $\dfrac{dy}{dx}$ is undefined which means the tangent line at $\theta = \pi$ is vertical so the slope is $\boxed{\text{undefined}}$.

27. It is given that $e = \dfrac{4}{5}$ and $p = 3$, so using Table 9, we have

$$\boxed{r = \frac{\frac{4}{5}\cdot 3}{1-\frac{4}{5}\cos\theta} = \frac{12}{5-4\cos\theta}}.$$

29. It is given that $e = 1$ and $p = 1$, so using Table 9, we have

$$\boxed{r = \frac{1\cdot 1}{1+\sin\theta} = \frac{1}{1+\sin\theta}}.$$

31. It is given that $e = 6$ and $p = 2$, so using Table 9, we have

$$\boxed{r = \frac{6\cdot 2}{1-6\sin\theta} = \frac{12}{1-6\sin\theta}}.$$

33. (a) $\boxed{e = 0.967}$.

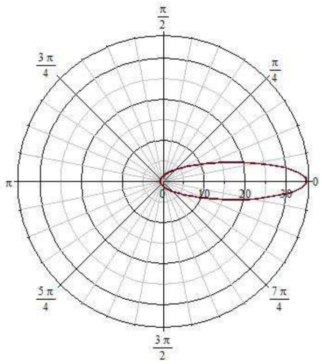
(b) The shortest distance occurs when r is a minimum which occurs when $\cos\theta = -1$. The shortest distance is then

$$r = \frac{1.155}{1-0.967(-1)}\boxed{\approx 0.587 \text{ AU}}.$$

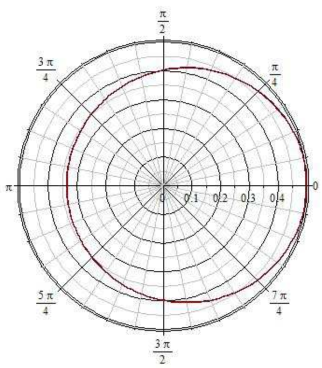
(c) The greatest distance occurs when r is a maximum which occurs when $\cos\theta = 1$. The shortest distance is then

$$r = \frac{1.155}{1-0.967(1)} = \boxed{35 \text{ AU}}.$$

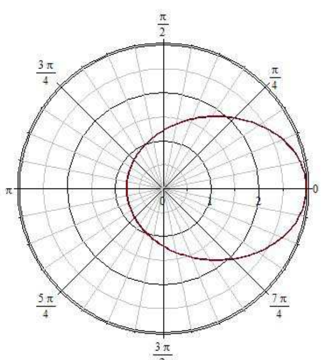
(d)


π/2
3π/4
π/4
π
0
10
20
30
5π/4
7π/4
3π/2

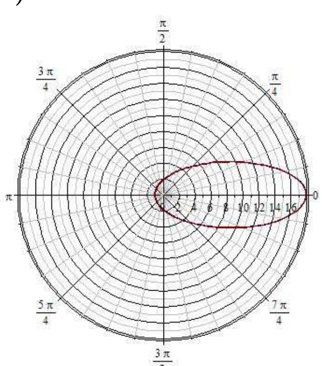
35. (a)
(i)


π/2
3π/4
π/4
π
0
0.1
0.2
0.3
0.4
5π/4
7π/4
3π/2

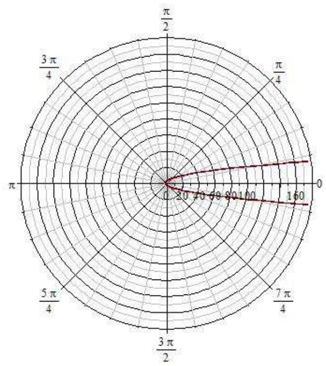
(ii)


π/2
3π/4
π/4
π
0
1
2
5π/4
7π/4
3π/2

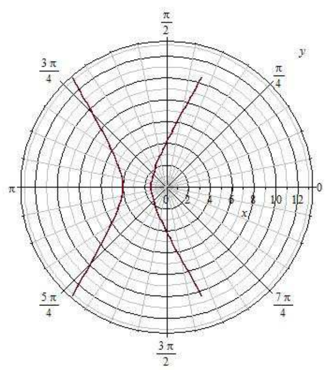
(iii)


π/2
3π/4
π/4
π
0
2
4
6
8
10
12
14
16
5π/4
7π/4
3π/2

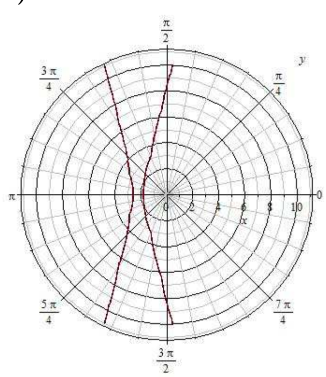
(iv)



(v)

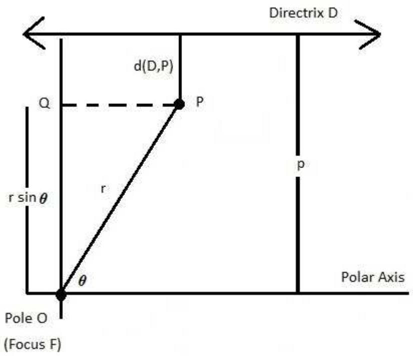


(vi)



(b) Answers will vary.
(c) Answers will vary.

37.



If we drop a perpendicular from point P to the ray $\theta = \dfrac{\pi}{2}$, the intersection point of the perpendicular and the ray is labeled Q. The distance from the pole to Q is $r\sin\theta$. Since the distance from the pole to the directrix D is p, the distance $d(D,P) = p - r\sin\theta$. Then since $d(F,P) = r$, the equation of the conic is

$$\begin{aligned}
\frac{d(F,P)}{d(D,P)} &= e \\
\frac{r}{p - r\sin\theta} &= e \\
r &= ep - re\sin\theta \\
r(1 + e\sin\theta) &= ep \\
r &= \frac{ep}{1 + e\sin\theta}.
\end{aligned}$$

Challenge Problems

39. For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the value $c = \sqrt{a^2 - b^2}$ is the focal distance from the center and $e = \dfrac{c}{a}$ is the eccentricity. If we use the parametric equations $x = a\cos\theta$ and $y = b\sin\theta$ for $0 \le \theta \le \dfrac{\pi}{2}$, then this parametrizes the ellipse in the first quadrant. If we rotate this part of the ellipse about the x-axis to find the surface area S, then

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/2} y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 2\pi \int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta \\
&= 2\pi b \int_0^{\pi/2} \sin\theta\sqrt{a^2(1-\cos^2\theta) + b^2\cos^2\theta}\, d\theta \\
&= 2\pi b \int_0^{\pi/2} a\sin\theta\sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\cos^2\theta\right)}\, d\theta.
\end{aligned}$$

Now we use the fact that $e = \dfrac{\sqrt{a^2 - b^2}}{a}$. Then

$$\begin{aligned}
S &= 2\pi ab \int_0^{\pi/2} \sin\theta\sqrt{1 - e^2\cos^2\theta}\, d\theta \\
&= 2\pi ab \int_e^0 -\frac{1}{e}\sqrt{1 - u^2}\, du,
\end{aligned}$$

where we used the substitution $u = e\cos\theta$ with $du = -e\sin\theta\, d\theta$ and new limits of integration $u = e\cos(0) = e$ to $u = e\cos(\pi/2) = 0$. To calculate this integral, we use Table of Integrals 60 with $a = 1$.

$$\begin{aligned} S &= \frac{2\pi ab}{e}\int_0^e \sqrt{1-u^2}\, du \\ &= \frac{2\pi ab}{e}\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u\right]_0^e \\ &= \frac{2\pi ab}{e}\left(\frac{e}{2}\sqrt{1-e^2} + \frac{1}{2}\sin^{-1} e\right) \\ &= \pi b^2 + \frac{\pi ab}{e}\sin^{-1} e, \end{aligned}$$

where in the last step we used that fact that

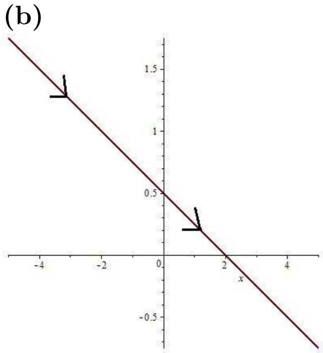
$$\sqrt{1-e^2} = \sqrt{1 - \frac{a^2-b^2}{a^2}} = \sqrt{\frac{a^2-a^2+b^2}{a^2}} = \frac{b}{a}.$$

Chapter 9 Review Exercises

1. (a) Solving for t in the y equation, we have $t = 1 - y$. Then

$$x = 4(1-y) - 2$$
$$\boxed{y = -\frac{1}{4}x + \frac{1}{2}}.$$

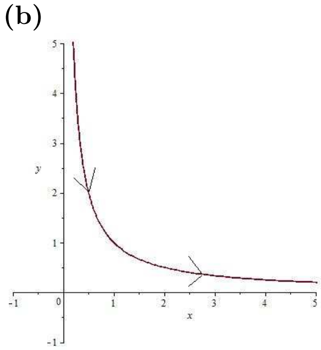
(b)



(c) There are $\boxed{\text{no restrictions}}$.

3. (a) Since $x = e^t$ and $y = (e^t)^{-1}$, we have $\boxed{y = \frac{1}{x}}$.

(b)

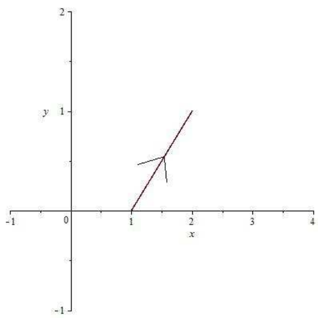


(c) Since $e^t > 0$ and $e^{-t} > 0$, we have $\boxed{x > 0}$ and $\boxed{y > 0}$.

5. (a) Since $\tan^2 t + 1 = \sec^2 t$, the rectangular equation is

$$\boxed{y + 1 = x}.$$

(b)



(c) On $0 \le t \le \frac{\pi}{4}$, $1 \le \sec t \le \sqrt{2}$ and $0 \le \tan t \le 1$ so

$$\begin{array}{rcl} 1 \le & \sec^2 t & \le 2 \\ 0 \le & \tan^2 t & \le 1. \end{array}$$

The restrictions on x and y are then

$$\boxed{1 \le x \le 2}$$
$$\boxed{0 \le y \le 1}.$$

7. (a)

$$\begin{array}{rcl} \frac{dy}{dt} & = & 1 \\ \frac{dx}{dt} & = & 2t \end{array}$$

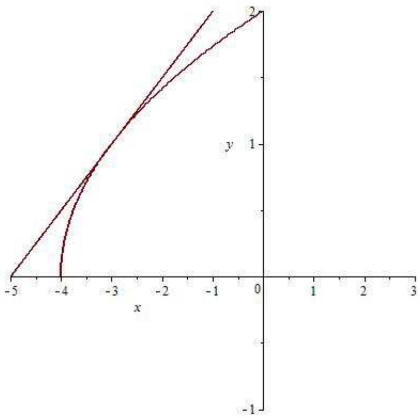
The slope of the tangent line is

$$\left[\frac{dy}{dx}\right]_{t=1} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=1} = \left[\frac{1}{2t}\right]_{t=1} = \frac{1}{2}.$$

The point of tangency is $(x(1), y(1)) = (-3, 1)$. The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x + 3)$$
$$\boxed{y = \frac{1}{2}x + \frac{5}{2}}.$$

(b)



9. (a)

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{2}(t^2+1)^{-1/2}(2t) = t(t^2+1)^{-1/2}\\ \frac{dx}{dt} &= -2t^{-3}\end{aligned}$$

The slope of the tangent line is

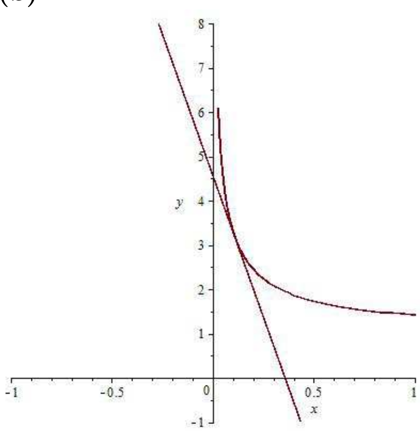
$$\left[\frac{dy}{dx}\right]_{t=3} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=3} = \left[\frac{t(t^2+1)^{-1/2}}{-2t^{-3}}\right]_{t=3} = \frac{3/\sqrt{10}}{-2/27} = -\frac{81\sqrt{10}}{20}.$$

The point of tangency is $(x(3), y(3)) = \left(\frac{1}{9}, \sqrt{10}\right)$. The equation of the tangent line is

$$y - \sqrt{10} = -\frac{81\sqrt{10}}{20}\left(x - \frac{1}{9}\right)$$

$$\boxed{y = -\frac{81}{2\sqrt{10}}x + \frac{9}{2\sqrt{10}} + \sqrt{10}}.$$

(b)



11. Answers will vary. Here are two parameterizations.

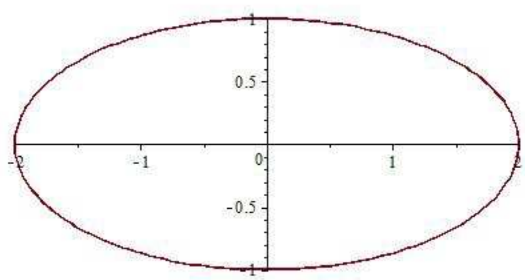
$$\boxed{x(t) = t,\ y(t) = -2t + 4, \quad -\infty < t < \infty}$$

$$\boxed{x(t) = t - 2,\ y(t) = -2t + 8, \quad -\infty < t < \infty}.$$

13. We eliminate the parameter t by using the Pythagorean Identity $\cos^2 t + \sin^2 t = 1$. Since $\cos t = \dfrac{x}{2}$ and $\sin t = y$, we have

$$\boxed{\frac{x^2}{4} + y^2 = 1}.$$

The plane curve is below.



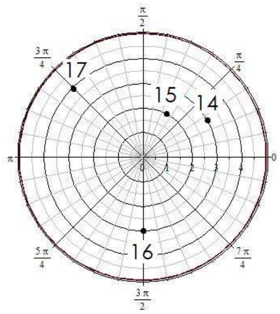
When $t = 0$, the object is at the point $(2, 0)$. As t increases, the object moves around the ellipse in a counterclockwise direction, taking $t = 2\pi$ seconds to complete one revolution.
For the graphs of Problems 14-17, see the graph below Problem 17.
15. Converting to rectangular coordinates

$$\begin{aligned} x &= r\cos\theta = -2\cos\left(\frac{4\pi}{3}\right) = 1 \\ y &= r\sin\theta = -2\sin\left(\frac{4\pi}{3}\right) = \sqrt{3}. \end{aligned}$$

The rectangular coordinates are $\boxed{\left(1, \sqrt{3}\right)}$.
17. Converting to rectangular coordinates

$$\begin{aligned} x &= r\cos\theta = -4\cos\left(-\frac{\pi}{4}\right) = -2\sqrt{2} \\ y &= r\sin\theta = -4\sin\left(-\frac{\pi}{4}\right) = 2\sqrt{2}. \end{aligned}$$

The rectangular coordinates are $\boxed{\left(-2\sqrt{2}, 2\sqrt{2}\right)}$.



19. Converting to polar coordinates

$$r = \sqrt{3^2+4^2} = 5$$
$$\theta = \tan^{-1}\left(\frac{4}{3}\right).$$

One pair of polar coordinates is $\boxed{\left(5, \tan^{-1}\left(\frac{4}{3}\right)\right) \approx (5, 0.927)}$. Another pair is $\boxed{\left(-5, \tan^{-1}\left(\frac{4}{3}\right)+\pi\right) \approx (-5, 4.068)}$.

21. Noting that the given point is in quadrant II, when we convert to polar coordinates we have

$$r = \sqrt{(-3)^2+3^2} = 3\sqrt{2}$$
$$\theta = \tan^{-1}\left(-\frac{3}{3}\right)+\pi = \frac{3\pi}{4}.$$

One pair of polar coordinates is $\boxed{\left(3\sqrt{2}, \frac{3\pi}{4}\right)}$. Another pair is $\boxed{\left(-3\sqrt{2}, \frac{7\pi}{4}\right)}$.

23. Using the formulas $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the equation becomes

$$\sqrt{x^2+y^2} = e^{\tan^{-1}(y/x)/2}$$
$$x^2+y^2 = e^{\tan^{-1}(y/x)}$$
$$\frac{y}{x} = \tan\left(\ln\left(x^2+y^2\right)\right)$$
$$\boxed{y = x\tan\left(\ln\left(x^2+y^2\right)\right)}.$$

25. Multiply the equation through by r.

$$r^2 = ar - r\sin\theta$$
$$\boxed{x^2+y^2 = a\sqrt{x^2+y^2} - y}.$$

27.

$$\sqrt{x^2+y^2} = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\boxed{y = x\tan\left(\sqrt{x^2+y^2}\right)}.$$

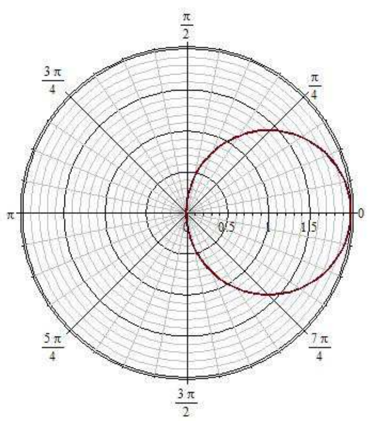
29. Using the formulas $r^2 = x^2 + y^2$, $x = r\cos\theta$, and $y = r\sin\theta$, the equation becomes

$$\begin{aligned}(r^2)^2 &= (r\cos\theta)^2 - (r\sin\theta)^2\\ r^4 &= r^2(\cos^2\theta - \sin^2\theta) \quad (\text{for } r \neq 0)\\ &\boxed{r^2 = \cos^2\theta - \sin^2\theta \quad (\text{for } r \neq 0)}.\end{aligned}$$

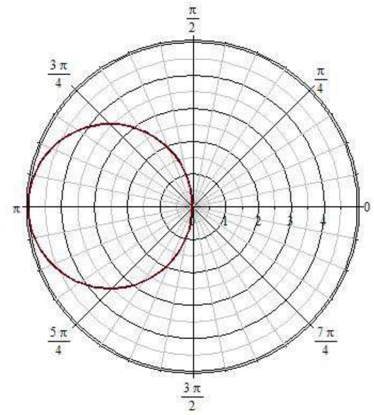
31. Using the formulas $x = r\cos\theta$ and $y = r\sin\theta$, the equation becomes

$$\begin{aligned}\frac{(r\cos\theta)^2}{4} + \frac{(r\sin\theta)^2}{9} &= 1\\ r^2\left(\frac{\cos^2\theta}{4} + \frac{\sin^2\theta}{9}\right) &= 1\\ r^2\left(\frac{9\cos^2\theta + 4\sin^2\theta}{36}\right) &= 1\\ r &= \frac{6}{\sqrt{9\cos^2\theta + 4\sin^2\theta}}\\ &\boxed{r = \frac{6\sqrt{9\cos^2\theta + 4\sin^2\theta}}{9\cos^2\theta + 4\sin^2\theta}}.\end{aligned}$$

33. Multiplying by $r\cos\theta$, the equation becomes $r^2 = 2r\cos\theta$ which in rectangular coordinates is $x^2 + y^2 = 2x$. Upon completing the square, the equation becomes $(x-1)^2 + y^2 = 1$. We recognize this as a circle centered at $(1, 0)$ with radius 1.



35. Multiplying by r, the equation becomes $r^2 = -5r\cos\theta$ which in rectangular coordinates is $x^2 + y^2 = -5x$. Upon completing the square, the equation becomes $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$. We recognize this as a circle centered at $\left(-\frac{5}{2}, 0\right)$ with radius $\frac{5}{2}$.

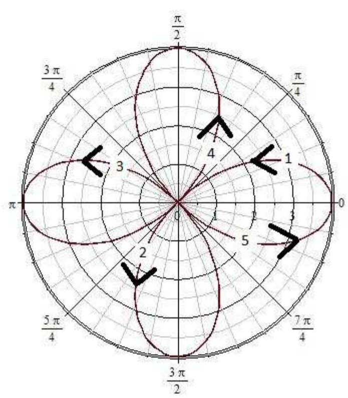


37. (a) The polar equation $r = 4\cos(2\theta)$ contains $\cos(2\theta)$, which has period π. We construct a table of common values of θ that range from 0 to 2π noting that the values from π to 2π duplicate those from 0 to π.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(4,0)$	$\left(2,\frac{\pi}{6}\right)$	$\left(0,\frac{\pi}{4}\right)$	$\left(-2,\frac{\pi}{3}\right)$	$\left(-4,\frac{\pi}{2}\right)$	$\left(-2,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(0,\frac{3\pi}{4}\right)$	$\left(2,\frac{5\pi}{6}\right)$	$(4,\pi)$	$\left(2,\frac{7\pi}{6}\right)$	$\left(0,\frac{5\pi}{4}\right)$	$\left(-2,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(-4,\frac{3\pi}{2}\right)$	$\left(-2,\frac{5\pi}{3}\right)$	$\left(0,\frac{7\pi}{4}\right)$	$\left(2,\frac{11\pi}{6}\right)$	$(4,2\pi)$



(b) Parametric equations for $r = 4\cos(2\theta)$:

$$\boxed{x = r\cos\theta = 4\cos(2\theta)\cos\theta \qquad y = r\sin\theta = 4\cos(2\theta)\sin\theta}$$

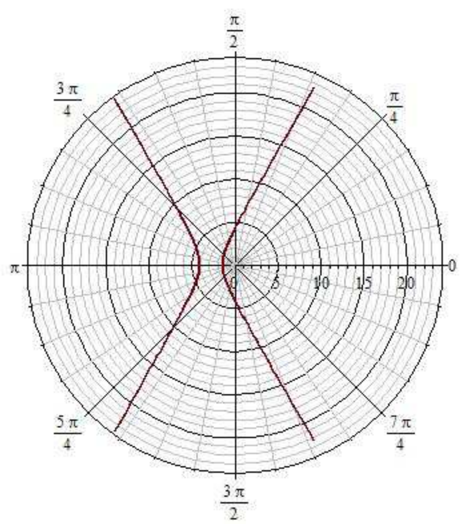
where θ is the parameter, and if $0 \le \theta \le 2\pi$, then the graph is traced out exactly once starting at $(4,0)$ and following the arrows in increasing order.

39. (a) The polar equation $r = \dfrac{4}{1-2\cos\theta}$ contains $\cos\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π (excluding the values $\dfrac{\pi}{3}$ and $\dfrac{5}{3}\pi$ since r is not defined there) and plot the points (r,θ).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(-4,0)$	$\left(\frac{4}{1-\sqrt{3}},\frac{\pi}{6}\right)$	$\left(\frac{4}{1-\sqrt{2}},\frac{\pi}{4}\right)$	undefined	$\left(4,\frac{\pi}{2}\right)$	$\left(2,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(\frac{4}{1+\sqrt{2}},\frac{3\pi}{4}\right)$	$\left(\frac{4}{1+\sqrt{3}},\frac{5\pi}{6}\right)$	$\left(\frac{4}{3},\pi\right)$	$\left(\frac{4}{1+\sqrt{3}},\frac{7\pi}{6}\right)$	$\left(\frac{4}{1+\sqrt{2}},\frac{5\pi}{4}\right)$	$\left(2,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(4,\frac{3\pi}{2}\right)$	undefined	$\left(\frac{4}{1-\sqrt{2}},\frac{7\pi}{4}\right)$	$\left(\frac{4}{1-\sqrt{3}},\frac{11\pi}{6}\right)$	$(-4,2\pi)$



(b) Parametric equations for the polar equation $r = \dfrac{4}{1-2\cos\theta}$ are

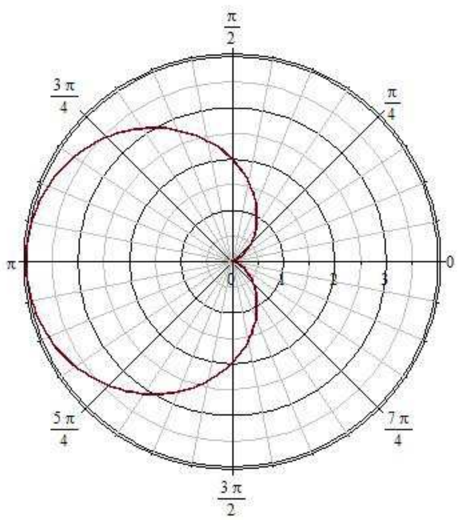
$$\boxed{x = \frac{4\cos\theta}{1-2\cos\theta}}$$
$$\boxed{y = \frac{4\sin\theta}{1-2\cos\theta}}.$$

41. (a) The polar equation $r = 2-2\cos\theta$ contains $\cos\theta$, which has the period 2π. We construct a table of common values of θ that range from 0 to 2π, plot the points $(r,\theta) = (2-2\cos\theta,\theta)$, and trace out the graph, beginning at the point $(0,0)$ and ending at $(0,2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(0,0)$	$\left(2-\sqrt{3},\frac{\pi}{6}\right)$	$\left(2-\sqrt{2},\frac{\pi}{4}\right)$	$\left(1,\frac{\pi}{3}\right)$	$\left(2,\frac{\pi}{2}\right)$	$\left(3,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(2+\sqrt{2},\frac{3\pi}{4}\right)$	$\left(2+\sqrt{3},\frac{5\pi}{6}\right)$	$(4,\pi)$	$\left(2+\sqrt{3},\frac{7\pi}{6}\right)$	$\left(2+\sqrt{2},\frac{5\pi}{4}\right)$	$\left(3,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(2,\frac{3\pi}{2}\right)$	$\left(1,\frac{5\pi}{3}\right)$	$\left(2-\sqrt{2},\frac{7\pi}{4}\right)$	$\left(2-\sqrt{3},\frac{11\pi}{6}\right)$	$(0,2\pi)$



(b) Parametric equations for $r = 2 - 2\cos\theta$:

$$\boxed{x = r\cos\theta = (2-2\cos\theta)\cos\theta \quad y = r\sin\theta = (2-2\cos\theta)\sin\theta}$$

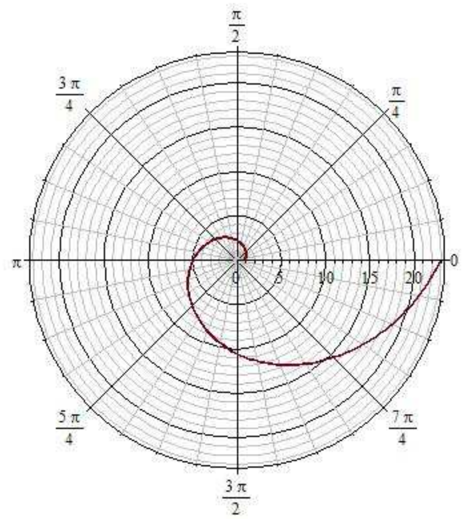
where θ is the parameter, and if $0 \le \theta \le 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

43. (a) The polar equation $r = e^{0.5\theta}$ increases in radius as the angle increases. We construct a table of common values of θ that range from 0 to 2π, plot the points (r,θ), and trace out the graph. All decimals in the table are approximate values of r.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r,θ)	$(1,0)$	$\left(1.299,\frac{\pi}{6}\right)$	$\left(1.481,\frac{\pi}{4}\right)$	$\left(1.688,\frac{\pi}{3}\right)$	$\left(2.193,\frac{\pi}{2}\right)$	$\left(2.850,\frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r,θ)	$\left(3.248,\frac{3\pi}{4}\right)$	$\left(3.702,\frac{5\pi}{6}\right)$	$(4.810,\pi)$	$\left(6.250,\frac{7\pi}{6}\right)$	$\left(7.124,\frac{5\pi}{4}\right)$	$\left(8.121,\frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r,θ)	$\left(10.551,\frac{3\pi}{2}\right)$	$\left(13.708,\frac{5\pi}{3}\right)$	$\left(15.625,\frac{7\pi}{4}\right)$	$\left(17.811,\frac{11\pi}{6}\right)$	$(23.141,2\pi)$



(b) Parametric equations for $r = e^{0.5\theta}$:

$$\boxed{x = r\cos\theta = e^{0.5\theta}\cos\theta \qquad y = r\sin\theta = e^{0.5\theta}\sin\theta}$$

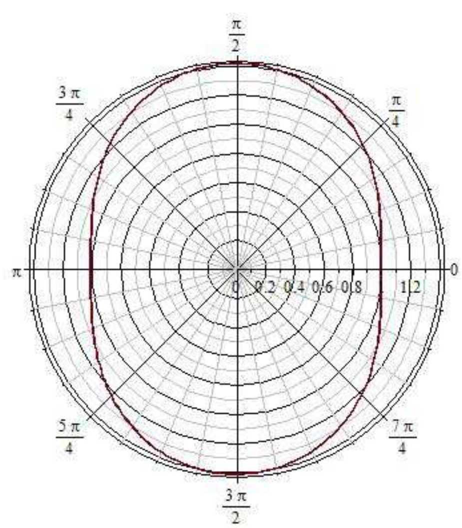
where θ is the parameter.

45. (a) The equation $r^2 = 1 + \sin^2\theta$, or $r = \sqrt{1+\sin^2\theta}$, contains $\sin\theta$ which has the period 2π. We construct a table of common values of θ that range from 0 to 2π, plot the points (r, θ).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(1, 0)$	$\left(\frac{\sqrt{5}}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{6}}{2}, \frac{\pi}{4}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{\pi}{3}\right)$	$\left(\sqrt{2}, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{\sqrt{6}}{2}, \frac{3\pi}{4}\right)$	$\left(\frac{\sqrt{5}}{2}, \frac{5\pi}{6}\right)$	$(1, \pi)$	$\left(\frac{\sqrt{5}}{2}, \frac{7\pi}{6}\right)$	$\left(\frac{\sqrt{6}}{2}, \frac{5\pi}{4}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(\sqrt{2}, \frac{3\pi}{2}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{5\pi}{3}\right)$	$\left(\frac{\sqrt{6}}{2}, \frac{7\pi}{4}\right)$	$\left(\frac{\sqrt{5}}{2}, \frac{11\pi}{6}\right)$	$(1, 2\pi)$

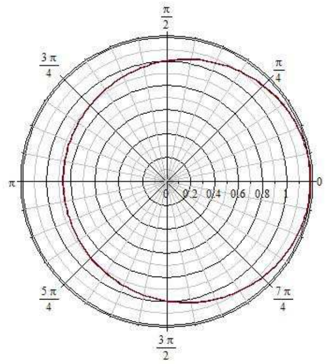


(b) Parametric equations for $r = \sqrt{1+\sin^2\theta}$:

$$\boxed{x = r\cos\theta = \sqrt{1+\sin^2\theta}\cos\theta \qquad y = r\sin\theta = \sqrt{1+\sin^2\theta}\sin\theta}$$

where θ is the parameter.

47. (a) Since $e = \dfrac{1}{6} < 1$, this is an $\boxed{\text{ellipse}}$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
r &= \frac{1}{1-\frac{1}{6}\cos\theta} \\
r - \frac{1}{6}r\cos\theta &= 1 \\
r^2 &= \left(1+\frac{1}{6}r\cos\theta\right)^2 \\
x^2+y^2 &= \left(1+\frac{1}{6}x\right)^2 \\
x^2+y^2 &= 1+\frac{1}{3}x+\frac{1}{36}x^2 \\
\frac{35}{36}\left(x^2-\frac{12}{35}x\right)+y^2 &= 1 \\
\frac{35}{36}\left(x-\frac{6}{35}\right)^2+y^2 &= 1+\frac{35}{36}\left(\frac{6}{35}\right)^2 \\
\boxed{35\left(x-\frac{6}{35}\right)^2+36y^2 = \frac{1296}{35}}&.
\end{aligned}$$

(c) Parametric equations are

$$\boxed{x = \frac{6\cos\theta}{6-\cos\theta}, \quad y = \frac{6\sin\theta}{6-\cos\theta}}.$$

49. For clockwise orientation, the ellipse can be parametrized by $x(t) = 4\sin(\omega t), y(t) = 3\cos(\omega t)$. If it takes 5 seconds for one revolution, then the period is $\dfrac{2\pi}{\omega} = 5$, or $\omega = \dfrac{2\pi}{5}$. The parametrization is then

$$\boxed{x(t) = 4\sin\left(\frac{2\pi}{5}t\right), y(t) = 3\cos\left(\frac{2\pi}{5}t\right), \ 0 \le t \le 5}$$

which has clockwise orientation, and does indeed start at $(0,3)$ when $t=0$.

51. First we find the derivatives.

$$\begin{aligned}\frac{dx}{dt} &= -\cos t\\ \frac{dy}{dt} &= -3\sin t\end{aligned}$$

To find the horizontal tangent lines, we set $\frac{dy}{dt} = 0$.

$$\begin{aligned}-3\sin t &= 0\\ t &= 0, \pi, 2\pi.\end{aligned}$$

None of these values make $\frac{dx}{dt} = 0$, so we have horizontal tangent lines at each value. The horizontal tangent lines occur at the points

$$\begin{aligned}(x(0), y(0)) = (x(2\pi), y(2\pi)) &= \boxed{(1,5)}\\ (x(\pi), y(\pi)) &= \boxed{(1,-1)}.\end{aligned}$$

To find the vertical tangent lines, we set $\frac{dx}{dt} = 0$.

$$\begin{aligned}-\cos t &= 0\\ t &= \frac{\pi}{2}, \frac{3\pi}{2}.\end{aligned}$$

None of these values make $\frac{dy}{dt} = 0$, so we have vertical tangent lines at each value. The vertical tangent lines occur at the points

$$\begin{aligned}\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) &= \boxed{(0,2)}\\ \left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right) &= \boxed{(2,2)}.\end{aligned}$$

53. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = \sec^2 t \quad \text{and} \quad \frac{dy}{dt} = \frac{2}{3}\sec t \sec t \tan t = \frac{2}{3}\sec^2 t \tan t$$

The curve is smooth for $0 \le t \le \frac{\pi}{4}$. Using the arc length formula, we have

$$\begin{aligned}s &= \int_0^{\pi/4} \sqrt{\left(\sec^2 t\right)^2 + \left(\frac{2}{3}\sec^2 t \tan t\right)^2}\, dt\\ &= \int_0^{\pi/4} \sqrt{\sec^4 t \left(1 + \frac{4}{9}\tan^2 t\right)}\, dt\\ &= \int_0^{\pi/4} \sec^2 t \sqrt{1 + \frac{4}{9}\tan^2 t}\, dt.\end{aligned}$$

Using the substitution $u = \tan t$, then $du = \sec^2 t$ and the new limits are $u = \tan(0) = 0$ to $u = \tan\left(\frac{\pi}{4}\right) = 1$, then

$$\begin{aligned} s &= \int_0^1 \sqrt{1 + \frac{4}{9}u^2}\, du = \int_0^1 \frac{2}{3}\sqrt{\frac{9}{4} + u^2}\, du \\ &= \frac{2}{3}\left[\frac{u}{2}\sqrt{\frac{9}{4} + u^2} + \frac{9/4}{2}\ln\left|u + \sqrt{\frac{9}{4} + u^2}\right|\right]_0^1 \\ &= \boxed{\frac{1}{3}\frac{\sqrt{13}}{2} + \frac{9}{8}\ln\left(1 + \frac{\sqrt{13}}{2}\right) - \frac{9}{8}\ln\left(\frac{\sqrt{13}}{2}\right)}, \end{aligned}$$

where to compute the integral we used Table of Integrals 47 with $a = \frac{3}{2}$.

55. The given curve can be parametrized by

$$\begin{aligned} y(t) &= t \\ x(t) &= \frac{1}{2}t^2 - \frac{1}{4}\ln t \end{aligned}$$

on $1 \le t \le 2$. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = t - \frac{1}{4t} \quad \text{and} \quad \frac{dy}{dt} = 1$$

The curve is smooth for $1 \le t \le 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_1^2 \sqrt{[1]^2 + \left[t - \frac{1}{4t}\right]^2}\, dt = \int_1^2 \sqrt{\frac{1}{16}t^{-2} + \frac{1}{2} + t^2}\, dt \\ &= \int_1^2 \sqrt{\left[\frac{1}{4}t^{-1} + t\right]^2}\, dt = \int_1^2 \left(\frac{1}{4}t^{-1} + t\right)\, dt \\ &= \left[\frac{1}{4}\ln|t| + \frac{1}{2}t^2\right]_1^2 = 2 + \frac{1}{4}\ln(2) - \frac{1}{2} \\ &= \boxed{\frac{3}{2} + \frac{1}{4}\ln(2)}. \end{aligned}$$

57. Using the arc length formula $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ from Section 9.5 with $r = e^{-\theta}$ and $\frac{dr}{d\theta} = -e^{-\theta}$, the arc length is

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{\left(e^{-\theta}\right)^2 + \left(-e^{-\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2}e^{-\theta}\, d\theta \\ &= \left[-\sqrt{2}e^{-\theta}\right]_0^{2\pi} = \boxed{\sqrt{2}\left(1 - e^{-2\pi}\right)}. \end{aligned}$$

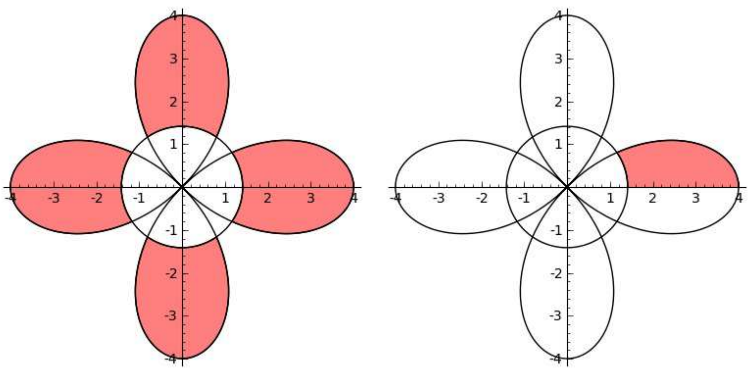
59. Using the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ from Section 9.5 with $r = 2\sin^2 \frac{\theta}{2}$ and
$\frac{dr}{d\theta} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, we will exploit symmetry and find the arc length s of the curve from $\theta = 0$ to $\theta = \frac{\pi}{2}$ and then double it. Then

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{\left(2\sin^2\frac{\theta}{2}\right)^2 + \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta = \int_0^{\pi/2} \sqrt{4\sin^4\frac{\theta}{2} + \left(4\sin^4\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)}\, d\theta \\ &= \int_0^{\pi/2} 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{\pi/2} \\ &= 4 - 2\sqrt{2}. \end{aligned}$$

The full arc length is then

$$2\left(4 - 2\sqrt{2}\right) = \boxed{8 - 4\sqrt{2}}.$$

61.



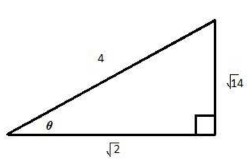
Exploiting symmetry, we will find the area A of the half-petal in quadrant I (see second figure), and then multiply it by 8 to find the full shaded region (see first figure). The area of the half-petal is swept out starting at $\theta = 0$ and ending at the first intersection point of the curves.

$$\begin{aligned} 4\cos(2\theta) &= \sqrt{2} \\ 2\theta &= \cos^{-1}\left(\frac{\sqrt{2}}{4}\right) \\ \theta &= \frac{1}{2}\cos^{-1}\left(\frac{\sqrt{2}}{4}\right). \end{aligned}$$

Then

$$\begin{aligned} A &= \int_0^{1/2\cos^{-1}(\sqrt{2}/4)} \frac{1}{2}\left((4\cos(2\theta))^2 - (\sqrt{2})^2\right)\, d\theta = \int_0^{1/2\cos^{-1}(\sqrt{2}/4)} \left(8\cos^2(2\theta) - 1\right)\, d\theta \\ &= \left[3\theta + 2\cos(2\theta)\sin(2\theta)\right]_0^{1/2\cos^{-1}(\sqrt{2}/4)}. \end{aligned}$$

In the following picture, we have $\cos\theta = \frac{\sqrt{2}}{4}$ or, equivalently, $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{4}\right)$.



Then

$$\begin{aligned}\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)\right) &= \frac{\sqrt{2}}{4}\\ \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)\right) &= \frac{\sqrt{14}}{4}.\end{aligned}$$

With these values, we have

$$\begin{aligned}A &= \left[3\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)\right) + 2\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)\right)\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)\right)\right]\\ &= \frac{3}{2}\cos^{-1}\left(\frac{\sqrt{2}}{4}\right) + \frac{\sqrt{7}}{4}.\end{aligned}$$

Then the full shaded region has area

$$8\left(\frac{\sqrt{7}}{4} + \frac{3}{2}\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)\right) = \boxed{2\sqrt{7} + 12\cos^{-1}\left(\frac{\sqrt{2}}{4}\right)}.$$

63. We start by finding $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\sqrt{t^2+1}}\\ \frac{dy}{dt} &= \frac{t}{\sqrt{t^2+1}}\end{aligned}$$

Using formula (2) in Section 9.3, the surface area S is

$$\begin{aligned}S &= 2\pi\int_0^1 \sqrt{t^2+1}\sqrt{\left(\frac{1}{\sqrt{1+t^2}}\right)^2 + \left(\frac{t}{\sqrt{1+t^2}}\right)^2}\,dt = 2\pi\int_0^1 \sqrt{t^2+1}\,dt\\ &= 2\pi\left[\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right|\right]_0^1 = \boxed{\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)},\end{aligned}$$

where the integral was computed using the Table of Integrals 47.

65. Using formula (1) from Section 9.6, the surface area S is

$$\begin{aligned}S &= 2\pi\int_0^{\pi/3} 4\sin\theta\sqrt{[4]^2+[0]^2}\,d\theta = 32\pi\int_0^{\pi/3}\sin\theta\,d\theta\\ &= 32\pi\left[-\cos\theta\right]_0^{\pi/3} = \boxed{16\pi}.\end{aligned}$$

AP® Review Problems

1. For the pair of parametric equations in (B), eliminate the parameter t using a Pythagorean Identity.

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y-1}{-2}\right)^2 = 1$$

The rectangular equation represents a circle. In the parametric equations, $-\pi \le t \le \pi$, so the curve begins when $t = -\pi$ at the point $(-2, 1)$, and ends when $t = \pi$ at the point $(-2, 1)$. The curve traces out the circle exactly once.

The pair of equations in (A) traces out the ellipse $(x-2)^2 + \left(\dfrac{y-1}{2}\right)^2 = 1$.

The pair of equations in (C) traces out a circle three times.

The pair of equations in (D) traces out a line segment $y = x$ for $-\pi \le x \le \pi$.

The answer is B.

3. We obtain parametric equations for $r = 2\cos\theta$ by using the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x = r\cos\theta = 2\cos\theta\cos\theta = 2\cos^2\theta \quad \text{and} \quad y = r\sin\theta = 2\cos\theta\sin\theta.$$

Then

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\big(2\cos^2\theta\big) = 4\cos\theta(-\sin\theta) = -4\cos\theta\sin\theta,$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}(2\cos\theta\sin\theta) = (2\cos\theta)(\cos\theta) + (\sin\theta)(-2\sin\theta) = 2\big(\cos^2\theta - \sin^2\theta\big),$$

and $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{2\big(\cos^2\theta - \sin^2\theta\big)}{-4\cos\theta\sin\theta} = \dfrac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta}$.

At $\theta = \dfrac{\pi}{3}$, $\quad \dfrac{dy}{dx} = \dfrac{\left(\sin\frac{\pi}{3}\right)^2 - \left(\cos\frac{\pi}{3}\right)^2}{2\cos\frac{\pi}{3}\sin\frac{\pi}{3}} = \dfrac{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2}{2\cdot\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$.

The answer is C.

5. $x(t) = 2t^2 + 5 \qquad y(t) = 3t - t^3$

Begin by finding the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 3 - 3t^2$$

The curve has a horizontal tangent when $\dfrac{dy}{dt} = 3 - 3t^2 = 0$, but $\dfrac{dx}{dt} \ne 0$.

Note that $3 - 3t^2 = 0$ when $t = -1$ and 1.

When $t = -1, (x, y) = (7, -2)$. When $t = -1, (x, y) = (7, 2)$.

The curve has a vertical tangent when $\dfrac{dx}{dt} = 4t = 0$.

Note that $4t = 0$ when $t = 0$.

When $t = 0, (x, y) = (5, 0)$.

The answer is C.

7. To convert the equation $r\sin\theta = \dfrac{5}{4}$ to rectangular coordinates, use $y = r\sin\theta$ to obtain $y = \dfrac{5}{4}$.

This is a horizontal line $\dfrac{5}{4}$ units above the polar axis.

The answer is B.

9. Obtain parametric equations for $r = 2^{\theta/3}$ by using the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$x = r\cos\theta = 2^{\theta/3}\cos\theta \quad \text{and} \quad y = r\sin\theta = 2^{\theta/3}\sin\theta.$

The answer is A.

11. The region inside the limaçon $r = 2 - \cos\theta$ is swept out by the interval $0 \le \theta \le 2\pi$.

The area of the region inside the limaçon is $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (2 - \cos\theta)^2\, d\theta$.

The region inside the circle $r = \cos\theta$ is swept out by the interval $0 \le \theta \le \pi$.

The interval $0 \le \theta \le \dfrac{\pi}{2}$ sweeps out the upper half of the circle.

The area of the region inside the circle is $A = \dfrac{1}{2}\displaystyle\int_0^{\pi} (\cos\theta)^2\, d\theta = \dfrac{1}{2}\displaystyle\int_0^{\pi} \cos^2\theta\, d\theta$ or, equivalently, $A = 2 \cdot \dfrac{1}{2}\displaystyle\int_0^{\pi/2} \cos^2\theta\, d\theta = \displaystyle\int_0^{\pi/2} \cos^2\theta\, d\theta$.

The area inside the limaçon $r = 2 - \cos\theta$ and outside the circle $r = \cos\theta$ is

$A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (2 - \cos\theta)^2\, d\theta - \displaystyle\int_0^{\pi/2} \cos^2\theta\, d\theta.$

The quantity $\dfrac{1}{2}\displaystyle\int_0^{\pi} \cos^2\theta\, d\theta$ is the area enclosed by a circle of radius $\dfrac{1}{2}$.

So, $\dfrac{1}{2}\displaystyle\int_0^{\pi} \cos^2\theta\, d\theta = \pi\left(\dfrac{1}{2}\right)^2 = \dfrac{\pi}{4}$.

Therefore, the area is also given by $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (2 - \cos\theta)^2\, d\theta - \dfrac{\pi}{4}$.

Neither expression for the area A is equivalent to $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (2 - 2\cos\theta)^2\, d\theta$.

The answer is C.